# TWO APPROACHES TO MIN-MAX FEEDBACK CONTROL AND DESIGN OF UNCERTAIN SYSTEMS

by

İbrahim EKSİN



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C O N T E N T S

## A B S T R A C T

In designing control systems with optimal performance parameter variations pose a great problem. In principle, it is possible to attain an ideal performance by sensing the uncertain parameters and using an adaptive controller. But such controllers are usually difficult or expensive to realize and there is a strong motivation for considering simpler controllers.

The simpler control strategy under parameter uncertainty is the min-max controller which operate the system according to one control law that minimizes the maximum variations from the optima corresponding to each parameter. There exist various methods to find the parameters of the min-max controller.

In this study two approaches have been considered to find the min-max controller parameters: iterative or search procedure, and game theoretic approach.

# CHAPTER I

# INTRODUCTION

## 1.1 PROBLEM STATEMENT

Intelligent design of an optimum control system can be carried out only if the designer knows the dynamic characteristics of the plant or process to be controlled. The extent to which system design can proceed in a logical, systematic, and intelligent manner is, to a considerable degree, directly measured by the knowledge of the process dynamics. Thus, the first thing in control system design is to determine the dynamic characteristics of the process to be controlled. Figure 1-1. below illustrates the block diagram of a plant.

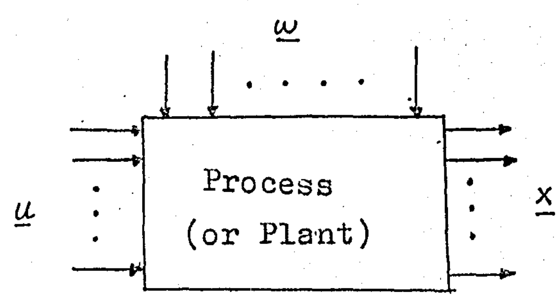



Fig. 1-1. A Multivariable process.

The dynamic characterization of a plant is usually described by a set of differential equations

$$\dot{X}_i = \frac{dX_i(t)}{dt} = f_i\left[X_1(t), X_2(t), \ldots X_n(t); U_1(t), \ldots, U_m(t); W_1(t), \ldots, W_s(t); t\right]$$

$$i = 1, 2, 3, \ldots, n$$

In vector-matrix notation, this set of equations may be written as

$$\dot{\underline{X}}(t) = \frac{d\underline{X}(t)}{dt} = \underline{f}[\underline{X}(t),\underline{U}(t),\underline{W}(t),t]$$

In above equations, the plant is assumed to be of n'th order; $\underline{X}(t)$ is defined as the state vector

$$\underline{X}(t) = [X_1(t),X_2(t),\ldots,X_n(t)]^T$$

whose n components are the state variables; $\underline{U}(t)$ is the m-dimensional control vector

$$\underline{U}(t) = [U_1(t),U_2(t),\ldots,U_m(t)]^T$$

whose m components are control signals; $\underline{W}(t)$ is an s-dimensional vector-valued random function known as the disturbance vector

$$\underline{W}(t) = [W_1(t),W_2(t),\ldots,W_s(t)]$$

whose s components represent the random disturbances, and $\underline{f}$ is a known vector function.

At each moment, the control signals must satisfy the inequalities

$$g_j(U_1,U_2,\ldots,U_m) \leq 0 \qquad j = 1,2,\ldots,m$$

or in vector notation

$$g(\underline{U}) \leq 0 \qquad \text{(where g is a general function)}$$

which reflects the restrictions imposed upon the control system The control vector $\underline{U}$ which satisfies the above inequality is referred to as the admissible control vector. For instance, in many practical situations the values of the control signals can not exceed certain upper bounds because of saturation effects or physical limitations. Under such circumstances, the admissible control signals must satisfy the inequalities

$$|U_j| \leq M_j \qquad j = 1,2,..,m$$

Now, considering the design of an optimum control system, if the designer does not know the dynamic characteristics (parameters) of the plant or process exactly, he can proceed in the following two ways:

A) System Identification

One tries to estimate the parameters of the system (thus identify the system) by gathering some statistical data about the system and using certain techniques (such as Least Square, Maximum Likelihood, Gradient Method, MAP, etc.) Afterwards, the optimum control law is found out in a conventional manner, with known plant dynamics.

B) Min-Max Approach

In this case, the designer considers the system parameters as uncertain over a range. He assumes that the worst will happen; and accordingly, he tries to choose a controller which results in the least drastic effect on the system performance.

In this study the second approach (Min-Max approach) will be used. Just like in parameter estimation there are various methods to solve min-max problem and here in this study the two most common methods will be used to solve it.

The uncertainty in the system parameters may arise because of lack of precise information about a specific system to be controlled or the requirement to determine a fixed controller suitable for use with an ensemble of systems differing in the values by some group of parameters. In such situations two reasonable design criteria can be immediately conjectured: 1) Minimize the maximum deviation from the optimal behaviour; and 2) Minimize the average deviation from optimal behaviour. Either of these criteria might be applied when the designer is attempting to find a single controller for a number of similar plants or when he is attempting to find a controller for a single changeable plant. The first criterion is, of course, the more meaningful when critical tolerances are present, the second, however, would probably find more "production line" use.

Although the methods which will be used in this thesis are applicable to nonlinear systems, here an uncertain linear autonomous system will be considered which may be modelled by the vector differential equation.

$$\dot{\underline{X}}(t) = \underline{\underline{A}}(\underline{v})\underline{X}(t) + \underline{\underline{B}}(\underline{v})\underline{U} \qquad \underline{X}(t_o) = \underline{X}_o$$

where $\underline{X}$ is an nx1 state vector, A is an nxn matrix, B is an nxm matrix, and U is an mx1 control vector, and v is a set of p time-invariant parameters. The uncertainty is introduced by assuming that the actual value of v is not known but may be assumed to lie in the compact; i.e., closed and bounded set v. The performance index to be minimized is

$$J(u,v) = \int_0^{\infty} (\underline{x}^T \underline{\underline{Q}}\, \underline{x} + \underline{u}^T \underline{\underline{P}}\, \underline{u})\, dt$$

where P is a positive definite n x n matrix and Q is a positive semidefinite n x n matrix.

Let $\mathcal{F}$ be the class of all admissible controls and $F^o$ be the subset of the set of all admissible controls which are optimal for some parameter $\underline{v} \in V$.

Then, find a control $u^o$ and a parameter $v^o$ such that

$$\left| J(u^o,v^o) - J(u,v) \right| \leq \max \left| J(u,v^o) - J(u,v) \right| \quad \text{for all } u \in$$

## 1.2 MIN-MAX PHILOSOPHY

A min-max controller is one which is least sensitive to parameter variations, within an assumed range in terms of the changes in the performance index. We know that an optimal control is evaluated at a particular set of parameters, and the variation in parameters requires a fresh determination of the "control law" to ensure optimality. One solution would be to operate the system according to one control law that minimizes the maximum variations from the respective optima; i.e. min-max controller. The philosophy may be explained with the aid of the figure 1-2.

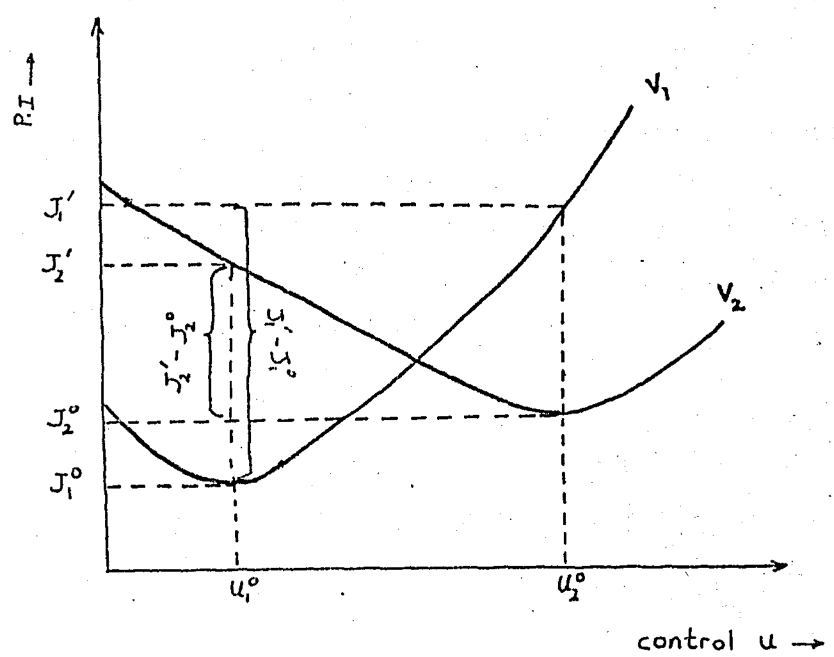



Fig. 1-2. Illustration of min-max concept.

Performance index $J$ is represented as a function of a single variable $u$. But this is adequate for the sake of explanation. Figure 1.2. shows the plots of the cost functionals for two values of parameter $v_1$ and $v_2$. For parameter value $v_1$, the optimal control is $u_1^o$ and the optimal value of the cost functional is $J_1^o$. For parameter value $v_2$, the optimal control is $u_2^o$ and the optimal value of the cost functional is $J_2^o$. If the system initially has the parametric value $v_1$, its control is set at $u_1^o$ resulting in optimal performance. If the parametric value changes to $v_2$, the value of the cost functional would b $J_2'$, the resulting difference from the optimal value being $(J_2' - J_2^o)$. Similarly, if the control is set at $u_2^o$ correspondi to the parameter value $v_2$, and if the parameter assumes a val $v_1$, the resulting perturbation from the optimal value would be $(J_1' - J_1^o)$. As in Fig 1.2. one has

$$J_1' - J_1^o > J_2' - J_2^o \; .$$

That is to say $u_1$ is a better control, since it result in a lesser value of the maximum deviation from the optimum If $v_1$ and $v_2$ are the only parameter values for the system, $u_1^o$ is clearly the min-max control. The argument can be extend to any number of parameter values.

## 1.3 METHODS OF SOLVING MIN-MAX PROBLEMS

The available methods for solving min-max problems ar

1) To find an analytic solution to the maximization step, if it exists, and then minimize

2) To locate a saddle point if it exists and show that it represents a global solution

3) To search or use an iterative procedure

4) To use a game theoretic approach.

Experience has shown that the first two methods are seldom useful in paractical controller design. For instance, the maximizing solution can not be found in a tractable form, or it is not unique and thus, the minimization step is not possible in the first method. In the second method, a saddle point solution does not exist in general. That is why, in this study to find the solution of the min-max problem the last two methods are considered.

A brief overview of the remaining parts of this study could be summarized as follows:

In chapter 2 a brief summary of some important concepts in game theory and their relation to control problems will be given. In chapters 3 and 4 the two methods, namely the game theoretic approach and the grid-point search methods will be introduced, respectively. In chapter 5 three specific problems which are solved by the two methods will be given. These three specific problems involve from one to three parameter uncertainties. Finally, chapter 6 will be a conclusion and comparison chapter.

In this study the numerical solutions of two famous matrix equations were needed. One of them was Algebraic Matrix Riccati Equation (ARE) and the other was Liapunov Equation. In Appendix A Algebraic Riccati Equation (ARE) solution with an example problem is given and in Appendix B Liapunov Equation solution with an example problem is given. In Appendix C the proof of the min-max theorem is given.

# CHAPTER II

# G A M E  T H E O R Y

## 2.1 DISCRETE GAMES AND SOME DEFINITIONS

Elementary game theory is concerned with discrete optimization problems involving two players with conflicting interests. In a typical matrix game there are two players, u and v, and a selection of strategies, $u_i$, i = 1,2,..m and $v_j$, j = 1,2,..n for each player.

DEFINITION:

By a <u>strategy</u> for u in a matrix game it is meant a decision by u to play the various rows with a given probability distribution, say to play row one with probability $P_1$, to play row two with probability $P_2$.

This strategy for u is formally denoted by the probability vector $\underline{P} = (P_1, P_2, P_3, \ldots P_m)$, For example, if the matrix game has two rows and u tosses a coin to decide which row to play, then this strategy is the probability vector $\underline{P} = (1/2 \quad 1/2)$. A strategy which contains a 1 as a component and 0, i.e., where u decides to play a given column is called PURE STRATEGY; otherwise, it is called MİXED STRATEGY.

For each pair of strategies there is a corresponding pay off, $J = L_{ij}$ where $L_{ij}$ are elements of a matrix game.

DEFINITION:

A game is said to be ZERO-SUM GAME if and only if the pay off function $(l_1, l_2, \ldots l_n)$ satisfies

$$\sum_{i=1}^{n} l_i = 0$$

In general, a zero-sum game represents a closed system: everything that one wins must be lost by someone else. Two person zero-sum games are sometimes called strictly competitive games. Player u attempts to minimize his pay off, while v attempts to maximize his pay off. This is a "perfect information game", in the sense that each player has all the information above and that each player knows the other's choice of strategies. Now, if V (the maximizer) plays first he should obviously pick the column with largest minimum since he knows u will subsequently pick the row with the minimum. Similarly, if u (the minimizer) plays first he should pick the row with the smallest maximum, since he knows v will subsequently pick the column with the maximum.

Example 2.1

Consider the following matrix game:

| | | V: $v_1$ | V: $v_2$ | |
|---|---|---|---|---|
| U | $u_1$ | $L_{11}=2$ | $L_{12}=7$ | ← row with smaller maximum |
| U | $u_2$ | $L_{21}=5$ | $L_{22}=9$ | |
| | | | ↑ column with larger minimum | |

Figure 2-1. A simple discrete game with a saddle point.

The optimal choices for the above game are $u_1$ and $v_2$, with payoff 7, regardless of who plays first, i.e., we have

$$\max_{v_j} \min_{u_i} L_{ij} = 7 = \min_{u_i} \max_{v_j} L_{ij}$$

(V plays first) (U plays first)

$$L(u_1,v_j) \leq L(u_1,v_2) \leq L(u_i,v_2)$$

The choice $u_1,v_2$ is called the minimax solution of the game. And the payoff element $L_{12}$ is called <u>saddle point</u>.

However, the choice is not always so simple; suppose that the value of $L_{11}$ is changed from 2 to 11.

Example 2-2.

V is maximizing

| | $v_1$ | $v_2$ |
|---|---|---|
| $u_1$ | $L_{11}=11$ | $L_{12}=7$ |
| $u_2$ | $L_{21}=5$ | $L_{22}=9$ |

U

U is minimizing

Figure 2-2. A discrete game where order of play makes a difference.

Then we have

$$\max_v \min_u = 7 \qquad \min_u \max_v = 9$$

(V plays first) (u plays first)

If v (the maximizer) plays first, he should pick $v_2$, since this is the column with the larger minimum, namely, 7. If u (the minimizer) plays first, he should pick $u_2$, since this is the row with the smaller maximum, namely, 9. Thus, it makes a difference who plays first, and we say that there is not a "saddle point" (or a minimax solution). This dilemna may be resolved by having each side make a random selection of strategies on each play according to some fixed probability. Since we are considering probabilistic strategies, it is clear that we will deal with mixed strategies.

## 2.2 MIXED STRATEGIES

In mixed strategy the strategy should be chosen at random but the randomization scheme should be chosen using a specific technique. Before going on further, let's give the definition of mixed strategy.

DEFINITION:

A mixed strategy for a player is a probability distribution on the set of his pure strategies. For simplicity, let's consider a 2 x 2 matrix game, say

$$\underline{\underline{L}} = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix}$$

Suppose player u adopts the strategy $\underline{P} = (P_1 \quad P_2)$ and player v adopts the strategy $\underline{q} = (q_1 \; q_2)$. Then U plays row 1 with probability $p_1$ and V plays column 1 with probability $q_1$, and so the entry $l_{11}$ will occur with probability $p_1 q_1$. Hence, the expected winning of U is

Expected payoff = $E(p,q) = p_1q_1l_{11} + p_1q_2l_{12} + p_2q_1l_{21} + p_2q_2l_{22}$

$$= \begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \underline{p}\,\underline{\underline{L}}\,\underline{q}^t = \underline{q}\,\underline{\underline{L}}^t\underline{p}$$

Player V will fear that U will discover his choice of strategy. If this should happen, then U will certainly choose p so as to minimize E(p,q); i.e., V's expected <u>gain-floor</u> assuming he uses q, will be

$$V(q) = \min_{p \in P} \underline{p}\,\underline{\underline{L}}\,\underline{q}^t = \min_{p \in P} \underline{q}\,\underline{\underline{L}}^t\underline{p}^t$$

Thus, the minimum will be attained by a pure strategy, i

$$V(q) = \min_i \underline{L}_{i\cdot}\underline{q}^t \quad (\underline{L}_{i\cdot} \text{ is the } i^{th} \text{ row of the matrix } \underline{\underline{L}})$$

Hence, player V should choose q so as to maximize V(q)

$$V_I = \max_{q \in Q} \min_i \underline{L}_{i\cdot}\underline{q}^t$$

One should prove that a maximum exists; however, since Q is compact and the function V(q) is continuous, a maximum exists. Such a $\underline{q}$ is V's max-min strategy.

