A SURVEY ON ENDO-TRIVIAL MODULES

by

Gözde Sert

B.S., Mathematics, Boğaziçi University, 2016

Submitted to the Institute for Graduate Studies in
Science and Engineering in partial fulfillment of
the requirements for the degree of
Master of Science

Graduate Program in Mathematics

Boğaziçi University

2018

ACKNOWLEDGEMENTS

Foremost, I would like to thank my thesis advisor Prof. Olcay Coşkun of the Department of Mathematics at Boğaziçi University for the continuous support of my master study and for his patience, motivation, enthusiasm, and immense knowledge. The door to Prof. Coşkun office was always open whenever I ran into a trouble spot or had a question about my study or writing. He definitely provided me with the tools that I needed to choose the right direction and successfully complete my dissertation. I could not have imagined having a better advisor and mentor for my master study.

I would like to thank Prof. Kazım İlhan İkeda and Assoc. Prof. Ergün Yaraneri for participating in my thesis committee. I am indebted to all of them for their valuable comments and advises.

I also thank my whole family members for supporting me spiritually during my thesis. Especially I want to express my sincere gratitude to my mother, Elif Sert, for unfailing support throughout my whole life.

I would like to thank my friends Deniz Yılmaz, Doğancan Karabaş, Ella Hiesmayr, Hatice Baba, Humeyra Nur Kaleli, Yusuf Hakan Kalaycı, and all other my friends for being beside me whenever I need their helps and advices in a number of ways. Specifically, I would like to present my thanks to Mehmet Arslan for not only writing this thesis but also helping for correction of latex codes' mistakes.

Finally, I wish to express my sincere thanks to Assoc. Prof. Ümit Işlak for his wise counsel concerning my academic carrier and continuous encouragement throughout the process of writing this thesis.

This thesis is supported by Tubitak 1001-115f156.

ABSTRACT

A SURVEY ON ENDO-TRIVIAL MODULES

In this thesis, we investigate endo-trivial modules and their classification endo-trivial modules over p-groups. Endo-trivial modules were introduced by Dade [1] and appear naturally in moduler representation theory. Several contributions towards the general aim of classifying those modules have already obtained and the classification of endo-trivial modules was completed in 2004 by Carlson and Thevenaz. Our aim is to see what are the endo-trivials modules over p-groups, especially over abelian groups.

ÖZET

TERSİNİR MODÜLLER ÜZERİNE BİR ARAŞTIRMA

Bu savda, tersinir modülleri ve p-grupları üzerindeki tersinir modüllerin sınıflandırılmasını araştıracağız. Tersinir modüller ilk olarak Dade tarafından tanıtıldı ve doğal olarak da modüler temsil teorisinde göründü. Bu modullerin genel sınıflandırılması amacıyla birçok katkı çoktan elde edildi ve tersinir modüllerin sınıflandırılması 2004 yılında tamamlandı. Bu tezdeki amacımız p-gruplar özellikle değişmeli gruplar üzerindeki tersinir modüllerin ne olduğunu anlamaktır.

TABLE OF CONTENTS

LIST OF SYMBOLS

cok	cokernel
C_n	cyclic group of order n
dim	dimension
$\mathrm{End_k}$	k-endomorphism ring
FP	group algebra of the group G over the field F
ker	kernel
M*	F-dual of the FG-module M
$\Omega^n(M)$	n-the syzygy module of the module M, the Heller operator
$\lvert G : H\rvert$	index of subgroup H of the group G in the group G
$\mathrm{GL}(n, R)$	general linear group of invertible $n \times n$ R-matrices
$\mathrm{Hom}_k(M, N)$	k-homomorphism from M to N
$\mathrm{Inf}_{G/N}^G$	inflation from G to G/N, inflation map
$\mathrm{Ind}_{G/N}^G$	induction from H to G
$\mathrm{mod}(FG)$	category of finitely generated left FG-modules
$\mathrm{Mod}(FG$	category of all left FG-modules
$\mathbb{N}$	the natural numbers , 0 included
$N_G(H)$	normaliser of the subgroup H in G
$\mathrm{Proj}(V)$	subcategory of V-projective modules
$\mathrm{Res_H^G}$	restriction from G to H, restriction map
$\mathrm{stmod}(FG)$	stable category of finitely generated left FG-modules
$\mathrm{tr}(A)$	trace, sum of all diagonal entries, of the square matrix A
Tr_Q	trace map of the module Q
$T(G)$	group of endo-trivial modules of G
$\mathbb{Z}$	the Integers
${}^gV$	conjugate of the module V by the element g
${}^gH$	conjugate of the subgroup H by the element g