Similarly, if U chooses p he will obtain the expected <u>loss-ceiling</u>.

$$V(p) = \max_{q \in Q} \underline{p}\,\underline{\underline{L}}\,\underline{q}^t = \max_{q \in Q} \underline{q}\,\underline{\underline{L}}^t\underline{p}^t$$

The maximum will be obtained by a pure strategy, j

$$V(p) = \max_j \underline{p}\,\underline{L}_{\cdot j} \quad (\underline{L}_{\cdot j} \text{ is the } j^{th} \text{ column of the matrix } \underline{\underline{L}}$$

and the player U should choose p so as to obtain

$$V_{II} = \min_{p \in P} \max_{j} \underline{p}\,\underline{L}_{\cdot j}$$

Such a p is U's min-max strategy.

Thus we obtain the two numbers $V_I$ and $V_{II}$. These two numbers are called the values of the game to V and U, respectively.

2.3 THE MIN-MAX THEOREM

It is easily proved that, for any function $F(x,y)$ defined on any cartesian product X x Y

$$\max_{x \in X} \min_{y \in Y} F(x,y) \leq \min_{y \in Y} \max_{x \in X} F(x,y) .$$

Hence we have $V_I \leq V_{II}$

It is very natural that V's gain-floor can not exceed U's loss-ceiling.

THE MIN-MAX THEOREM :

The min-max and the max-min values are equal in expected value basis, i.e.,

$$V_I = V_{II}$$

Proof : Reference 12

Let's try to illustrate this theorem by the example 2.3

V is maximizer

| | $v_1$ | $v_2$ |
|---|---|---|
| $u_1$ | $L_{11}=11$ | $L_{12}=7$ |
| $u_2$ | $L_{21}=5$ | $L_{22}=9$ |

U

U is minimizer

If V plays a fixed choice while U uses a random choice, the expected payoffs for various probability mixes of $U_1$, and $U_2$ is shown in the figure 2.3.

Similarly, if U plays a fixed choice while V uses a random choice, the expected payoffs for various probability mixes of $v_1$ and $v_2$ is shown in the figure 2.3'.

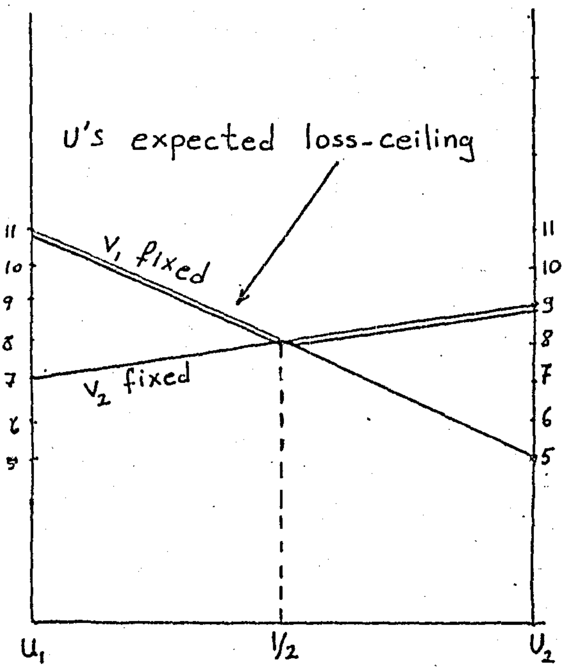



Figure 2.3. min-max-u plays first

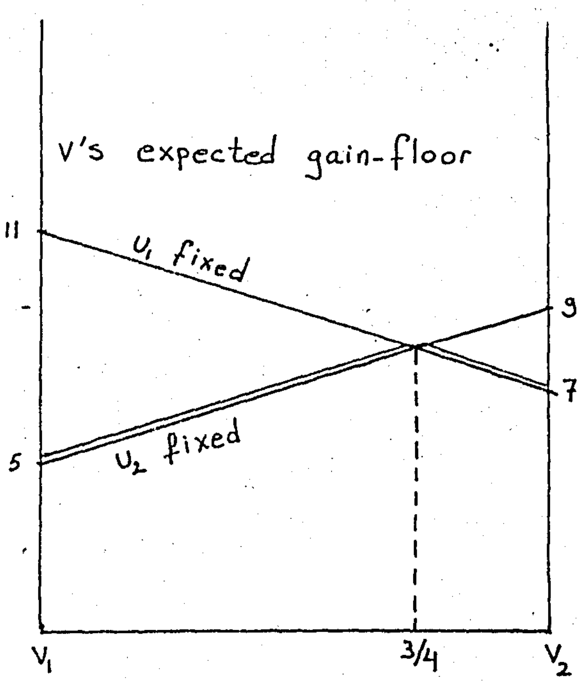



Figure 2.3'. max-min-v plays first

In figure 2.3. if U plays any probability mix other than half of time $u_1$, half of time $u_2$, V can obtain a higher average payoff by playing the fixed strategy indicated by the solid line. Similarly, we see that V must play the probability mix one-quarter of the time $v_1$, three quarters of the time $v_2$ to realize the maximal expected payoff. It is no accident that

$$E \min_p \max_q L_{ij} = 8 = E \max_q \min_p L_{ij}$$

That is, by randomization the difference between minimax and maximin can be equalized on the expected value basis. In this case $p^o = 1/2$, $q^o = 3/4$.

## 2.4 COMPUTATION OF OPTIMAL STRATEGIES

Minimax theorem assures us that every two-person zero-sum game will have optimal strategies, but it does not give a hint as how to compute these optimal strategies.

Saddle points :

The simplest case occurs if a saddle point exists, i.e., if there exists an entry $L_{ij}$ which is both the maximum entry in its column and the minimum in its row. In this case, the pure strategies i and j, or equivalently, the mixed strategies p and q with $p_i = 1$ $q_j = 1$ and all other components equal to zero, will be optimal strategies for players u and v, respectively.

Domination :

In a matrix L, we say the $i^{th}$ row dominates the $k^{th}$ row if

$$L_{ij} \geq L_{kj} \quad \text{for all } j$$

and

$$L_{ij} > L_{kj} \quad \text{for at least one } j$$

Similarly, we say the $j^{th}$ column dominates the $l^{th}$ column if

$$L_{ij} \leq L_{il} \quad \text{for all } i$$

and

$$L_{ij} < L_{il} \quad \text{for at least one } i.$$

THEOREM :

Any optimal strategy for the game obtained by removing the dominated rows (or columns) will also be an optimal strategy for the original game.

2 x 2 Games: Suppose we are given the 2 x 2 matrix game

$$\underline{\underline{L}} = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix}$$

It may be that this game has a saddle point; if so, there is no problem. Suppose, however, that the game has no saddle point. It follows that the optimal strategies $\underline{p} = (p_1\ p_2)$ and $\underline{q} = (q_1\ q_2)$ must have positive components. Now, if the value of the game $(V_I = V_{II} = V)$ is V we have

$$l_{11}p_1q_1 + l_{12}p_1q_2 + l_{21}p_2q_1 + l_{22}p_2q_2 = V \qquad 2.1$$

or

$$p_1(l_{11}q_1 + l_{12}q_2) + p_2(l_{21}q_1 + l_{22}q_2) = V \qquad 2.2$$

The two terms in parenthesis are both less than or equal to V, since $\underline{q}$ is by hypothesis an optimal strategy. Suppose one of them were less than V; i.e., suppose

$$l_{11}q_1 + l_{12}q_2 < V \qquad 2.3$$

$$l_{21}q_1 + l_{22}q_2 \leq V \qquad 2.4$$

Then, since $p_1$ 0 and $p_1+p_2 = 1$ it follows that Eq. 2.2 would be strictly smaller than V. Then, it follows that the terms in parenthesis must be equal to V. Hence

$$\begin{aligned} l_{11}q_1 + l_{12}q_2 &= V \\ l_{21}q_1 + l_{22}q_2 &= V \end{aligned} \quad \text{or} \quad \underline{\underline{L}}\,\underline{q}^t = \begin{bmatrix} V \\ V \end{bmatrix} \qquad 2.5$$

Similarly it can be seen that

$$\begin{aligned} l_{11}p_1 + l_{12}p_2 &= V \\ l_{21}p_1 + l_{22}p_2 &= V \end{aligned} \quad \text{or} \quad \underline{p}\,\underline{\underline{L}} = \begin{bmatrix} V & V \end{bmatrix} \qquad 2.6$$

These equations together with the equations

$$p_1 + p_2 = 1 \quad \text{and} \quad q_1 + q_2 = 1 \qquad 2.7$$

allows us to solve for $\underline{p}$, $\underline{q}$, and V.

THEOREM :

Let $\underline{\underline{L}}$ be a 2 x 2 matrix game. Then, if $\underline{\underline{L}}$ does not have a saddle point, its unique optimal strategies and valve will be given by

$$\underline{q} = \frac{\underline{\underline{L}}^* \underline{J}^t}{\underline{J}\,\underline{\underline{L}}^* \underline{J}^t} \qquad 2.8$$

$$\underline{p} = \frac{\underline{J}\,\underline{\underline{L}}^*}{\underline{J}\,\underline{\underline{L}}^* \underline{J}^t} \qquad 2.9$$

$$V = \frac{|\underline{\underline{L}}|}{\underline{J}\ \underline{\underline{L}}^{*}\ \underline{J}^{t}} \qquad 2.10$$

where $\underline{\underline{L}}^*$ is the adjoint of $\underline{\underline{L}}$, $|\underline{\underline{L}}|$ is the determinant of $\underline{\underline{L}}$, and $\underline{J}$ is the vector $[1 \quad 1]$ .

## 2.5 CONTINUOUS GAMES

If the choices of U and V are continuous instead of discrete, there must be a continuous payoff function, L(u,v) instead of a payoff matrix $L_{ij}$. We look for a pair of choices, $u^o$, $v^o$, such that

$$L(u^o,v) \leq L(u^o,v^o) \leq L(u,v^o) \quad \text{for all } u,v$$

It is claimed that necessary conditions for $u^o$ and $v^o$ are

$$\frac{\partial L}{\partial u} = 0 \quad , \quad \frac{\partial L}{\partial v} = 0 \qquad 2.11$$

$$\frac{\partial^2 L}{\partial u^2} \geq 0 \quad , \quad \frac{\partial^2 L}{\partial v^2} \leq 0 \qquad 2.12$$

and sufficient conditions are Eq. 2.11 and Eq. 2.12 with the equalities changed to inequalities. Any $u^o$, $v^o$ satisfying the sufficient conditions is called a "game-theoretic saddle point". It should be pointed out that the two equations are not equivalent to the usual conditions for a "calculus saddle point", which are

$$\frac{\partial L}{\partial u} = 0 \quad , \quad \frac{\partial L}{\partial v} = 0 \qquad 2.11'$$

$$\frac{\partial^2 L}{\partial u^2}\frac{\partial^2 L}{\partial v^2} - \left(\frac{\partial^2 L}{\partial u \partial v}\right)^2 \leq 0 \qquad 2.12'$$

# CHAPTER III

# G A M E  T H E O R E T I C  A P P R O A C H

## 3.1 FORMULATION IN GAME SETTING

We can view the determination of a control law for on uncertain system as a zero - sum two-person game. The first player is the DESIGNER who must choose a feedback control law which generates the control function based on instantaneous observations of the states of the uncertain system. His opponent, referred to as NATURE, chooses the system parameters. When system performance is measured by a cost functional, the value of the cost functional is viewed as the designer's loss and nature's gain resulting from operation of the system viewed as the play of the game. The min-max criterion by which the control is chosen in a game formulation yields the smallest guaranteed upper bound on the cost. In the game formulation, the criterion used by the controller as a basis for determining the value of control to be applied not knowing $\underline{v}$, is

$$\min_{u \in U} \max_{v \in V} J(\underline{u},\underline{v})$$

After this formulation in game setting the general min-max feedback control problem will be stated and formulated; then the formulas evolued will be applied to the specific case - linear, autonomous uncertain system.

## 3.2 STATEMENT OF THE MIN-MAX FEEDBACK CONTROL PROBLEM

In general, the uncertain dynamical system to be controlled may be modelled by the vector differential equation

$$\dot{\underline{x}}(t) = \underline{f}\,(\,\underline{x}(t),\,\underline{u}(t),\,\underline{v};\,t) \qquad ,\; v \in V \qquad 3.1$$

Here, $\underline{x}(t)$ is an n-vector referred to as the state at time t, $\underline{u}(t)$ represents a vector of m forcing functions, and $\underline{v}$ a set of p time invariant parameters.

The first participant in the game is the DESIGNER who is to choose a control law which generates $\underline{u}(t)$ based on instantaneous observations of the state and time. Such a control is referred to as a memoryless or instantaneous feedback control and may be represented in the form

$$\underline{u}(t) = \underline{k}\,(\underline{x}(t),t) \qquad 3.2$$

Denote by U the set of permissible feedback controls of the form Eq. 3.2 with elements $\underline{u}$ mapping $R^n \times [t_o,T]$ into $R^m \times [t_o,T]$ .

The designer's opponent, referred to as NATURE, chooses the vector of parameters $\underline{v}$ from the set V.

Associated with an initial condition

$$\underline{x}(t_o) = \underline{x}_o \qquad 3.3$$

for the system in Eq. 3.1 is a performance functional, denoted by $J(\underline{u},\underline{v},\underline{x}_o,t_o)$ of the fixed time, free endpoint form

$$J(\underline{u},\underline{v};\underline{x}_o,t_o) = h(\underline{x}(T)) + \int_{t_o}^{T} l(\underline{x}(t),\underline{u}(t),t)dt \qquad 3.4$$

$J(\underline{u},\underline{v},\underline{x}_o,t)$ is viewed as the controller's loss and nature's gain as a result of a play of the game corresponding to operation of the system. The criterion used by the controller, in the game theory formulation, was

$$\min_{\underline{u}\in U}\ \max_{\underline{v}\in V}\ J(\underline{u},\underline{v},\underline{x}_o,t) \qquad 3.5$$

Because a dynamic programming approach will be used, in order to avoid technical difficulties, it is convenient to assume that all functions involved in Eq. 3.1, Eq. 3.2 and Eq. 3.4 are continuous with respect to their arguments.

Note that the initial conditions $\underline{x}_o$, and to appear in Eq. 3.5 as specified constants. Hence, a solution to Eq. 3.5, denoted by the pair $u^o$, $v^o$ will depend parametrically upon $\underline{x}_o$, and $t_o$. From a game viewpoint the initial conditions is treated as having been chosen by a neutral agent. Then Eq. 3.5 corresponds to the case where both the designer and nature base their decision upon knowledge of $\underline{x}_o$.

In seeking to apply techniques of deterministic optimal control to the min-max criterion used as a basis of choosing a control law for uncertain systems in the game setting, the major obstacle to be overcome is that the order of minimization and maximization cannot, in general, be interchanged, i.e., a min-max solution may exist but not a saddle point. In the theory of games (Ch. 2), the introduction of mixed strategies is used to create a saddle point solution.

## 3.3 INTRODUCTION OF MIXED STRATEGIES OVER THE UNCERTAINTY SET

Viewing the direct solution of the min-max problem (Eq. 3.5) as the minimization with respect to u of the functional ∅ defined as

$$\phi(\underline{u}) = \max_{\underline{v} \in V} J(\underline{u},\underline{v}) \qquad 3.6$$

indicates two sources of difficulty. First, evaluation of the functional ∅ by maximization with respect to $\underline{v}$ cannot usually be done in a convenient form. Secondly, if the maximizing $\underline{v}$ is not unique, then ∅ may not differentiable and hence the minimum cannot be characterized as a stationary point of ∅ with respect to $\underline{u}$.

Consider the interchange of the order of the minimization and maximization operations. The max-min problem

$$\max_{\underline{v} \in V} \min_{\underline{u} \in U} J(\underline{u},\underline{v}) \qquad 3.7$$

is readily solved by using deterministic optimal control to minimize with respect to u and, noting the parametric dependence of the optimal control on v, then maximizing over V.

From the game theory we know that

$$\max_{\underline{v} \in V} \min_{\underline{u} \in U} J(\underline{u},\underline{v}) \leq \min_{\underline{u} \in U} \max_{\underline{v} \in V} J(\underline{u},\underline{v}) \qquad 3.8$$

For equality to hold in Eq. 3.8 a saddle point should exist. If it doesn't exist it is generated by introducing mixed strategies defined as the set of probability measures over the pure strategies, which are U and V in our case. Because of the measurability and implementation problems associated with introduction of randomized feedback, we will introduce mixed strategies only over the uncertainty set, V. Let $\hat{V}$ denote the set of all probability measures over the compact set V. Elements in $\hat{V}$ are denoted by $\vartheta$. Note that the pure strategies are a subset of the mixed strategies correcponding to point distributions.

In the case of a finite uncertainty set

$$V = v_i : i = 1, \ldots N \qquad 3.9$$

then the mixed strategies $\hat{V}$ correspond to the simplex in $R^N$ consisting of the set of all probability weights which satisfy the linear constraints.

$$\lambda_i \geq 0 \;, \; i = 1, \ldots . N \qquad \sum_{i=1}^{N} \quad = 1 \qquad 3.10$$

At this point one should state a very important lemma, called as equivalence lemma which states that min-max problem with mixed strategies over the uncertainty set is equivalent to the original min-max problem without mixed strategies.

Equivalence Lemma :

$$\min_{\underline{u} \in U} \max_{\underline{v} \in V} J(\underline{u},\underline{v}) = \min_{\underline{u} \in U} \max_{\underline{\hat{v}} \in V} J(\underline{u},\underline{\hat{v}})$$

Proof : Reference 1

## 3.4 MIN-MAX THEOREM AND SOLUTION OF MODIFIED PROBLEM

The equivalence lemma assures that the min-max problem with mixed strategies over the uncertainty set is equivalent to the original min-max problem. Now a min-max theorem is needed which provides sufficient conditions such that solution of the modified problem

$$\max_{\hat{v} \in \hat{V}} \min_{\underline{u} \in U} J(\underline{u},\underline{\hat{v}})$$

MIN-MAX THEOREM Assume

1) $J(\underline{u},\underline{v})$ is continuous with respect to $\underline{u}$ and $\underline{v}$ for each $\underline{u} \in U$ and $\underline{v} \in V$.

2) V is compact, i.e., closed and bounded.

3) For each $\hat{v} \in \hat{V}$ there exists a unique minimizing $\underline{u} \in U$ for the problem with random parameters

$$\min_{\underline{u} \in U} J(\underline{u},\underline{\hat{v}}) \qquad 3.11$$

denoted by $\underline{u}^{o}(\underline{\hat{v}})$ which depends continuously on $\hat{v}$.

Then there exists a saddle point $\underline{u}^* \in U$ and $\underline{\vartheta}^* \in \hat{V}$ for $J(\underline{u},\underline{\vartheta})$ over $U \times \hat{V}$ and

$$\min_{\underline{u} \in U} \max_{\underline{v} \in V} J(\underline{u},\underline{v}) = \max_{\underline{\vartheta} \in V} \min_{\underline{u} \in U} J(\underline{u},\underline{\vartheta}) \qquad 3.12$$

$$= J(\underline{u}^*,\underline{\vartheta}^*)$$

Proof : Appendix C.

We can now examine the properties of the modified problem

$$\max_{\underline{\vartheta} \in V} \min_{\underline{u} \in U} J(\underline{u},\underline{\vartheta}) \qquad 3.13$$

It is clear that min-max theorem gives sufficient conditions for both existence of a solution to this problem and its equivalence to the min-max problem defined at Eq. 3.5. The dynamic programming viewpoint is applied to determine a necessary condition for a feedback control to solve the problem

$$\min_{\underline{u} \in U} J(\underline{u},\hat{\underline{\vartheta}}) \qquad 3.14$$

viewed as an optimal control problem of minimizing the expected cost for a system with constant random parameters described by the arbitrary but fixed probability distribution $\underline{\vartheta}$. As a result an integro-differential equation which is analogous to the Hamilton-Jacobi equation for the corresponding deterministic control to be a solution of Eq. 3.14. in terms of this equation.

To obtain the necessary condition, we assume existence of a minimizing feedback control $\underline{u}^o \in U$ of the form

$$\underline{u}^o = \dot{\underline{k}}(\underline{x}(t),t) \qquad 3.15$$

for the random parameter expected cost problem for all $t_o \leq t \leq T$ and all $\underline{x}_o$. Next, we define a real valued function $S(\underline{x},t;\underline{v})$ for all $t_o \leq t \leq T$, all $\underline{x}$, and all $\underline{v} \in V$ by

$$S(\underline{x},t;\underline{v}) = J(\underline{u}^o, \underline{v};\underline{x},t) \qquad 3.16$$

The interpretation of $S(\underline{x},t,\underline{v})$ is that it denotes the value of the cost functional resulting from using the feedback control which is optimal for the expected cost, $u^o$, with a system whose parameters are actually equal to $\underline{v}$ over the interval $[t,T]$ and starting at point $\underline{x}$. Note that $S(\underline{x},t,\underline{v})$ does not therefore denote an optimal value function for any system. The optimal value function for the expected cost functional is however given in terms of S by expectation of S, i.e.,

$$\int_V S(\underline{x}_o,t_o;\underline{v})d\hat{\underline{v}} = \min_{\underline{u}\in U} J(\underline{u},\hat{\underline{v}};x_o,t_o) \qquad 3.17$$

This may be verified from Eq. 3.4. and assumption that $\underline{u}^o$ is optimal.

For arbitrary points $t_1$ and $t_2$ satisfying $t_o \leq t_1 \leq t_2 \leq T$ the dynamic programming argument applied to the minimum cost function for the expected cost problem (Eq. 3.16) implies:

$$\int_V S(\underline{x}_1,t_1;\underline{v})d\hat{\underline{v}} = \min_{\underline{k}(\underline{x}(t),t)} \int_V \left\{ \int_{t_1}^{t_2} l(\underline{x},\underline{k})d\tau + S(\underline{x}(t_2),t_2;\underline{v} \right. \qquad 3.18$$

where $\underline{k}(\underline{x}(t),t)$ is written in place $\underline{u}$ to emphasize a feedback control, as indicated in Eq. 3.15, is being sought. $\underline{x}(t_2)$ is the result of the transition of the system from the state $x_1$ at $t_1$ via the equation.

$$\dot{\underline{x}} = f(\underline{x},\underline{k},\underline{v},t) \qquad 3.19$$

Integrating along the trajectory determined by Eq. 3.19 we may write

$$S(x(t_2),t_2;v) = S(\underline{x}_1,t_1,\underline{v}) + \int_{t_1}^{t_2} \left\{ \frac{\partial S}{\partial \underline{x}} \underline{f} + \frac{\partial S}{\partial \underline{t}} \right\} d\tau \qquad 3.20$$

Substituting Eq. 3.20 into Eq. 3.19 we get

$$\int_V S(\underline{x}_1,t_1,\underline{v})d\underline{\hat{v}} = \min_{\underline{k}(\underline{x}(t),t)} \int_V \left\{ S(\underline{x}_1,t_1;\underline{v}) + \int_{t_1}^{t_2} (1+\frac{\partial S}{\partial \underline{x}}\underline{f}+\frac{\partial S}{\partial \underline{t}})d\tau \right\} d\underline{\hat{v}}$$

3.21

From Eq. 3.17 we conclude that the term on the right side of Eq. 3.21 given by

$$S(\underline{x}_1,t_1,\underline{v})d\underline{\hat{v}}$$

is independent of $\underline{k}$ and hence may be cancelled from both sides of Eq. 3.20 to leave

$$0 = \min_{\underline{k}} \int_V \int_{t_1}^{t_2} \left\{ 1 + \frac{\partial S}{\partial \underline{x}} \underline{f} + \frac{\partial S}{\partial t} \right\} d\tau \, d\underline{\hat{v}} \qquad 3.22$$

Under general conditions of integrability the order of integration in Eq. 3.22 may be reversed by Fubini's theorem to obtain

$$0 = \min_{\underline{k}} \int_{t_1}^{t_2} \int_V \left\{ 1 + \frac{\partial S}{\partial \underline{x}} \underline{f} + \frac{\partial S}{\partial t} \right\} d\underline{\hat{v}} \, d\tau \qquad 3.23$$

1, $\underline{f}$, and $\underline{k}$ with respect to all their arguments are continuous which implies the continuity of the integrand with respect to along trajectories associated with the optimal feedback for all $\underline{v}$. This implies continuity with respect to of the inner integral and hence applying mean value theorem to eliminate the integral over $[t_1,t_2]$ we obtain that for some $t \in [t_1,t_2]$

$$0 = \min_{\underline{k}} \int_V \left\{ 1 + \frac{\partial S}{\partial \underline{x}}\underline{f} + \frac{\partial S}{\partial t} \right\} d\underline{\hat{v}} \qquad 3.24$$

Since $t_1$ and $t_2$ were arbitrary, we conclude that Eq. 3.24 is a necessary condition which the optimal control $\underline{u}^o$ must satisfy for all t in $[t_o,T]$

If the expression

$$\int_V \left\{ l(\underline{x},\underline{k},t) + \frac{\partial S}{\partial \underline{x}} (\underline{x},t,\underline{v})\ f(\underline{x},\underline{k},\underline{v},t) \right\} d\underline{\hat{v}} \qquad 3.25$$

which is the expected Hamiltonian has the property of having a unique minimum with respect to $\underline{k}$ for all $\underline{x}$ and all $t \in [t_o,T]$, then the problem is said to be normal in analogy with the deterministic case. Let $\underline{k}^o$ denote the unique minimizer in the normal case. Then by substituting $\underline{k}^o$ into Eq. 3.24 we obtain the analogous of Hamilton-Jacobi equation.

$$\int_V \left\{ l(\underline{x},\underline{k}^o(\underline{x},t)) + \frac{\partial S}{\partial \underline{x}}(\underline{x},t;\underline{v})\ f(\underline{x},\underline{k}^o(\underline{x},t,\underline{v}) + \frac{\partial S}{\partial t}(\underline{x},t,\underline{v}) \right\} d\underline{\hat{v}} = 0 \qquad 3.26$$

which is a integro-differential equation which must be satisfied by $S(\underline{x},t,\underline{v})$

The sufficient condition for a control $\underline{k}^o(\underline{x},t)$ to be optimal can now be stated in terms of Eq. 3.26 in analogy with the Hamilton. Jacobi sufficiency condition for deterministic optimal control problem as follows. If Eq. 3.25 is normal and a unique solution $S(\underline{x},t;\underline{v})$ can be found satisfying the integro-differential equation (Eq.3.26), then a solution to problem

$$\min_{\underline{u} \in U} J(\underline{u},\underline{\hat{v}})$$

is found which is unique except on a set of measure zero.

In the following section these results are applied to the linear system quadratic criterion case where $S(\underline{x},t;\underline{v})$ separates into the quadratic form $\underline{x}^T\ \underline{\underline{R}}(t,\underline{v})\underline{x}$

## 3.5 THE LINEAR SYSTEM, QUADRATIC CASE

We begin by specializing the results of the previous section to the case of a linear system, described by setting

$$f(\underline{x},\underline{u},\underline{v}) = \underline{\underline{A}}(\underline{v})\underline{x} + \underline{\underline{B}}(\underline{v})\underline{u} \qquad 3.27$$

and quadratic criterion given by

$$l(\underline{x},\underline{u}) = 1/2 \ \underline{x}^T \underline{\underline{Q}} \ \underline{x} + 1/2 \ \underline{u}^T \underline{\underline{P}} \ \underline{u} \qquad 3.28$$

where $\underline{\underline{Q}}$ is nonnegative definite and $\underline{\underline{P}}$ is positive definite.

As in the deterministic case assume a quadratic form for S given by

$$S(\underline{x},\underline{v}) = 1/2 \ \underline{x}^T \underline{\underline{R}}(\underline{v})\underline{x} \qquad 3.29$$

Note that the assumptions on $\underline{\underline{Q}}$ and $\underline{\underline{P}}$ imply that $J(\underline{u},\underline{v})$ is nonnegative for all $\underline{v} \in V$, $\underline{u} \in U$, and hence the matrix $\underline{\underline{R}}(\underline{v})$ may be assumed symmetric and nonnegative definite.

On analogy to the deterministic case, the following Riccati integro-differential equation for $\underline{\underline{R}}(\underline{v})$ is obtained, where the bar notation denotes expectation with respect to $\vartheta$ over V.

$$\dot{\underline{\underline{R}}}(\underline{v}) = \underline{\underline{A}}^T\underline{\underline{R}} - \underline{\underline{R}} \ \underline{\underline{A}}^T + (\underline{\underline{B}}^T\underline{\underline{R}})^T \ \underline{\underline{P}}^{-1} \ \overline{\underline{\underline{B}}^T\underline{\underline{R}}} + \overline{\underline{\underline{B}}^T\underline{\underline{R}}} \ \underline{\underline{P}}^{-1}\underline{\underline{B}}^T\underline{\underline{R}} \qquad 3.30$$
$$- \underline{\underline{Q}} - (\overline{\underline{\underline{B}}^T\underline{\underline{R}}})^T \ \underline{\underline{P}}^{-1} \ \overline{\underline{\underline{B}}^T\underline{\underline{R}}} \qquad \underline{v} \in V$$

Minimization of the Hamiltonian yields a unique solution

$$\underline{u}^o = - \underline{\underline{P}}^{-1} \ \overline{\underline{\underline{B}}^T\underline{\underline{R}}} \ \underline{x}(t) \qquad 3.31$$

By normality of the problem, the existence and uniqueness of the minimizing feedback control is equivalent to existence and uniqueness of a solution to Riccati integro-differential equation.

In case of a finite uncertainty, set $\hat{v}$ is represented by probability weights satisfying

$$\lambda_i > 0 \ , \ \sum_{i=1}^{N} \lambda_i = 1$$

Then Riccati - Integro - differential equation (Ex.3.30) reduces to a set of N-coupled Riccati Differential equations with parameters .

Now two theorems about existence and uniqueness of solution will be stated without proof. For proofs one may refer to reference 1.

THEOREM :

The minimum expected cost feedback control for the case of a linear system, quadratic criterion finite uncertainty set problem exists, is unique and depends continuously on $\hat{v}$. It is given by

$$\underline{u}^{o} = - \underline{\underline{P}}^{-1} \overline{\underline{\underline{B}}^{T}\underline{\underline{R}}} \ \underline{x}(t)$$

in terms of a solution of the coupled Riccati-differential equations (Eq. 3.30).

THEOREM :

In the case of a finite uncertainty set, a solution exists to the linear system quadratic criterion min-max control problem. The feedback law and min-max value are given by Eq. 3.31 and Eq. 3.29, respectively, and in terms of the solution of the coupled Riccati differential equations (3.30). The min-max feedback law is linear except for parametric dependence upon initial state $\underline{x}_o$.

In the problem considered in this study

$t_o = 0$ and $T \rightarrow \infty$

and that's why, differential term $\dot{R}(\underline{v})$ drops and Eq. 3.30 becomes simpler, called Algebraic Riccati Equation (ARE).

## 3.6 A SADDLE POINT TEST

In many min-max problems the hypotheses of a min-max theorem providing sufficient conditions for existence of a saddle point cannot be verified. Especially in such situations it should be pointed out that if a max-min solution is obtainable, then it may be possible to apply the following test giving necessary and sufficient conditions for the solution to be a saddle point and hence also a min-max solution.

Saddle Point Test:

Let $\hat{V}_1$ denote an arbitrary subset of $\hat{V}$ and assume $\underline{u}' \in U$ and $\underline{\hat{v}}' \in \hat{V}_1$ are a solution to the max-min problem

$$\max_{\underline{\hat{v}} \in \hat{V}_1} \min_{\underline{u} \in U} J(\underline{u}, \underline{\hat{v}}) \qquad 3.32$$

Then $\underline{u}', \underline{\hat{v}}'$ is a saddle point for $J(\underline{u}, \underline{\hat{v}})$ over $U \times \hat{V}$ and hence also a solution to min-max problem

$$\min_{\underline{u} \in U} \max_{\underline{v} \in V} J(\underline{u}, \underline{v}) \qquad 3.33$$

if and only if

$$\max_{\underline{v} \in V} J(\underline{u}', \underline{v}) = J(\underline{u}', \underline{\hat{v}}') \qquad 3.34$$

PROOF : Reference 1

## 3.7 PROCEDURE FOR SOLVING THE MIN-MAX PROBLEM IN GAME THEORETIC APPROACH

In this thesis study the following procedure has been used to solve the min-max feedback control problem in a game theory setting.