$a \mid b$	a divides b
$a \nmid b$	a does not divide b
$M \mid N$	M is a direct summand of N
$\oplus$	direct sum
$\otimes$	tensor product
$\cong$	isomorphism
$\downarrow_H^G$	restriction from G to H
$\uparrow_H^G$	induction from H to G
$\twoheadrightarrow$	surjective morphism
$\hookrightarrow$	injective morphism
$\forall$	universal symbol "for all"
$\exists$	universal symbol "there exists"
$\exists!$	universal symbol "there exists a unique"

1. INTRODUCTION

1.1. Notations

Throughout this text we denote by F an algebraically closed field with characteristic p where p is a prime number, by G a finite group with order divisible by p and by P a finite p-group often Sylow p-subgroup of G. Furthermore, we symbolize FG as a group algebra and we assume that all FG-modules are finitely generated left modules.

1.2. Background Material

Before we start to interest in the $endo-trivial$ modules, we give some basic information about groups, rings and category theory, homomorphisms and group algebra.

1.2.1. Group Algebra

In this section, we assume that the reader is familiar with some elementary knowledge about rings and modules.
Let R be a ring with an identity element, usually denoted as 1_R, which act on R-modules as the identity operator.

Definition 1.1. *Let A be a ring and R be a commutative ring. If there exists a homomorphism ϕ: $A \to R$ from A to center of R satisfying ϕ $(1_A)= 1_R$, then we say that the ring A is algebra over a ring R or an R-algebra.*

1.2.1.1. Examples.

(i) Every ring is a $\mathbb{Z}$-algebra.

Next example is most useful algebra for modular representation theory:

(ii) Let G be a group and R be a ring with identity . The *group algebra* RG is the set of all finite formal sums

$$\left\{\sum_{x\in G} a_x x : a_x \in R\right\} \tag{1.1}$$

with addition and multiplication are defined by

$$\left(\sum a_x x\right) + \left(\sum b_x x\right) = \left(\sum (a_x + b_x)x\right) \tag{1.2}$$

and

$$\left(\sum a_x x\right)\left(\sum b_y y\right) = \sum_{x,y} (a_x b_y)xy \tag{1.3}$$

This group ring RG is an R-algebra by using the map $R \to RG$ is defined by $r \to r \cdot 1_G$ where $r \in R$ and 1_G is the identity element of G.

In the special case where we start with field F, we can construct group ring FG which is usually called *group algebra* of G relative to F. For the modular representation theory this type of group algebra has very crucial role since the information of an FG-module is equivalent to the information of a representation of the group G over F. The next two theorems are fundamental theorems of the representation theory. The reader can read their proofs from any representation theory textbook, we take them from [2]

Theorem 1.1 (Krull-Schmidt)**.** *Let A be an F-algebra (finitely generated as an F-module), where F is a field or a complete discrete valuation ring. Then each finitely generated left A-module U can be written as a finite direct sum of indecomposable submodules. Furthermore, if* $U = V_1 \oplus \cdots \oplus V_k \cong W_1 \oplus \cdots \oplus W_l$ *are two expressions, then* $k = l$ *and there is a reordering so that* $V_i \cong W_i$ *for all* $1 \leq i \leq k$.

Theorem 1.2 (Maschke)**.** *The group algebra FG is semisimple if and only if the characteristic of the field F does not divide the order of the group G.*

1.2.2. Group Theory

In this subsection we give to reader an essential theorem of group theory and some definitions which we use in the later chapters.

Before we start to write the theorem we give two definition:

Definition 1.2. *Let G and H be two groups and let ϕ: $G \to H$ be a homomorphism of groups. Then the image of ϕ and the kernel of ϕ (usually denoted as* im ϕ *and* ker ϕ *respectively) are defined by* im $\phi = \{\phi(x)\colon x \in G\}$ *and* ker $\phi = \{x \in G\colon \phi(x)=1_H\}$.