1) Solve the max-min problem without introduction of mixed-strategies.

To solve the max-min problem, one first finds the solution to the deterministic optimal control problem with parameter $\underline{v}$ obtained by solution of the Riccati equation. Then one maximizes the quadratic cost functional J which is given by

$$J = \underline{x}_o^T \underline{R}(\underline{v}) \underline{x}_o \qquad 3.35$$

over the set of parameter $\underline{v}$; and note the max-min value.

2) Apply Saddle-Point Test

a) Stability test

First, to test the condition of the saddle point test, it is noted that for an arbitrary constant feedback matrix $\underline{K}$ such that closed loop system

$$\dot{\underline{x}} = (\underline{A} + \underline{B}\ \underline{K})\ \underline{X} \qquad 3.36$$

is stable. In this case, the quadratic cost is given by Ex. 3.35, but now, R is the symmetric nonnegative definite solution of the equations (Liapunov Equations) below

$$(\underline{A} + \underline{B}\ \underline{K})^T \underline{R} + \underline{R}(\underline{A} + \underline{B}\ \underline{K}) = \underline{Q} - \underline{K}^T \underline{P}\ \underline{K} \qquad 3.37$$

b) In place of $\underline{K}$ matrix in above equation the max-min control vector (or matrix) is replaced, and Liapunov equation is solved. The solution of Liapunov equations depend parametrically on v; and maximization of the cost functional J over the set of parameter v will give us a new value. This new value is compared with the max-min value found at step 1 of the procedure.

3) If the max-min value of step 1 and the new value found at step 2 are equal, it means that there exists a saddle point and max-min value equals min-max value.

If, however, those two values are not equal, it means that mixed strategies should be introduced or as it is done in this study successive search method can be used.

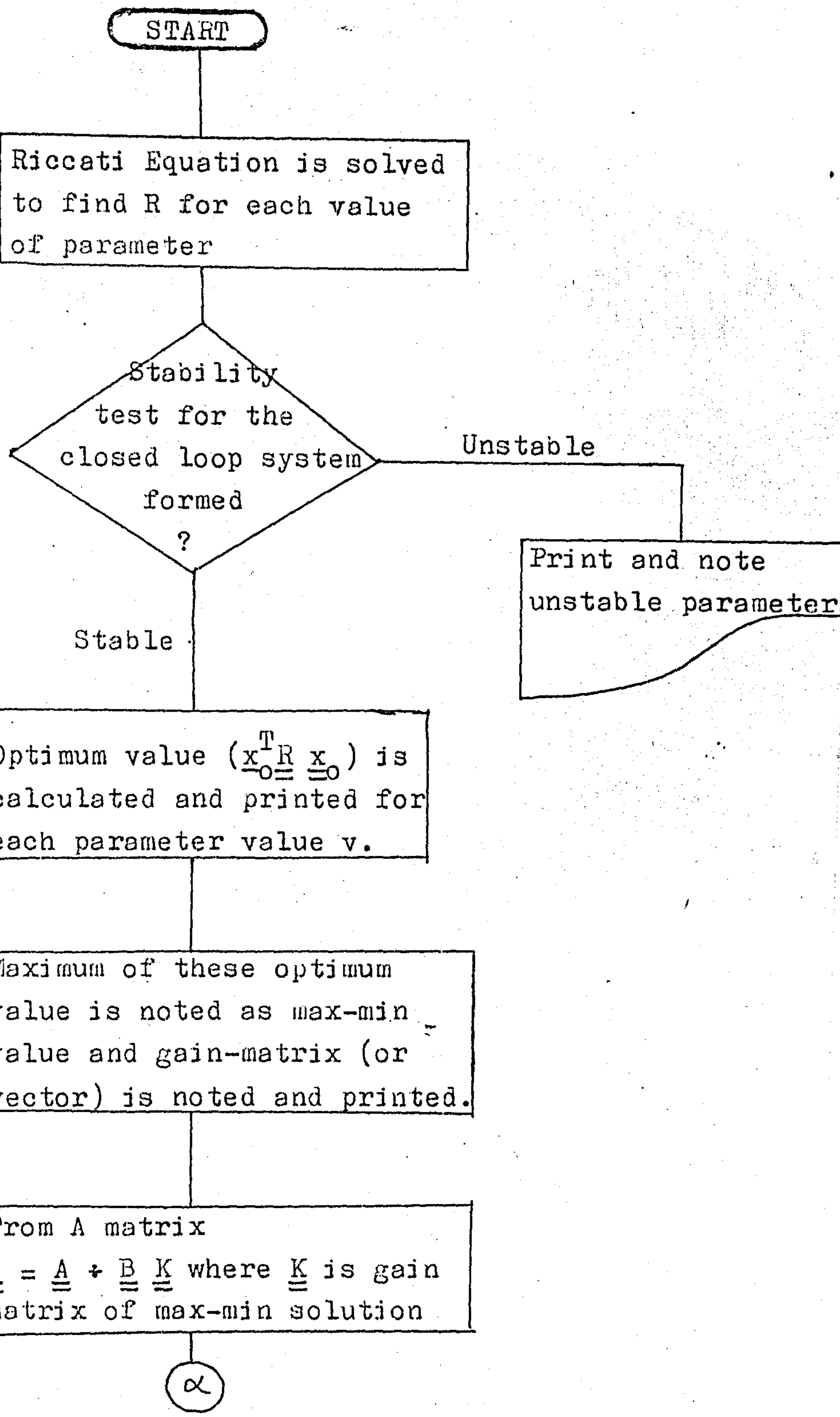
As it is clear from the above procedure the two matrix equations, namely, Algebraic Riccati and Liapunov equations, have to be solved in computer in order to find the max-min solution and make the saddle-point test. Another point which should be pointed out in the above procedure is that the cost functional J in step 2b is again given by

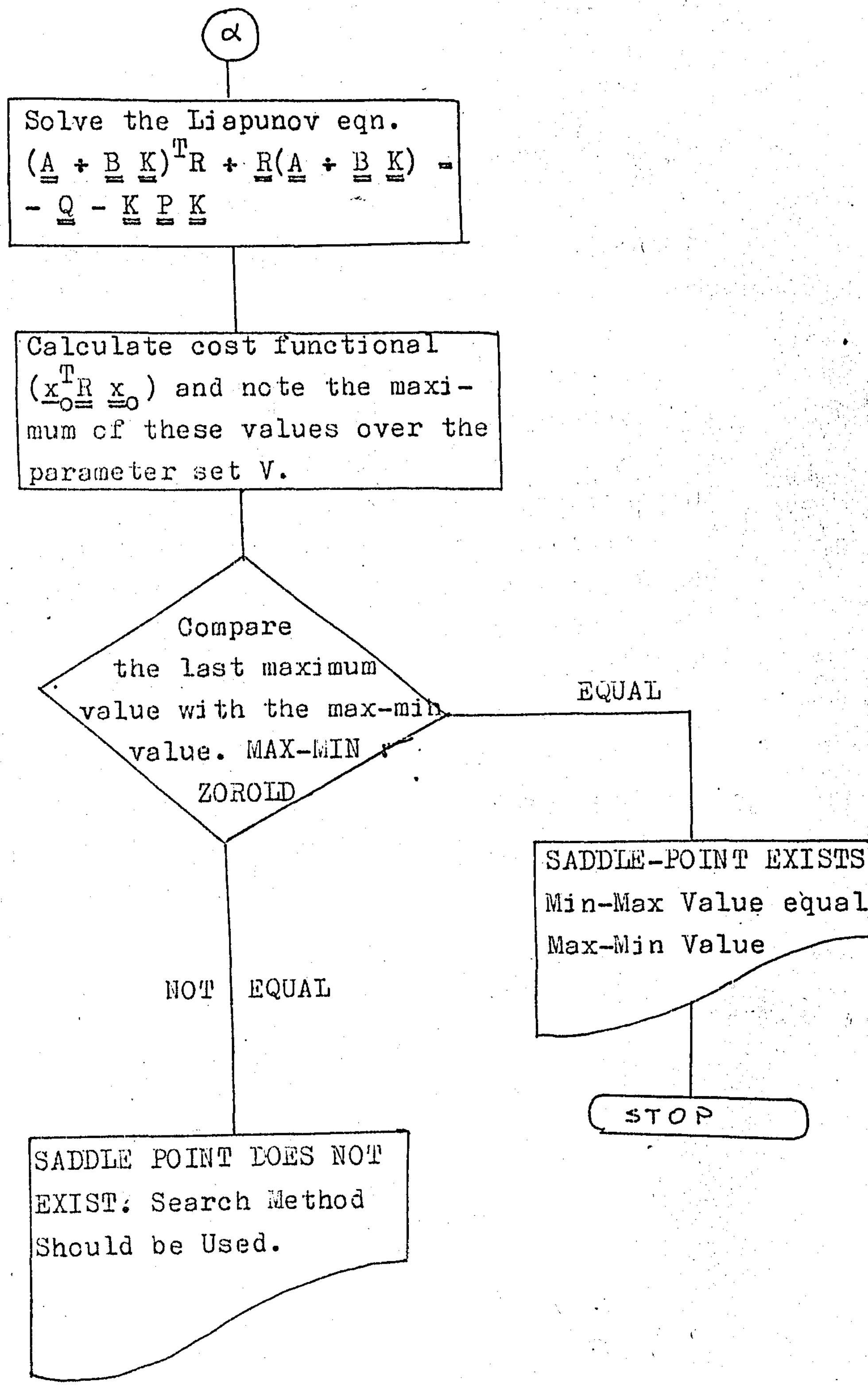
$$J = \underline{x}_0^T \, \underline{\underline{R}} \, \underline{x}$$

as in step 1; however, R, in this case, is the Liapunov equation solution. This point is a claim which needs at least a justification. This justification will be given in the appendix part of this thesis.

Now, a flow-chart of this game theoretic approach method will be given in order to give an overview of the computer program.

## 3.8 FLOW - CHART OF THE METHOD




α
Solve the Liapunov eqn.
(A + B K)^T R + R(A + B K) = - Q - K P K
Calculate cost functional (x_o^T R x_o) and note the maximum of these values over the parameter set V.
Compare the last maximum value with the max-min value. MAX-MIN ← ZOROLD
EQUAL
SADDLE-POINT EXISTS Min-Max Value equal Max-Min Value
STOP
NOT EQUAL
SADDLE POINT DOES NOT EXIST. Search Method Should be Used.

## 3.9 DISCUSSION

The most important aspect of this method is that if there exists a saddle point the solution is found out rather rapidly. For instance, the program runs for 1 minute 38 seconds for one parameter uncertainty (11 grid points), and it runs for 1 minute 42 seconds for two parameter (16 grid points), and it runs for 3 minutes 7 seconds for three parameters (64 grid points). If, however, the saddle point does not exist, then it is faced with a rather difficult problem of introduction of mixed strategies. In this study, this problem has been skipped and search method has been used since there had been a difficulty in choosing the parameter set to introduce mixed strategies. Thus, mixed strategies would be a new area for those who are interested; and hopefully this thesis will be a good reference and hopefully a basis for those people.

# CHAPTER IV

# GRID POINT SEARCH METHOD

## 4.1 PHILOSOPHY OF THE METHOD

One classical approach to the problem of uncertainty in the design of control systems is to introduce a sufficient amount of feedback so as to make the system insensitive to variations in plant parameters or controllers. However, the mere incorporation of negative feedback around the plant does not guarantee for which the controlled quantity is unaffected by the uncertainty.

Since our aim is to design an insensitive system against various variations in plant we have to have a definition of sensitivity. The sensitivity of a control system is usually taken to be the normalized variation of some desired characteristic with the variation of plant or controller parameters. Let us assume that T is desired quantity and v is the parameter; then the sensitive is defined to be

$$S_v^T = \frac{\Delta T}{\Delta v} \frac{v}{T} \qquad 4.1$$

This form of sensitivity gives a measure of the normalized deviation from desired value T. The above description is an absolute sensitivity definition. In this search method the absolute sensitivity measure.has been used. As it is discussed in chapter 1 section 2 it has been searched for $u^o$ and a parameter $v^o$ such that

$$\left|J(u^o,v^o) - J(u,v)\right| \leq \max\left|J(u,v^o) - J(u,v)\right| \text{ for all } u \in F^o \qquad 4.2$$

Since the whole idea behind this method is presented in section 2 of Chapter 1 now the procedure will be considered.

## 4.2 PROCEDURE FOR GRID-POINT SEARCH

Let us assume that there are two parameters $v_1, v_2$ subject to the variations

$$0.8 \leq v_1, v_2 \leq 1.2$$

The two-dimensional parameter space is divided into twenty-five grid points. The spacing is 0.1 on each axis, that is, we will allow for 0.1 changes in the two parameters $v_1, v_2$.

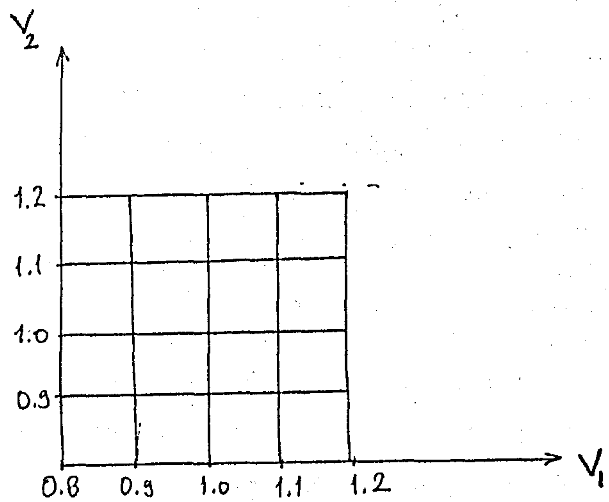


Figure 4.2. Parameter grid

The procedure consists of the following steps.

1) The optimal control law for a particular grid point is evaluated using the principles of optimal control theory. For optimality, u is given as a function of the state variables

$$\underline{u} = - \underline{\underline{P}}^{-1} \underline{\underline{B}}^T \underline{\underline{R}} \underline{x}$$

where $\underline{\underline{R}}$ satisfies the matrix equation (ARE).

$$\underline{\underline{A}}^T \underline{\underline{R}} + \underline{\underline{R}}\,\underline{\underline{A}} - \underline{\underline{R}}\,\underline{\underline{B}}\,\underline{\underline{P}}^{-1} \underline{\underline{B}}^T \underline{\underline{R}} + \underline{\underline{Q}} = 0$$

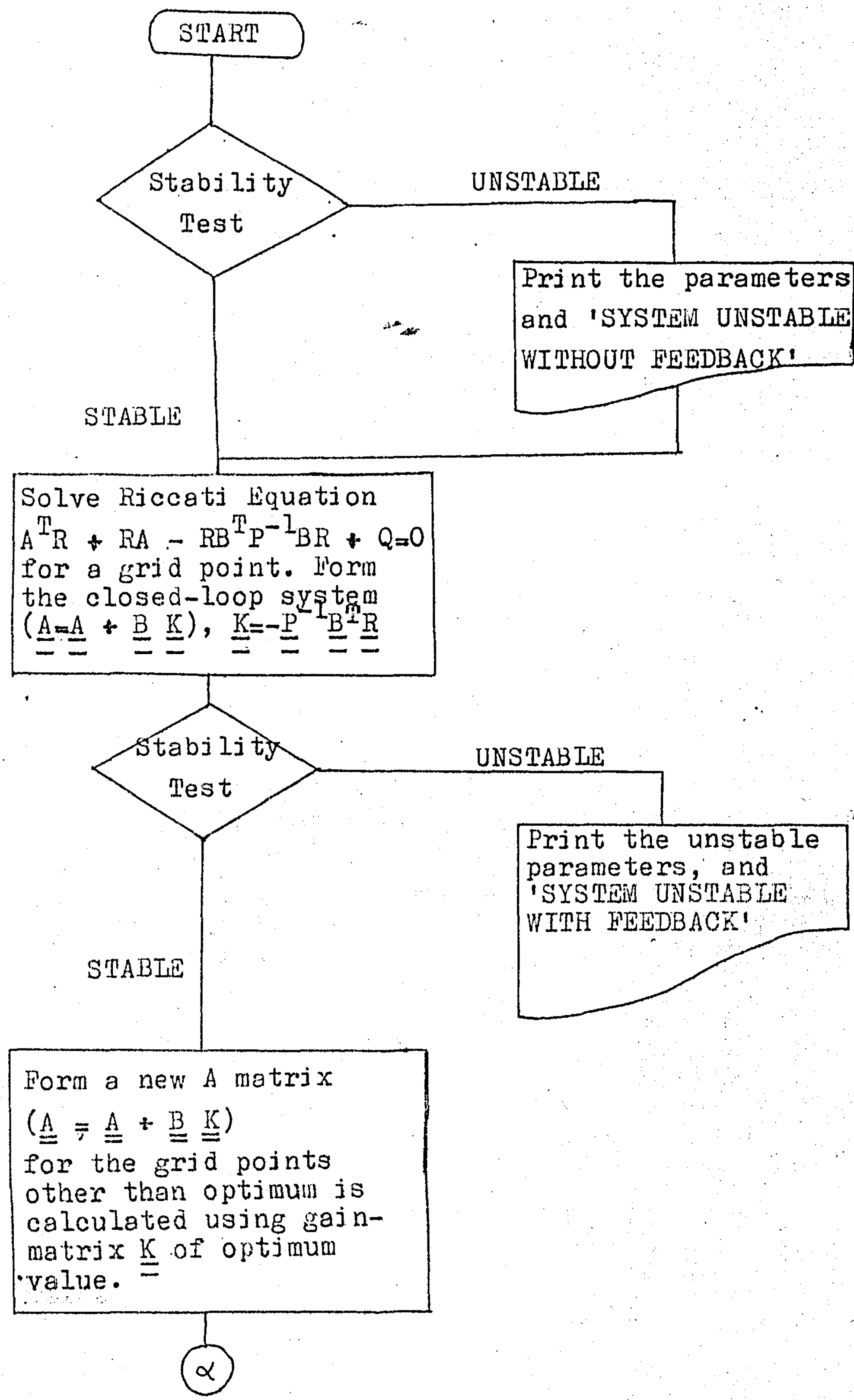
(The equation is algebraic, since the system is autonomous.)