Now we can state the theorem:

Theorem 1.3 (Fundamental Theorem on Homomorphism)**.** *(i) Let G be a group and N is a normal subgroup of G. Then the cosets of N in G form a group G/N and there is a natural epimorphism π of groups: π: $G \to G/N$, which is defined by $\pi(x)= xN$ for $x \in G$.*

(ii) Let G and H be two groups and let ϕ: $G \to H$ be a homomorphism of groups. Then im ϕ *is a subgroup of H and* ker ϕ *is a normal subgroup of G. Moreover, there exists a factorization*

$$\phi = \alpha \circ \pi \tag{1.4}$$

where π: $G \to G/\ker\phi$ is the natural epimorphism in (i) above, and where α: $G/\ker\phi \to \operatorname{im}\phi$ is the isomorphism given by $\alpha(x \ker \phi)= \phi(x)$ for all $x \in G$.

(iii) Let ϕ: $G \to H$ an epimorphism of groups. There is a bijection between the family of all subgroups Q of H, and the family of all subgroups P of G such that ker ϕ is a subgroup of P. This bijection is defined by $P \to \phi(P)=Q$, and $Q \to \phi^{-1}(Q)=P$, for subgroups P of G that P contains ker ϕ*, and subgroups Q is subgroup of H. We have that Q is a normal subgroup of H if and only if $\phi^{-1}(Q)$ is a normal*

subgroup of G, and if this occurs, then

$$G/\phi^{-1}(Q) \cong H/Q. \tag{1.5}$$

The proof of the theorem can be found in any algebra book so we skip the proof of the theorem.

1.2.3. Homomorphisms

Homomorphisms are central tool for representation theory. In this subsection we briefly give information about module homomorphisms and their properties. We assume that the reader is familiar with the definition of modules and some basic idea about modules (We come later to the modules to get deep understanding).

Let R be a ring, and let M,N be R-modules. Consider the set of all R-homomorphisms from M into N, it is denoted by $\mathrm{Hom}_R(M,N)$. It forms an additive group. For $\alpha \in \mathrm{Hom}_R(M,N)$, we have the following sets:

$$\ker\alpha = \text{kernel of } \alpha = \{m \in M : \alpha(m) = 0\},$$

$$\mathrm{im}\ \alpha = \text{image of } \alpha = \alpha(M) = \{\alpha(m) : m \in M\},$$

$$\mathrm{cok}\ \alpha = \text{cokernel of } \alpha = N/\alpha(M).$$

A sequence of R-modules and R-homomorphisms

$$M_1 \xrightarrow{\alpha_1} M_2 \xrightarrow{\alpha_2} M_3 \xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_{n-1}} M_n \tag{1.6}$$

is *exact* at M_i if $\ker\alpha_i = \mathrm{im}\ \alpha_{i-1}$. The sequence is *exact* if it is exact at each M_i, in other words, $\ker\alpha_i = \mathrm{im}\ \alpha_{i-1}$ for $2 \le i \le n-1$. Moreover, if we have an exact sequence of the form

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0 \tag{1.7}$$

then it is called a short exact sequence. In other terms we say that the above sequence is a short exact sequence if the following three conditions are satisfied:

(i) f is injective,

(ii) im f = ker g,

(iii) g is surjective.

In this case, we deduce that g induces an R-isomorphism $M/f(L) \cong N$.

Lastly, we mention the commutativity of homomorphisms: Let M_1, M_2, N_1, N_2, L be R-modules and the maps f_1: $N_1 \to N_2$, f_2: $N_2 \to M_2$, f_3 :$N_1 \to M_1$ and f_4: $M_1 \to M_2$ be homomorphism. The diagram of R-modules and R-homomorphisms

$$\begin{array}{ccc} N_1 & \xrightarrow[f_1]{} & N_2 \\ {\scriptstyle f_3}\downarrow & & \downarrow{\scriptstyle f_2} \\ M_1 & \xrightarrow[f_4]{} & M_2 \end{array} \tag{1.8}$$

commutes if $f_1 f_2 = f_3 f_4$.