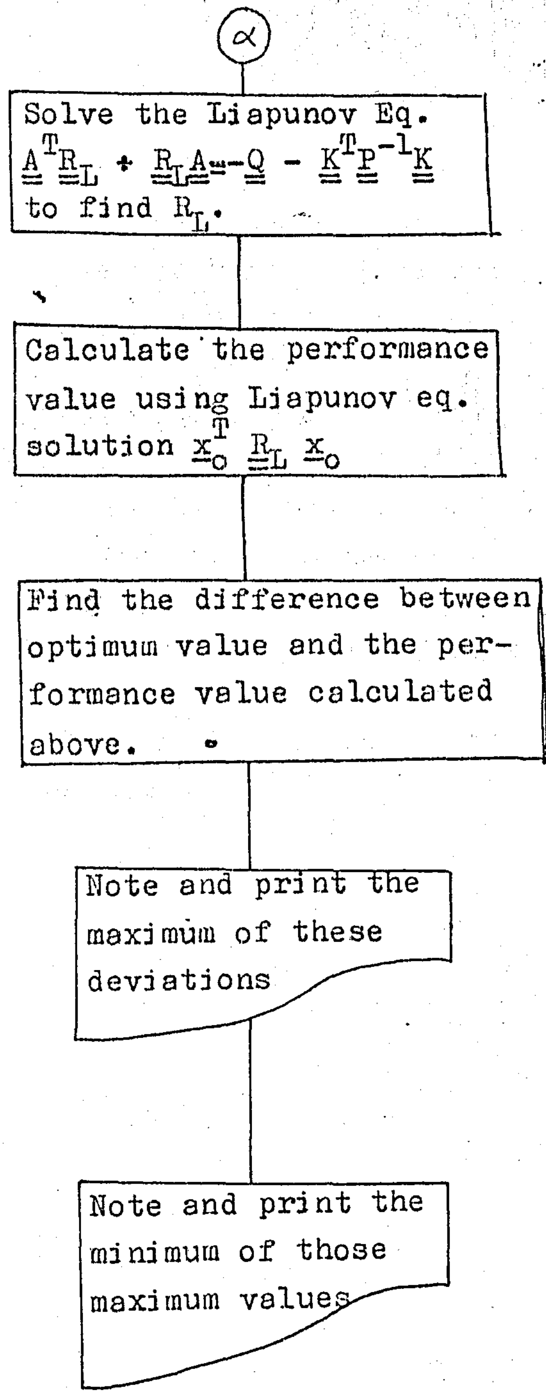
2) Taking the optimal control law for the grid point (1,1), the performance indices at all other points are evaluated (this corresponds $J_2'$ in Fig. 1.2.1) and deviations from their respective optima are calculated (corresponding to the difference $(J_2' - J_2^o)$). The maximum of these deviations is noted. Now taking the optimal control for the grid point (1,2), the calculations are repeated and the maximum deviation is found again. The procedure is repeated for all the other grid points. The control vector that yields the minimum of these maximum deviations is the one that will be chosen for the system, since it ensures the least dractic effects on the performance.

In the study of Sudhakar and Eyman (Ref 2) the procedure was developed for two parameter uncertainty. Here in this study a program has been developed up to five parameter uncertainty case.

The flowchart of the computer program of this method is given on the next page.

## 4.3 FLOW-CHART OF GRID-POINT SEARCH




α
Solve the Liapunov Eq.
$A^T R_L + R_L A = -Q - K^T P^{-1} K$
to find $R_L$.
Calculate the performance value using Liapunov eq. solution $x_o^T R_L x_o$
Find the difference between optimum value and the performance value calculated above.
Note and print the maximum of these deviations
Note and print the minimum of those maximum values

## 4.4 DISCUSSION AND COMPARISON

This grid-point search method is used when the saddle point solution does not exist; since it directly gives the min-max solution. This is the main advantage of this method.

However, one great disadvantage of this method when compared to game theoretic approach is that run-time (CPU) of the computer program of this search method is almost twice of the game theoretic approach method.

For instance, it takes 7 minutes 33 seconds for three parameter uncertainty problem (64 grid points); whereas the same problem is solved in 3 minutes 7 seconds by the game theoretic approach method.

# CHAPTER V

## EXAMPLE PROBLEM SOLUTIONS IN TWO METHODS

### 5.1 ONE PARAMETER UNCERTAINTY CASE

Consider the determination of the min-max feedback control for the uncertain second order stationary linear system described by

$$x + 2w\zeta\dot{x} + w^2 x = u$$

We know from control theory if $0 < \zeta < 1$, the closed-loop poles are complex conjugates and lie in the left-half S-plane. The system is then called underdamped, and transient response is oscillatory.

Suppose that $\zeta = 0.5$, and the variation in the natural frequency term w is as follows

$$0.2 \leq w \leq 2.2$$

with initial conditions

$$x_o = 1 \ , \ \dot{x}_o = 0$$

and the quadratic cost functional

$$J(u,v) = \int_0^\infty (x^2 + \dot{x}^2 + u^2)\, dt$$

In the state-variable formulation corresponding to letting

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

we have

$$\underline{A} = \begin{bmatrix} 0 & 1 \\ -v_1^2 & -v_1 \end{bmatrix} \quad \underline{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad P=1 \quad \underline{x}(0)=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

that is,

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -v_1^2 & v_1 \end{bmatrix}\underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad \underline{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$P.I = \int_0^{\infty} (\underline{x}^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\underline{x} + u^2)\, dt$$

A) SOLUTION OF GAME THEORETIC APPROACH METHOD

MAX-MIN VALUE = 31.6

PARAMETER VALUE = 0.2

SADDLE POINT DOES NOT EXIST

B) SOLUTION OF GRID POINT SEARCH

PARAMETER VALUE = 1.6

5.2 TWO PARAMETER UNCERTAINTY CASE

In this case I have considered a second order plant with two parameter uncertainty described by

Plant:

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -v_1 & -v_2 \end{bmatrix}\underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad , \ \underline{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Performance Index :

$$J(u,v) = \int_0^{\infty} (\underline{x}^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\underline{x} + u^2)\, dt$$

Variation in parameters

$$1.0 \leq v_1, v_2 \leq 2.0$$

grid is set up as in figure 5.2. below

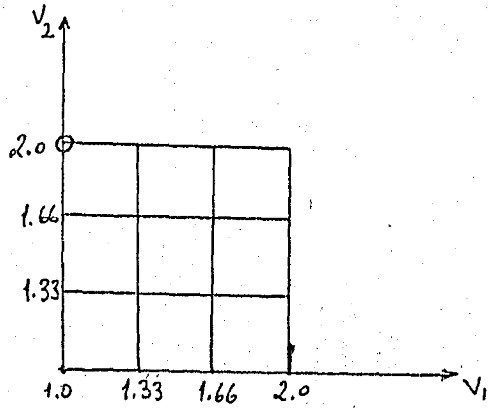



Figure 5.2. Parameter grid.

A) SOLUTION OF GAME THEORETIC APPROACH METHOD

MAX-MIN VALUE = 1.41

PARAMETER VALUES :

$v_1 = 1.0$ , $v_2 = 2.0$

GAIN VECTOR $\underline{K} = \begin{bmatrix} 0.41 \\ -0.41 \end{bmatrix}$

SADDLE POINT DOES NOT EXIST

B) SOLUTION OF GRID-POINT SEARCH METHOD

PARAMETER VALUES

$v_1 = 1.0$ , $v_2 = 2.0$

Eventhough, those values for the parameters coincide in min-max and max-min solution it does not require a saddle-point.

## 5.3 THREE PARAMETER UNCERTAINTY CASE

In this case I have considered again a second order system with three parameter uncertainty which is described by

plant: $$\dot{\underline{x}} = \begin{bmatrix} 0 & v_3 \\ -v_1 & -v_2 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad \underline{x}_o = \underline{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Performance Index given by

$$J(u,v) = \int_0^{\infty} (\underline{x}^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underline{x} + u^2)\, dt$$

and parameters variation

$$1.0 \leq v_1, v_2, v_3 \leq 2.0$$

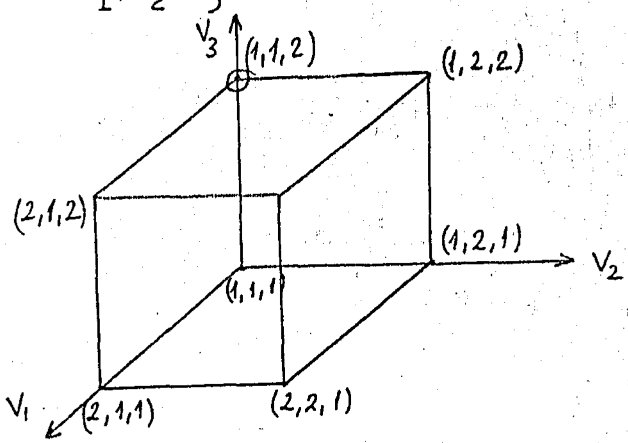



Fig. 5.3 Grid-points for three parameters

Grid is set up as in figure 5.3 above.

A) SOLUTION OF GAME THEORETIC APPROACH METHOD

MAX-MIN VALUE = 1.51

PARAMETER VALUES

$v_1 = 1.0$ , $v_2 = 1.0$ , $v_3 = 2.0$

GAIN VECTOR $\underline{K} = \begin{bmatrix} 0.23 \\ -0.57 \end{bmatrix}$

B) SOLUTION OF GRID-POINT SEARCH METHOD

PARAMETER VALUES

$v_1 = 1.0$ , $v_2 = 1.0$ , $v_3 = 2.0$

# CHAPTER VI

## CONCLUSION AND COMPARISON OF TWO METHODS

This study has revealed that when the saddle point does not exist it is better to apply a search method over the uncertain parameter set; since search methods directly gives the min-max solution. When the saddle point exists, it is wiser to apply the game theoretic approach method. However, if one shows that even after the introduction of the mixed strategies over the uncertain parameter set computer program of the game theoretic approach method takes shorter time than the search method, then it may be concluded that game theoretic approach is preferable for the two cases - saddle - point exist or saddle - point does not exist.

If the system under consideration is a non-linear one, then it is again preferable to use game theoretic approach method since integro-differential equation (Eq. 3.24) which is analogous to Hamilton-Jacobi Equation of Control theory is ready for use. To apply the search method in the case of a nonlinear plant requires further development.

Thus, introducing mixed strategies over the uncertain parameter set in case of nonexistence of saddle-point will be a further study area. The contribution of such a study will be great since it will give an answer to question of computer time after the introduction of mixed strategies.

For both grid-point search or game theoretic approach it is true that as one wants to be more accurate one needs more grids points which in return implies more computer time. In this study no criteria has been developed in choosing the number of grid points. This may also be a further study area together with introduction of mixed strategies over the uncertain parameter set.

It is also true as the number of uncertain parameters increases it becomes harder and harder to be accurate because run-time (CPU) in computer increases very rapidly. In going from 2 parameter uncertainty case to 3 parameter uncertainty case with the same accuracy one needs four times as much computer time, because the number of grid-points increases geometrically.

# APPENDIX A

## NUMERICAL SOLUTION OF ALGEBRAIC MATRIX RICCATI EQUATION

### A.1 METHODS OF SOLVING (ARE).

The quadratic matrix equation

$$\underline{\underline{A}}\,\underline{\underline{X}} + \underline{\underline{X}}\,\underline{\underline{A}}^T - \underline{\underline{X}}\,\underline{\underline{B}}\,\underline{\underline{X}} + \underline{\underline{C}} = \underline{\underline{O}} \qquad \text{Eq.A1}$$

arises in many diverse application - steady state optimal control and filtering, radiative transfer and neutron diffusion, and economic equilibria, to name a few. Eq. A1, widely known as the Algebraic Riccati Equation (ARE), is deceptively simple in appearance; however, considerable effort has gone into development of effective numerical procedures for solving the Riccati equation A1. Upto now six solution methods actually have been used.

(i) Integrate the Riccati differential equation

$$\dot{\underline{x}} = \underline{A}\,\underline{X} + \underline{X}\,\underline{A}' - \underline{X}\,\underline{B}\,\underline{X} + \underline{C} \quad \text{to steady state}$$

(ii) Integrate the "Hamiltonian" system of linear differential equations $\begin{bmatrix} \dot{X} \\ Y \end{bmatrix} = \underline{\underline{\mathcal{H}}} \begin{bmatrix} X \\ Y \end{bmatrix}$

where

$$\underline{\underline{\mathcal{H}}} = \begin{bmatrix} \underline{\underline{A}}^T & -\underline{\underline{B}} \\ -\underline{\underline{C}} & -\underline{\underline{A}} \end{bmatrix}$$

to steady-state (Kalman and Englar (1966))

(iii) Solve the linear system $p(\mathcal{H}) \begin{bmatrix} I \\ X \end{bmatrix} = 0$ where $\mathcal{H}$ is the "Hamiltonian" matrix defined above and p(s) is the polynomial whose zeros are the n eigenvalues of $\mathcal{H}$ having negative real parts (Bucy and Joseph (1968)).

(iv) Transform equation A1 to a (discrete Riccati) quadratic matrix equation and solve that (Hitz and Anderson (1972))

(v) Calculate $\underline{\underline{X}} = \underline{\underline{U}}\, \underline{\underline{V}}^{-1}$ where $\begin{bmatrix} \underline{\underline{U}} \\ \underline{\underline{V}} \end{bmatrix}$ consists of the eigenvectors of , corresponding to the n eigenvalues with negative real parts (Mc Farlane (1963) and Potter (1964)).

(vi) Convert the quadratic matrix problem to a sequence of linear problems $\underline{\underline{A}}\, \underline{\underline{X}} + \underline{\underline{X}}\, \underline{\underline{A}}^T + \underline{\underline{C}} = 0$ where $\underline{\underline{A}} = \underline{\underline{A}} - \underline{\underline{X}}\, \underline{\underline{B}}$ and $\underline{\underline{C}} = \underline{\underline{C}} + \underline{\underline{X}}\, \underline{\underline{B}}\, \underline{\underline{X}}$ and solve these iteratively (Kleinman (1968)).

Recent investigations suggest that method (v), eigenvalue-eigenvector method and method (vi), quasilinearization method, are effective general-purpose algorithms for solving the Riccati Matrix equation A1.

## A.2 SOME PROPERTIES OF SOLUTIONS TO (ARE).

In this section let us quate relevant results concerning properties to the Algebraic Riccati Equation (ARE). The basic existence theorem was established by Kalman. It is given below as theorem A1. First, we restate the Riccati equation A1 in a more suitable form. It is

$$\underline{\underline{A}}\, \underline{\underline{X}} + \underline{\underline{X}}\, \underline{\underline{A}}^T - \underline{\underline{X}}\, \underline{\underline{B}}^T \underline{\underline{B}}\, \underline{\underline{X}} + \underline{\underline{C}}\, \underline{\underline{C}}^T = 0 \qquad \text{Eq. A2}$$

where A,B and C are given matrices - (n x n), (m x n) and (n x 1), respectively, such that $B^T B$ is positive definite and $C C^T$ is nonnegative definite. Depending upon the conditions which A, B and C are required to satisfy there may be a unique solution, a finite number of solutions or infinitely many solutions.

Definition 1 An eigenvalue is Stable if its real part is negative and it is unstable otherwise.

Definition 2 The matrix pair $(A,B)$ is stabilizable if there exists an $X$ such that all eigenvalues of $A - X B^T B$ are stable.

THEOREM A1 A necessary and sufficient condition for the Riccati matrix equation (A2) to possess a unique solution $X$, for which all eigenvalues of $A - X B^T B$ are stable, is that $(A,B)$ be stabilizable and $(C,A)$ be detectable.

Definition 3 An eigenvalue of $A$ is $(A,B)$ - controllable if $X B = 0$ implies $x = 0$ for all (left) eigenvectors of $A$ ; $xA=$ $x$ ; is $(A,B)$ - uncontrollable if $x B = 0$ for a nontrivial eigenvector $x$ of $A$.

Definition 4 The matrix pair $(A,B)$ is controllable if all unstable eigenvalues of $A$ are $(A,B)$ - controllable

Definition 5 The matrix pair $(C,A)$ is detectable if $(A^T, C^T)$ is controllable.

Detectability is thus the dual condition to controllability.

THEOREM A2 A sufficient condition for $(\underline{\underline{A}},\underline{\underline{B}})$ to be stabilizable is that $(\underline{\underline{A}},\underline{\underline{B}})$ be controllable.

In this thesis study I have used method (vi), quasilinearization method of Kleinman (1968).

## A.3 QUASİLİNEARIZATION METHOD

This method is essentially a Newton-raphson procedure. It proceeds as follows

(i) Choose an initial $\underline{\underline{X}}$ such that $\underline{\underline{A}} = \underline{\underline{A}} - \underline{\underline{X}}\,\underline{\underline{B}}$

(ii) Solve the Liapunov matrix equation

$$\underline{\underline{A}}\,\underline{\underline{X}} + \underline{\underline{X}}\,\underline{\underline{A}}^T + \underline{\underline{C}} = 0$$

where $\underline{\underline{C}} = \underline{\underline{C}} + \underline{\underline{X}}\,\underline{\underline{B}}\,\underline{\underline{X}}$ using the previous $\underline{\underline{X}}$.

(iii) If the relative change in $\underline{\underline{X}}$ is sufficiently small and if the error $\underline{\underline{A}}\,\underline{\underline{X}} + \underline{\underline{X}}\,\underline{\underline{A}}^T - \underline{\underline{X}}\,\underline{\underline{B}}\,\underline{\underline{X}} + \underline{\underline{C}}$ is sufficiently small, then exist. Otherwise, return to step (ii).

Notice that, under the assumptions necessary to assure the existence of a unique solution to the Riccati matrix equation, an initial stabilizing $\underline{\underline{X}}$ exists. Moreover, the sequence of solutions to the Liapunov equation in step (ii) converges monotonically to the solution of the Riccati Equation (A1).

To become a competitive numerical method for solving the (ARE), the quasilinearization algorithm requires efficient numerical procedures for stabilization, that is, computing an initial $\underline{\underline{X}}$ in step (i), and for solving the Liapunov matrix equation in step (ii).

The computer program for solving Riccati Matrix equations employs a stabilization procedure as follows. If (A1) is controllable, then an initial stabilizing $\underline{\underline{X}}$ is given by $\underline{\underline{X}} = \underline{\underline{W}}^{-1}\underline{\underline{B}}$ where $\underline{\underline{W}}$ is the "observability" matrix

$$\underline{\underline{W}} = \int_0^T e^{-\underline{\underline{A}}s}\, \underline{\underline{B}}\, e^{-\underline{\underline{A}}^T s}\, ds$$

for some $T > 0$. A Simpson quadrature algorithm is used to calculate $\underline{\underline{W}}$. This involves choosing a stepsize h and approximating the integral by

$$\underline{\underline{W}} = \left(\frac{h}{3}\right)\left[\underline{\underline{B}} + 4\underline{\underline{\Psi}}^T\, \underline{\underline{B}}\,\underline{\underline{\Psi}} + 2(\underline{\underline{\Psi}}^T)^2\underline{\underline{B}}\,\underline{\underline{\Psi}}^2 + 4(\underline{\underline{\Psi}}^T)^3\underline{\underline{B}}\,\underline{\underline{\Psi}}^3 + \ldots\right]$$

where $\underline{\underline{\Psi}} = e^{-\underline{\underline{A}}h}$

This approach requires the user to furnish two parameters T and h, for which he may have little feeling or concern. But the typical valve of h that was found satisfactory to use in the algorithm was

$$h = \frac{1}{20\,|\lambda_{dom}(\underline{\underline{A}})|}$$

where $\lambda_{dom}(\underline{\underline{A}})$ is the dominant eigenvalue of $\underline{\underline{A}}$, i.e.,

$$\lambda_i(\underline{\underline{A}}) \leq \lambda_{dom}(\underline{\underline{A}}) \quad i = 1,2,\ldots n$$

In this study one example problem is solved by hand calculation and computer solution of the same problem is compared with it; it is seen that with this value of h solution converged to at least 6 significant figure accuracy.

## A.4 EXAMPLE SOLUTION OF ALGEBRAIC RICCATI EQUATION (ARE)

The following Algebraic Riccati Equation is considered

$$\underline{\underline{A}}^T\underline{\underline{R}} + \underline{\underline{R}}\,\underline{\underline{A}} - \underline{\underline{R}}\,\underline{\underline{B}}^T\underline{\underline{P}}^{-1}\underline{\underline{B}}\,\underline{\underline{R}} + \underline{\underline{Q}} = 0$$

with

$$\underline{\underline{A}} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \qquad \underline{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad P = 2 \quad \underline{\underline{Q}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and the following result has been got from the computer in 1 minute 11 seconds

$$\underline{\underline{R}} = \begin{bmatrix} 1.420079 & 0.4494897 \\ 0.4494897 & 0.7924826 \end{bmatrix}$$

# APPENDIX B

## NUMERICAL SOLUTION OF THE LIAPUNOV MATRIX EQUATION

### B.1 THE LIAPUNOV MATRIX EQUATION

The Linear Matrix Equation

$$\underline{\underline{A}}\,\underline{\underline{X}} + \underline{\underline{X}}\,\underline{\underline{A}}^T + \underline{\underline{B}} = \underline{\underline{0}} \qquad \text{B1}$$

where $\underline{\underline{A}}$ and $\underline{\underline{B}}$ are given (n x n) matrices and $\underline{\underline{B}}$ is symmetric, is known as the Liapunov matrix equation. Note that $\underline{\underline{X}}$ must be symmetric as well.

A number of procedures have been proposed for solving the Liapunov Matrix Equation (B1). These include direct solution, iterative procedures, finite series approximations, and transformation to various canonical forms. In this study, the following iterative procedure is used.

### B.2 METHOD OF SOLUTION

Let q be a positive parameter, let $\underline{\underline{I}}$ be an (n x n) identity matrix and let

$$\underline{\underline{U}} = (q\,\underline{\underline{I}} - \underline{\underline{A}})^{-1} \qquad \text{B2}$$

$$\underline{\underline{V}} = \underline{\underline{U}}\,(q\underline{\underline{I}} + \underline{\underline{A}}) \qquad \text{B3}$$

$$\underline{\underline{W}} = 2q(\underline{\underline{U}}\,\underline{\underline{B}}\,\underline{\underline{U}}^T) \qquad \text{B4}$$

If all eigenvalues of $\underline{\underline{A}}$ have negative real parts (which means a stable system),

$$\underline{\underline{Y}} = \sum_{i=1}^{L} \underline{\underline{V}}^{i-1} \, \underline{\underline{W}} \, (\underline{\underline{V}}^{i-1})^T \qquad B5$$

converges and is the solution of Eq.(B1). L may go to infinity for exact solution.

The rate of convergence can be improved by using the sequence of partial sums

$$\underline{\underline{Y}}_V = \sum_{i=1}^{2^V} \underline{\underline{V}}^{i-1} \, \underline{\underline{W}} \, (\underline{\underline{V}}^{i-1})^T \qquad B6$$

which can be obtained recursively from

$$\underline{\underline{Y}}_o = \underline{\underline{W}} \qquad B7$$

$$\underline{\underline{Y}}_{V+1} = \underline{\underline{Y}}_V + \underline{\underline{V}}^{2^V} \, \underline{\underline{Y}}_V \, (\underline{\underline{V}}^{2^V})^T \qquad B8$$

Ten application of the Recursion formula B8 yield the first 1024 terms of the series solution B5.

# APPENDIX C

## PROOF OF THE MIN-MAX THEOREM

### C.1 DEFINITIONS OF NOTATION

U denotes an arbitrary set with elements u

V denotes a set from a metric space with elements v

$\mathcal{J}(\underline{u},\underline{v})$ denotes a real valued mupping of UXV into R

$\hat{V}$ denotes the set of probability measures over V with elements $\hat{v}$.

Assuming $J(\underline{u},\underline{v})$ is measurable with respect to elements of $\hat{v}$ then by $J(\underline{u},\underline{\hat{v}})$ we denote

$$J(\underline{u},\underline{\hat{v}}) = \int_V \mathcal{J}(\underline{u},\underline{v})\, d\underline{\hat{v}} \qquad \underline{u} \in U, \underline{\hat{v}} \in \hat{V} \qquad \text{C1}$$

THE MIN-MAX THEOREM Assume

H1 $J(\underline{u},\underline{v})$ is continuous with respect to $\underline{u}$, and $\underline{v}$ for each $\underline{u} \in U$ and $\underline{v} \in V$.

H2 V is compact

H3 For each $\hat{v} \in \hat{V}$ there exists a unique element, denoted $\underline{u}^o(\underline{\hat{v}}) \in U$ such that

$$\min_{\underline{u} \in U} J(\underline{u},\underline{\hat{v}}) = J(\underline{u}^o(\underline{\hat{v}}),\underline{\hat{v}}) \qquad \text{C2}$$

and $\underline{u}^o(\underline{\hat{v}})$ is a continuous mapping from $\hat{V}$ into U.

Then there exists a saddle point $\underline{u} \in U$ and $\underline{\hat{v}} \in \hat{V}$ for $J(\underline{u},\underline{\hat{v}})$ over U x $\hat{V}$ and

$$\min_{\underline{u}\in U}\ \max_{\underline{v}\in V}\ J(\underline{u},\underline{v}) = \max_{\hat{\underline{v}}\in \hat{V}}\ \min_{\underline{u}\in U}\ J(\underline{u},\hat{\underline{v}}) = J(\underline{u}\ ,\hat{\underline{v}}\ ) \qquad C3$$

PROOF : The proof will proceed in three steps. First, we show existence of a pair $\underline{u} \in U$ and $\hat{\underline{v}} \in \hat{V}$ which satisfy

$$\max_{\hat{\underline{v}}\in \hat{V}}\ \min_{\underline{u}\in U}\ J(\underline{u},\hat{\underline{v}}) = J(\underline{u}\ ,\ \hat{v}\ ) \qquad C4$$

Next we show that the pair $\underline{u}$ , $\hat{\underline{v}}$ are a saddle point for $J(\underline{u},\hat{\underline{v}})$ which implies

$$\max_{\hat{\underline{v}}\in \hat{V}}\ \min_{\underline{u}\in U}\ J(\underline{u},\hat{\underline{v}}) = \min_{\underline{u}\in U}\ \max_{\hat{\underline{v}}\in \hat{V}}\ J(\underline{u},\hat{\underline{v}}) \qquad C5$$

Finally, we must show

$$\min_{\underline{u}\in U}\ \max_{\underline{v}\in V}\ J(u,v) = \min_{\underline{u}\in U}\ \max_{\hat{\underline{v}}\in \hat{V}}\ J(\underline{u},\hat{\underline{v}}) \qquad C6$$

We begin with the proof of C4. Define

$$\psi(\hat{\underline{v}}) = \min_{u\in U}\ J(\underline{u},\hat{\underline{v}}) \qquad C7$$

Then $\psi(\hat{\underline{v}})$ is upper semicontinuous with respect to $\hat{\underline{v}}$. The set $\hat{V}$ is compact by H2 and by the following Lemma.

Lemma : Compactness of V implies that $\hat{V}$ is a compact set. Hence $\psi(\hat{\underline{v}})$ achieves a maximum over $\hat{V}$ for some element which we denote by $\hat{\underline{v}}^*$ which satisfies

$$\psi(\hat{\underline{v}}^*) \geqslant \psi(\hat{\underline{v}}) \qquad \hat{\underline{v}} \in \hat{V} \qquad C8$$

By C7,

$$\psi(\underline{\hat{\vartheta}}^*) = \max_{\underline{\hat{\vartheta}} \in V} \min_{\underline{u} \in U} J(\underline{u},\underline{\hat{\vartheta}}) \qquad C9$$

However, by H3 there exists an element $u^o(\hat{\vartheta}^*) \in U$ such that

$$\psi(\underline{\hat{\vartheta}}^*) = \min_{\underline{u} \quad U} J(u,\hat{\vartheta}^*) = J(u^o(\underline{\hat{\vartheta}}^*),\underline{\hat{\vartheta}}^*) \qquad C10$$

Setting $\underline{u}^* = \underline{u}^o(\underline{\hat{\vartheta}})$ C11

and combining C9 and C10 yields C4 .

Next we turn to the proof of C5 . By the definition of a saddle point, C5 is satisfied if existence of a saddle point for $J(\underline{u},\underline{\hat{\vartheta}})$ in U x $\hat{V}$ can be shown. We will show that the pair $\underline{u}^*$ , $\underline{\hat{\vartheta}}^*$ shown to exist and satisfy C4 constitutes such a saddle point. Let

$$J^* = U(\underline{u}^*,\underline{\hat{\vartheta}}^*) \qquad C12$$

Then we must show that

$$J(\underline{u}^*,\underline{\hat{\vartheta}}) \leq J^* \leq J(\underline{u},\hat{\vartheta}^*) \quad \forall\, \underline{u} \in U \text{ and } \underline{\hat{\vartheta}} \in \hat{V} \qquad C13$$

By the definition C1 of $J(\underline{u},\underline{\hat{\vartheta}})$ we have that $J(\underline{u},\underline{\hat{\vartheta}})$ is linear and hence concave with respect to $\underline{\hat{\vartheta}}$ for all $\underline{u} \in U$ Concavity of $J(\underline{u},\underline{\hat{\vartheta}})$ over the convex set $\hat{V}$ implies that for any $0 \leq \xi \leq 1$ all $\underline{u} \in U$ and all $\underline{\hat{\vartheta}} \in \hat{V}$.

$$J(\underline{u},(1-\xi)\underline{\hat{\vartheta}} + \xi\underline{\hat{\vartheta}}) \geq (1-\xi)\psi(\underline{\hat{\vartheta}}^*) + \xi J(\underline{u},\underline{\hat{\vartheta}}) \qquad C14$$

Using C7 and C14 yields

$$J(\underline{u},(1-\xi)\underline{\hat{\vartheta}} + \xi\underline{\hat{\vartheta}}) \geq (1-\xi)\psi(\underline{\hat{\vartheta}}^*) + \xi J(\underline{u},\underline{\hat{\vartheta}}) \qquad C15$$

Replace $\underline{u}$ in C15 by

$$\underline{u}^{o}_{\xi} = u^{o}((1-\xi)\underline{\vartheta}^{*} + \xi\underline{\vartheta}) \qquad C16$$

to obtain

$$\psi((1-\xi)\underline{\vartheta} + \xi\underline{\vartheta}) \geqslant (1-\xi)\,\psi(\underline{\vartheta}^{*}) + \xi J(\underline{u}^{o}_{\xi}, \underline{\vartheta}) \qquad C17$$

From C12 and C8

$$\psi(\underline{\vartheta}^{*}) = J^{*} \geqslant \psi((1-\xi)\,\underline{\vartheta}^{*} + \xi\underline{\vartheta}) \qquad C18$$

and therefore combining C18 with C17

$$J^{*} \geqslant (1-\xi)J + \xi J(\underline{u}^{o}_{\xi}, \underline{\vartheta}) \qquad C19$$

or for $0 \leq \xi \leq 1$

$$\xi\, J(\underline{u}^{o}_{\xi}, \vartheta) \leq \xi J^{*} \qquad C20$$

By continuity, setting $\xi = 0$ in C20 and using C11 yields the left inequality of C13. Setting $\xi = 0$ in C15 and noting C18 yields the right inequality of C13. This completes the proof of C13 which shows that $\underline{u}$, $\underline{\vartheta}$ is a saddle point and hence C5 holds.

To prove C16 we simply note that this corresponds to the equivalence lemma. The proof of this lemma also exists in the work of Howard Blum (R1).

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0123456789012345678901234567890123456789012345678901234567890123456789
EKSIN,111-13-207,THESIS,10,50                  .IBRAHIM EKSIN
    .MAIN
07/27/79-16:03(,0)
          DIMENSION A(10,10),B(10,10),P(10,10),Q(10,10),R(10,10),EI
         1,ABSEIG(10),GAIN(10,10),XIN(10),BARB(10,10),ARB(10,10)
          COMPLEX POL,EIGEN
          READ(5,7)N,M,NN,NPARAM
          DATA V1IN,V2IN,V1FIN,V2FIN/1.0,1.0,2.0,2.0/
          IST=0
          P(1,1)=1.
          Q(1,1)=1.
          Q(1,2)=0.
          Q(2,1)=0.
          Q(2,2)=1.
          XIN(1)=1.
          XIN(2)=0.
        7 FORMAT(4I5)
          NNM1=NN-1
          II=1
          IF(NPARAM.EQ.II)GO TO 701
          II=II+1
          IF(NPARAM.EQ.II)GO TO 702
          II=II+1
          IF(NPARAM.EQ.II)GO TO 703
          II=II+1
          IF(NPARAM.EQ.II)GO TO 704
          DELV5=(V5FIN-V5IN)/NNM1
      704 DELV4=(V4FIN-V4IN)/NNM1
      703 DELV3=(V3FIN-V3IN)/NNM1
      702 DELV2=(V2FIN-V2IN)/NNM1
      701 DELV1=(V1FIN-V1IN)/NNM1
          IC=1
          II=1
          IF(NPARAM.EQ.II)GO TO 801
          II=II+1
          IF(NPARAM.EQ.II)GO TO 802
          II=II+1
          IF(NPARAM.EQ.II)GO TO 803
          II=II+1
          IF(NPARAM.EQ.II)GO TO 804
          DO 588 MCEK=1,NN
      804 CONTINUE
          DO 688 LCEK=1,NN
```

```
  803 CONTINUE
      DO 788 KCEK=1,NN
  802 CONTINUE
      DO 888 JCEK=1,NN
  801 CONTINUE
      DO 999 ICEK=1,NN
  666 IF(IC.EQ.2)GO TO 552
      II=1
      IF(NPARAM.EQ.II)GO TO 901
      II=II+1
      IF(NPARAM.EQ.II)GO TO 902
      II=II+1
      IF(NPARAM.EQ.II)GO TO 903
      II=II+1
      IF(NPARAM.EQ.II)GO TO 904
      DO 577 MM=1,NN
  904 CONTINUE
      DO 677 L=1,NN
  903 CONTINUE
      DO 777 K=1,NN
  902 CONTINUE
      DO 877 J=1,NN
  901 CONTINUE
      DO 977 I=1,NN
      II=1
      IF(NPARAM.NE.II)GO TO 401
      IF(I.EQ.ICEK)GO TO 556
  401 II=II+1
      IF(NPARAM.NE.II)GO TO 402
      IF((I.EQ.ICEK).AND.(J.EQ.JCEK))GO TO 557
  402 II=II+1
      IF(NPARAM.NE.II)GO TO 403
      IF((I.EQ.ICEK).AND.(J.EQ.JCEK).AND.(K.EQ.KCEK))GO TO 558
  403 II=II+1
      IF(NPARAM.NE.II)GO TO 404
      IF((I.EQ.ICEK).AND.(J.EQ.JCEK).AND.(K.EQ.KCEK).AND.(L.EQ.LC
     1TO 559
  404 II=II+1
      IF(NPARAM.NE.II)GO TO 977
      IF((I.EQ.ICEK).AND.(J.EQ.JCEK).AND.(K.EQ.KCEK).AND.(L.EQ.LC
     1.(MM.EQ.MCEK))GO TO 560
  977 CONTINUE
  877 CONTINUE
  777 CONTINUE
  677 CONTINUE
  577 CONTINUE
  560 V5=V5IN+(MM-1)*DELV5
  559 V4=V4IN+(L-1)*DELV4
  558 V3=V3IN+(K-1)*DELV3
  557 V2=V2IN+(J-1)*DELV2
  556 V1=V1IN+(I-1)*DELV1
C   SYSTEM DESCRIBTION--- A,AND B MATRICES
      A(1,1)=0.
      A(2,1)=-V1**2
      A(2,2)=-V1
      A(1,2)=1.
      B(1,1)=0.
      B(2,1)=1.
      CALL EIGVAL(A,BB,C,N,POL,EIGEN)
      DO 201 J=1,N
      IF(REAL(EIGEN(J)).GT.0.)GO TO 58
```

```
      IF(NPARAM.EQ.II)GO TO 987
      V1=V1IN
      I=1
      V2=V2+DELV2
      J=J+1
      IF(J.LE.NN)GO TO 85
      II=II+1
      IF(NPARAM.EQ.II)GO TO 887
      V1=V1IN
      V2=V2IN
      J=1
      I=1
      V3=V3+DELV3
      K=K+1
      IF(K.LE.NN)GO TO 85
      II=II+1
      IF(NPARAM.EQ.II)GO TO 787
      V3=V3IN
      V2=V2IN
      V1=V1IN
      K=1
      J=1
      I=1
      V4=V4+DELV4
      L=L+1
      IF(L.LE.NN)GO TO 85
      II=II+1
      IF(NPARAM.EQ.II)GO TO 687
      V4=V4IN
      V3=V3IN
      V2=V2IN
      V1=V1IN
      L=1
      K=1
      J=1
      I=1
      V5=V5+DELV5
      MM=MM+1
      IF(MM.LE.NN)GO TO 85
      IF(ICEK.EQ.1.AND.JCEK.EQ.1.AND.KCEK.EQ.1.AND.LCEK.EQ.1.AND
     1.1)XMINMA=PEROLD
      IF(XMINMA.LT.PEROLD)GO TO 988
      XMINMA=PEROLD
      V1=V1IN+(ICEK-1)*DELV1
      V2=V2IN+(JCEK-1)*DELV2
      V3=V3IN+(KCEK-1)*DELV3
      V4=V4IN+(LCEK-1)*DELV4
      V5=V5IN+(MCEK-1)*DELV5
      PRINT*,,COORDINATES OF SUBOPTIMAL GRID POINT,
      PRINT*,ICEK,JCEK,KCEK,LCEK,MCEK
      PRINT*,,VALUES OF THE PARAMETERS FOR SUBOPTIMAL GRID POINT
      PRINT*,V1,V2,V3,V4,V5
      GO TO 988
  687 IF(ICEK.EQ.1.AND.JCEK.EQ.1.AND.KCEK.EQ.1.AND.LCEK.EQ.1)XMI
     1OLD
      IF(XMINMA.LT.PEROLD)GO TO 988
      XMINMA=PEROLD
      V1=V1IN+(ICEK-1)*DELV1
      V2=V2IN+(JCEK-1)*DELV2
      V3=V3IN+(KCEK-1)*DELV3
      V4=V4IN+(LCEK-1)*DELV4
```

```
      PRINT*,,COORDINATES OF SUBOPTIMAL GRID POINT,
      PRINT*,ICEK,JCEK,KCEK,LCEK
      PRINT*,,VALUES OF THE PARAMETERS FOR SUBOPTIMAL GRID POINT,
      PRINT*,V1,V2,V3,V4
      GO TO 988
  787 IF(ICEK.EQ.1.AND.JCEK.EQ.1.AND.KCEK.EQ.1)XMINMA=PEROLD
      PRINT*,,MINMAX=,,XMINMA
      IF(XMINMA.LT.PEROLD)GO TO 988
      XMINMA=PEROLD
      V1=V1IN+(ICEK-1)*DELV1
      V2=V2IN+(JCEK-1)*DELV2
      V3=V3IN+(KCEK-1)*DELV3
      PRINT*,,COORDINATES OF SUBOPTIMAL GRID POINT,
      PRINT*,ICEK,JCEK,KCEK
      PRINT*,,VALUES OF THE PARAMETERS FOR SUBOPTIMAL GRID POINT,
      PRINT*,V1,V2,V3
      GO TO 988
  887 IF(ICEK.EQ.1.AND.JCEK.EQ.1)XMINMA=PEROLD
      IF(XMINMA.LT.PEROLD)GO TO 988
      XMINMA=PEROLD
      V1=V1IN+(ICEK-1)*DELV1
      V2=V2IN+(JCEK-1)*DELV2
      PRINT*,,COORDINATES OF SUBOPTIMAL GRID POINT,
      PRINT*,ICEK,JCEK
      PRINT*,,VALUES OF THE PARAMETERS FOR SUBOPTIMAL GRID POINT,
      PRINT*,V1,V2
      GO TO 988
  987 IF(ICEK.EQ.1)XMINMA=PEROLD
      IF(XMINMA.LT.PEROLD)GO TO 988
      XMINMA=PEROLD
      V1=V1IN+(ICEK-1)*DELV1
         PRINT*,,COORDINATES OF SUBOPTIMAL GRID POINT,
      PRINT*,ICEK
      PRINT*,,VALUES OF THE PARAMETERS FOR SUBOPTIMAL GRID POINT,
      PRINT*,V1
  988 IC=1
  999 CONTINUE
  888 CONTINUE
  788 CONTINUE
  688 CONTINUE
  588 CONTINUE
      END




      SUBROUTINE RICATI(A,B,P,Q,N,M,H,R,GAIN,BARB)
      DIMENSION A(10,10),B(10,10),P(10,10),Q(10,10),EXPA(10,10),
     1),V(10),JC(10),BSQ(10,10),ASQ(10,10),QSQ(10,10),RB(10,10),
     1,10),AT(10,10),ERR(10,10),ET(10,10),R(10,10),ARB(10,10),
     1ARBY(10,10),EXPAT(10,10),BARB(10,10),GAIN(10,10)
      V(1)=1.
      CALL GJR(P,10,10,M,M,$50,JC,V)
      DO 181 I=1,N
      DO 181 J=1,M
  181 BARB(I,J)=B(I,J)
      CALL MXMLT(B,P,ARB,N,M,M,10,10)
```

```
SIN,111-13-207,THESIS,10,50                    .IBRAHIM EKSIN
   .MAIN
17/27/79-16:03(,0)
      DIMENSION A(10,10),B(10,10),P(10,10),Q(10,10),R(10,10),FTC
     1,ABSEIG(10),GAIN(10,10),XIN(10),BARB(10,10),ARB(10,10)
      COMPLEX POL,EIGEN
      READ(5,7)N,M,NN,NPARAM
      DATA V1IN,V2IN,V1FIN,V2FIN/1.0,1.0,2.0,2.0/
      IST=0
      P(1,1)=1.
      Q(1,1)=1.
      Q(1,2)=0.
      Q(2,1)=0.
      Q(2,2)=1.
      XIN(1)=1.
      XIN(2)=0.
    7 FORMAT(4I5)
      NNM1=NN-1
      II=1
      IF(NPARAM.EQ.II)GO TO 701
      II=II+1
      IF(NPARAM.EQ.II)GO TO 702
      II=II+1
      IF(NPARAM.EQ.II)GO TO 703
      II=II+1
      IF(NPARAM.EQ.II)GO TO 704
      DELV5=(V5FIN-V5IN)/NNM1
  704 DELV4=(V4FIN-V4IN)/NNM1
  703 DELV3=(V3FIN-V3IN)/NNM1
  702 DELV2=(V2FIN-V2IN)/NNM1
  701 DELV1=(V1FIN-V1IN)/NNM1
      IC=1
      II=1
      IF(NPARAM.EQ.II)GO TO 801
      II=II+1
      IF(NPARAM.EQ.II)GO TO 802
      II=II+1
      IF(NPARAM.EQ.II)GO TO 803
      II=II+1
      IF(NPARAM.EQ.II)GO TO 804
      DO 588 MCEK=1,NN
  804 CONTINUE
      DO 688 LCEK=1,NN
```

```
  803 CONTINUE
      DO 788 KCEK=1,NN
  802 CONTINUE
      DO 888 JCEK=1,NN
  801 CONTINUE
      DO 999 ICEK=1,NN
      II=1
      IF(NPARAM.EQ.II)GO TO 901
      II=II+1
      IF(NPARAM.EQ.II)GO TO 902
      II=II+1
      IF(NPARAM.EQ.II)GO TO 903
      II=II+1
      IF(NPARAM.EQ.II)GO TO 904
      V5=V5IN+(MCEK-1)*DELV5
  904 V4=V4IN+(LCEK-1)*DELV4
  903 V3=V3IN+(KCEK-1)*DELV3
  902 V2=V2IN+(JCEK-1)*DELV2
  901 V1=V1IN+(ICEK-1)*DELV1
C   SYSTEM DESCRIBTION--- A,AND B MATRICES
      A(1,1)=0.
      A(1,2)=1.
      A(2,1)=-V1**2
      A(2,2)=-V1
      B(1,1)=0.
      B(2,1)=1.
      CALL EIGVAL(A,BB,C,N,POL,EIGEN)
      DO 201 J=1,N
      IF(REAL(EIGEN(J)).GT.0.)GO TO 58
  201 CONTINUE
      GO TO 59
   58 PRINT*,,SYSTEM IS UNSTABLE FOR THESE PARAMETERS WITHOUT FEE
      IST=1
   59 CONTINUE
      DO 55 I=1,N
   55 ABSEIG(I)=CABS(EIGEN(I))
      BIGEIG=ABSEIG(1)
      DO 56 I=2,N
   56 BIGEIG=AMAX1(ABSEIG(I),BIGEIG)
      H=1./(20.*BIGEIG)
      CALL RICATI(A,B,P,Q,N,M,H,R,GAIN,BARB)
      IF(IST.EQ.1)GO TO 16
      GO TO 69
   16 CALL MXMLT(BARB,GAIN,ARB,N,M,N,10,10)
      CALL MXADD(A,ARB,A,N,N,10)
      CALL EIGVAL(A,BB,C,N,POL,EIGEN)
      IST=0
      DO 17 I=1,N
      IF(REAL(EIGEN(I)).GT.0.)GO TO 68
   17 CONTINUE
      GO TO 69
   68 PRINT*,,SYSTEM UNSTABLE FOR BELOW PARAMETERS WITH STATE FEE
      PRINT*,,PARAMETER 1---PARAMETER 2---PARAMETER 3--PARAMETER
      PRINT*,V1
   69 CONTINUE
      CALL OPTIMA(XIN,R,N,M,OPT)
      PRINT*,,OPTIMUM VALUE=,,OPT
      II=1
      IF(NPARAM.EQ.II)GO TO 987
      II=II+1
      IF(NPARAM.EQ.II)GO TO 887
```

```
      II=II+1
      IF(NPARAM.EQ.II)GO TO 787
      II=II+1
      IF(NPARAM.EQ.II)GO TO 687
      IF(ICEK.EQ.1.AND.JCEK.EQ.1.AND.KCEK.EQ.1.AND.LCEK.EQ.1.AND.M
     1.1)GO TO 500
      GO TO 600
  687 IF(ICEK.EQ.1.AND.JCEK.EQ.1.AND.KCEK.EQ.1.AND.LCEK.EQ.1)GO TO
      GO TO 600
  787 IF(ICEK.EQ.1.AND.JCEK.EQ.1.AND.KCEK.EQ.1)GO TO 500
      GO TO 600
  887 IF(ICEK.EQ.1.AND.JCEK.EQ.1)GO TO 500
      GO TO 600
  987 IF(ICEK.EQ.1)GO TO 500
  600 CONTINUE
      IF(OPT.LT.OPTOLD)GO TO 999
      IF(OPT.EQ.OPTOLD)GO TO 988
  500 OPTOLD=OPT
      DO 301 I=1,M
      DO 301 J=1,N
  301 OPGAIN(I,J)=GAIN(I,J)
      II=1
      IF(NPARAM.EQ.II)GO TO 501
      II=II+1
      IF(NPARAM.EQ.II)GO TO 502
      II=II+1
      IF(NPARAM.EQ.II)GO TO 503
      II=II+1
      IF(NPARAM.EQ.II)GO TO 504
      V5OPT=V5
  504 V4OPT=V4
  503 V3OPT=V3
  502 V2OPT=V2
  501 V1OPT=V1
      PRINT*,,MAXIMIN SOL-OPTOLD=,,OPTOLD
      PRINT*,,GAIN VECTOR FOR MAXMIN SOL,
      PRINT*,((GAIN(I,J),J=1,N),I=1,M)
      PRINT*,,PARAMETERS FOR MAXMIN SOL,,V1
      GO TO 999
  988 PRINT*,,THERE EXISTS ANOTHER MAXMIN SOL,,V1,V2
  999 CONTINUE
  888 CONTINUE
  788 CONTINUE
  688 CONTINUE
  588 CONTINUE
      II=1
      IF(NPARAM.EQ.II)GO TO 601
      II=II+1
      IF(NPARAM.EQ.II)GO TO 602
      II=II+1
      IF(NPARAM.EQ.II)GO TO 603
      II=II+1
      IF(NPARAM.EQ.II)GO TO 604
      DO 188 MK=1,NN
  604 CONTINUE
      DO 288 LK=1,NN
  603 CONTINUE
      DO 388 KK=1,NN
  602 CONTINUE
      DO 488 JK=1,NN
  601 CONTINUE
```

```
      DO 499 IK=1,NN
      II=1
      IF(NPARAM.EQ.II)GO TO 101
      II=II+1
      IF(NPARAM.EQ.II)GO TO 102
      II=II+1
      IF(NPARAM.EQ.II)GO TO 103
      II=II+1
      IF(NPARAM.EQ.II)GO TO 104
      V5=V5IN+(MK-1)*DELV5
  104 V4=V4IN+(LK-1)*DELV4
  103 V3=V3IN+(KK-1)*DELV3
  102 V2=V2IN+(JK-1)*DELV2
  101 V1=V1IN+(IK-1)*DELV1
C     SYSTEM DESCRIBTION--- A,AND B MATRICES
      A(1,1)=0.
      A(1,2)=1.
      A(2,1)=-V1**2
      A(2,2)=-V1
      B(1,1)=0.
      B(2,1)=1.
      CALL PERFOR(A,B,OPGAIN,P,Q,XIN,N,M,ZOR)
      II=1
      IF(NPARAM.EQ.II)GO TO 487
      II=II+1
      IF(NPARAM.EQ.II)GO TO 387
      II=II+1
      II=II+1
      IF(NPARAM.EQ.II)GO TO 287
      IF(NPARAM.EQ.II)GO TO 187
      IF(IK.EQ.1.AND.JK.EQ.1.AND.KK.EQ.1.AND.LK.EQ.1.AND.MK.EQ.1)(
     1400
      GO TO 399
  187 IF(IK.EQ.1.AND.JK.EQ.1.AND.KK.EQ.1.AND.LK.EQ.1)GO TO 400
      GO TO 399
  287 IF(IK.EQ.1.AND.JK.EQ.1.AND.KK.EQ.1)GO TO 400
      GO TO 399
  387 IF(IK.EQ.1.AND.JK.EQ.1)GO TO 400
      GO TO 399
  487 IF(IK.EQ.1)GO TO 400
  399 CONTINUE
      PRINT*,,ZOR VALUE=,,ZOR
      IF(ZOR.LE.ZOROLD)GO TO 499
  400 ZOROLD=ZOR
      II=1
      IF(NPARAM.EQ.II)GO TO 202
      II=II+1
      IF(NPARAM.EQ.II)GO TO 203
      II=II+1
      IF(NPARAM.EQ.II)GO TO 204
      II=II+1
      IF(NPARAM.EQ.II)GO TO 205
      V5ZOR=V5
  205 V4ZOR=V4
  204 V3ZOR=V3
  203 V2ZOR=V2
  202 V1ZOR=V1
      PRINT*,,ZOROLD=,,ZOROLD,,V1ZOR=,,V1
  499 CONTINUE
  488 CONTINUE
  388 CONTINUE
```

```
  288 CONTINUE
  188 CONTINUE
      IF(OPTOLD.EQ.ZOROLD)GO TO 215
      PRINT*,,SADDLE POINT DOES NOT EXIST,
      GO TO 216
  215 PRINT*,,SADDLE POINT EXISTS,
      PRINT*,V1OPT
  216 STOP
      END
```

```
      SUBROUTINE RICATI(A,B,P,Q,N,M,H,R,GAIN,BARB)
      DIMENSION A(10,10),B(10,10),P(10,10),Q(10,10),EXPA(10,10),W(
     1),V(10),JC(10),BSQ(10,10),ASQ(10,10),QSQ(10,10),RB(10,10),R(
     1,10),AT(10,10),ERR(10,10),ET(10,10),R(10,10),ARB(10,10),
     1ARBY(10,10),EXPAT(10,10),BARB(10,10),GAIN(10,10)
      V(1)=1.
      CALL GJR(P,10,10,M,M,$50,JC,V)
      DO 181 I=1,N
      DO 181 J=1,M
  181 BARB(I,J)=B(I,J)
      CALL MXMLT(B,P,ARB,N,M,M,10,10)
      CALL MXTRN(B,B,N,M,10,10)
      CALL MXMLT(ARB,B,BSQ,N,M,N,10,10)
      DO 88 I=1,N
      DO 88 J=1,N
      B(I,J)=BSQ(I,J)
   88 W(I,J)=BSQ(I,J)
      CALL MXSCA(A,N,N,10,-1.)
      CALL MATEXP(A,EXPA,N,H)
      CALL MXMLT(B,EXPA,ARB,N,N,N,10,10)
      CALL MXTRN(EXPA,EXPAT,N,N,10,10)
      CALL MXMLT(EXPAT,ARB,ARBY,N,N,N,10,10)
      CALL MXSCA(ARBY,N,N,10,4.)
      CALL MXADD(B,ARBY,W,N,N,10)
      CALL MXSCA(ARBY,N,N,10,0.5)
      CALL MXMLT(EXPAT,ARBY,ARB,N,N,N,10,10)
      CALL MXMLT(ARB,EXPA,ARBY,N,N,N,10,10)
      CALL MXADD(W,ARBY,W,N,N,10)
      CALL MXSCA(ARBY,N,N,10,2.)
      CALL MXMLT(EXPAT,ARBY,ARB,N,N,N,10,10)
      CALL MXMLT(ARB,EXPA,ARBY,N,N,N,10,10)
      CALL MXADD(W,ARBY,W,N,N,10)
      S=H/3.
      CALL MXSCA(W,N,N,10,S)
      V(1)=1.
      CALL GJR(W,10,10,N,N,$50,JC,V)
      GO TO 152
  155 CONTINUE
      DO 183 I=1,N
      DO 183 J=1,N
  183 W(I,J)=0.
      DO 302 I=1,N
  302 W(I,I)=1.
  152 CALL MXMLT(W,BSQ,R,N,N,N,10,10)
```

```
      DO 99 I=1,N
      DO 99 J=1,N
   99 ERR(I,J)=10.E-05
      CALL MXSCA(A,N,N,10,-1.)
  111 CALL MXMLT(R,BSQ,RB,N,N,N,10,10)
      CALL MXSUB(A,RB,ASQ,N,N,10)
   55 CALL MXMLT(RB,R,ARB,N,N,N,10,10)
      CALL MXADD(Q,ARB,QSQ,N,N,10)
      DO 98 I=1,N
      DO 98 J=1,N
   98 ROLD(I,J)=R(I,J)
      CALL LIAPUN(ASQ,QSQ,N,R)
      DO 78 I=1,N
      DO 77 J=1,N
      IF(ABS(R(I,J)-ROLD(I,J)).LE.ERR(I,J))GO TO 77
      GO TO 111
   77 CONTINUE
   78 CONTINUE
      CALL MXMLT(A,R,ET,N,N,N,10,10)
      CALL MXTRN(A,AT,N,N,10,10)
      CALL MXMLT(R,AT,ARB,N,N,N,10,10)
      CALL MXADD(ET,ARB,ET,N,N,10)
      CALL MXMLT(R,BSQ,ARB,N,N,N,10,10)
      CALL MXMLT(ARB,R,ARBY,N,N,N,10,10)
      CALL MXSUB(ET,ARBY,ET,N,N,10)
      CALL MXADD(ET,Q,ET,N,N,10)
      DO 73 I=1,N
      DO 72 J=1,N
      IF(ABS(ET(I,J)).LE.ERR(I,J))GO TO 72
      GO TO 111
   72 CONTINUE
   73 CONTINUE
      WRITE(6,64)
      PRINT*,((R(I,J),J=1,N),I=1,N)
      GO TO 51
   50 WRITE(6,140)
      GO TO 155
   51 CONTINUE
      CALL MXTRN(BARB,BARB,N,M,10,10)
      CALL MXMLT(BARB,R,ARB,M,N,N,10,10)
      CALL MXMLT(P,ARB,GAIN,M,M,N,10,10)
      CALL MXSCA(GAIN,M,N,10,-1.)
      V(1)=1.
      CALL GJR(P,10,10,M,M,$50,JC,V)
      PRINT*,,GAIN VECTOR K IS GIVEN BELOW,
      PRINT*,((GAIN(I,J),J=1,N),I=1,M)
   64 FORMAT(1X,,RICATI MATRIX IS GIVEN BELOW,,/)
  140 FORMAT(1X,,INVERSE DOES NOT EXIST,,/)
      RETURN
      END


      SUBROUTINE LIAPUN(ASQ,QSQ,N,R)
      DIMENSION ASQ(10,10),QSQ(10,10),R(10,10),SIDEN(10,10),U(10
     1,VV(10,10),UT(10,10),Y(10,10),Z(10,10),ZT(10,10),ARB(10,10
```

```
     1,JC(10),ZEN(10,10)
      S=0.1
      DO 101 J=1,N
  101 SIDEN(J,J)=S
      CALL MXSUB(SIDEN,ASQ,U,N,N,10)
      V(1)=1.
      CALL GJR(U,10,10,N,N,$50,JC,V)
      GO TO 105
  102 CONTINUE
      DO 103 I=1,N
      DO 103 J=1,N
  103 U(I,J)=0.
      DO 104 I=1,N
  104 U(I,I)=1.
  105 CONTINUE
      CALL MXADD(SIDEN,ASQ,VV,N,N,10)
      CALL MXMLT(U,VV,ARB,N,N,N,10,10)
      DO 181 I=1,N
      DO 181 J=1,N
  181 Z(I,J)=ARB(I,J)
      CALL MXMLT(U,QSQ,ARB,N,N,N,10,10)
      CALL MXTRN(U,UT,N,N,10,10)
      CALL MXMLT(ARB,UT,Y,N,N,N,10,10)
      S2=2.*S
      CALL MXSCA(Y,N,N,10,S2)
      DO 203 KK=1,10
      IF(KK.EQ.1)GO TO 190
      CALL MXMLT(Z,Z,ARB,N,N,N,10,10)
      DO 182 I=1,N
      DO 182 J=1,N
  182 Z(I,J)=ARB(I,J)
  190 CALL MXMLT(Z,Y,ARB,N,N,N,10,10)
      CALLMXTRN(Z,ZT,N,N,10,10)
      CALL MXMLT(ARB,ZT,ZEN,N,N,N,10,10)
      CALL MXADD(Y,ZEN,Y,N,N,10)
  203 CONTINUE
      DO 204 I=1,N
      DO 204 J=1,N
  204 R(I,J)=Y(I,J)
      GO TO 51
   50 WRITE(6,140)
      GO TO 102
   51 CONTINUE
  140 FORMAT(1X,'INVERSE DOES NOT EXIST,'/)
      RETURN
      END


      SUBROUTINE MATEXP(A,EXPA,N,T)
      DIMENSION A(10,10),EXPA(10,10),ST(10,10),ENTEXP(10,10)
      DO 1 I=1,N
      DO 1 J=1,N
      ST(I,J)=A(I,J)*T
    1 EXPA(I,J)=ST(I,J)
      IPOWER=100
```

```
      DO 7 I=2,IPOWER
      EPOWR=IPOWER-I+2
      DO 5 J=1,N
      DO 3 K=1,N
    3 ENTEXP(J,K)=EXPA(J,K)/EPOWR
    5 ENTEXP(J,J)=ENTEXP(J,J)+1.0
    7 CALL MXMLT(ST,ENTEXP,EXPA,N,N,N,15,15)
      DO 9 J=1,N
    9 EXPA(J,J)=EXPA(J,J)+1.0
      RETURN
      END
```

```
      SUBROUTINE PERFOR(A,B,GAIN,P,Q,XIN,N,M,PER)
      DIMENSION A(10,10),B(10,10),P(10,10),Q(10,10),GAIN(10,10),X
     1,ARB(10,10),ARBY(10,10),ASQ(10,10),QSQ(10,10)
      CALL MXMLT(B,GAIN,ARB,N,M,N,10,10)
      CALL MXADD(A,ARB,ASQ,N,N,10,10)
      CALL MXMLT(P,GAIN,ARB,M,M,N,10,10)
      CALL MXTRN(GAIN,GAIN,M,N,10,10)
      CALL MXMLT(GAIN,ARB,ARBY,N,M,N,10,10)
      CALL MXADD(Q,ARBY,QSQ,N,N,10,10)
      CALL LIAPUN(ASQ,QSQ,N,R)
      CALL OPTIMA(XIN,R,N,M,PER)
      RETURN
      END
```

```
      SUBROUTINE OPTIMA(XIN,R,N,M,OPT)
      DIMENSION XIN(10),RR(10),R(10,10)
      CALL MXMLT(R,XIN,RR,N,N,1,10,10)
      CALL MXTRN(XIN,XIN,N,1,10,10)
      CALL MXMLT(XIN,RR,OPT,1,N,1,10,10)
      RETURN
      END
```

```
      SUBROUTINE EIGVAL(A,BB,C,N,POL,EIGEN)
      DIMENSION A(10,10),BB(10,10),C(10,10),P(10),POL(10),EIGEN(1
      COMPLEX POL,EIGEN
      EPS=.000001
      KMAX=50
      DO 1 I=1,N
```

```
      DO 1 J=1,N
    1 BB(I,J)=A(I,J)
      DO 2 M=1,N
      P(M)=0.0
      FM=M
      DO 21 I=1,N
   21 P(M)=P(M)+BB(I,I)/FM
      DO 3 I=1,N
    3 BB(I,I)=BB(I,I)-P(M)
      CALL MXMLT(A,BB,C,N,N,N,10,10)
      DO 6 K=1,N
      DO 6 L=1,N
    6 BB(K,L)=C(K,L)
    2 CONTINUE
      NI=N
      DO 41 I=1,N
      L=N-I+1
      IF(P(L).NE.0.)GO TO 51
   41 NI=N-I
   51 CONTINUE
      IF(NI.EQ.0)GO TO 22
      DO 4 I=1,NI
      J=NI-I+1
    4 POL(J)=CMPLX(-P(I),0.0)
      NP1=NI+1
      POL(NP1)=CMPLX(1.0,0.0)
      CALL ROOTCP(POL,NI,EPS,KMAX,EIGEN,J,$66)
      GO TO 85
   22 NP1=NI+1
   85 CONTINUE
      DO 23 I=NP1,N,1
   23 EIGEN(I)=CMPLX(0.0,0.0)
      WRITE(6,11)(EIGEN(I),I=1,N)
      GO TO 31
   66 WRITE(6,77)J
   31 CONTINUE
   11 FORMAT(1X,,**EIGENVALUES ARE AS FOLLOWS**,,//,1X,F12.6,5X,F
   77 FORMAT(1X,I4,,TH ROOT DIVERGED,,//)
      RETURN
      END

 IBANK 5393 DBANK
  ,.MAIN
1 08/14/79 14:05:57

S     001000 042521      17234 IBANK WORDS DECIMAL
      043000 061020       7185 DBANK WORDS DECIMAL
ESS   035763


 SEGMENT $MAIN$              001000 042521      043000 061020

$(1)      002200 002134         $(2)      043000 043002
             $(3)    002132 002146
             $(5)    002147 002147
                                        $(2)      043003 043314
```