Similarly, the diagram

$$\begin{array}{ccc} L & \xrightarrow{f} & M_1 \\ & {\scriptstyle h}\searrow & \downarrow{\scriptstyle g} \\ & & M_2 \end{array} \tag{1.9}$$

is commutative if $gf = h$.

2. REPRESENTATIONS AND MODULES

In this chapter, we begin with giving the notions of representations of groups by linear transformations and by matrices. From these ideas we can go further to representations of algebras and, eventually to modules over algebras and rings. For the rest of the chapter we give some definitions and some fundamental theorems regarding modules over rings and algebras which are requirement for our final purpose.

2.1. Linear Transformations

Let G and H be abelian groups and as we defined before, $\mathrm{Hom}(G, H)$ be the set of all homomorphisms of G into H. If we define the sum of two homomorphism f and g as

$$(f + g)(x)= f(x)+g(x) \text{ where } x \in G,$$

then $\mathrm{Hom}(G, H)$ becomes an abelian group. Also, if $G = H$, then we can define multiplication of homomorphism by composition

$$(fg)(x) = f(g(x)) \tag{2.1}$$

where $x \in G$ so that the additive group $\mathrm{Hom}(G, G)$ becomes a ring with an identity element. We write $\mathrm{End}_(G) = \mathrm{Hom}(G, G)$.

Let V and W be vector spaces over a field F. Then $\mathrm{Hom}_F\ (V, W)$ denotes the subgroup of $\mathrm{Hom}(V, W)$ including in all mappings $f \in \mathrm{Hom}(V, W)$ such that

$$f(c \cdot v) = c \cdot f(v) \tag{2.2}$$

where c $\in$ F, x $\in V$. The mappings in $\mathrm{Hom}_F(V, W)$ are called *linear transformations* or *F-homomorphisms* or *F-linear transformations.* Let $f \in \mathrm{Hom}_F(V, W)$ and $c \in F$. If we define

$$(c \cdot f)(v) = c \cdot f(v) \tag{2.3}$$

where $v \in V$, then $\mathrm{Hom}_F(V, W)$ becomes a vector space over F. In particular, $\mathrm{End}_F(V, V)$ is at the same time a ring and a vector space over F and

$$c \cdot (fg) = (c \cdot f)g \tag{2.4}$$

where $f, g \in \mathrm{Hom}_F(V, V)$ and $c \in F$. Before we pass the next section, we give some definitions and notations which are essential for the representation.

Definition 2.1. *Let R be a ring with identity, say 1. A unit in R is any element $r \in R$ which has a two-sided multiplicative inverse s in R. In other words, r is a unit if and only if for some $s \in R$,*

$$r \cdot s = s \cdot r = 1. \tag{2.5}$$

We also say that r is an invertible element of R, and we denote the inverse of r by r^{-1}.

It is easy to observe that the set of all units in R is a multiplicative group. Now, the group of units in $\mathrm{End}_F(V, V)$ is called the *general linear group* denoted by $GL(V)$. So we can deduce that any element T in *GL(V)* is invertible. In other words, for some $T^{-1} \in \mathrm{End}_F(G, G)$ we get

$$TT^{-1} = T^{-1}T = 1 \tag{2.6}$$

where 1 denotes the identity operator on V.

2.1.1. Representations

In this subsection, we briefly introduce the concept of a representation of a group. The notion of representations is of fundamental importance for the study of abstract groups. We are generally interested in symmetries since they are observed and understood greatly.

In this section we take G as an arbitrary finite multiplicative group with identity element 1, F as a field. It is assumed that all vector spaces are finite dimensional over F. We call to mind that $GL(V)$ presents the group of all invertible linear transformations of a vector space V onto itself. Also we have the notation $GL(n, F)$ for the group of all invertible $n \times n$ matrices over F.

Definition 2.2. *Let G be a finite group and let R be a commutative ring with identity element 1. If V is an R-module then $GL(V)$ denote the group of all invertible R-module homomorphisms from V into V. A (linear) representation of G over R is a group homomorphism*

$$\rho : G \to GL(V).$$

We can write each element of $GL(V)$ as a matrix whose entries comes from R and we obtain for each $g \in G$ a matrix $\rho(g)$. Thus we have a *matrix representation* of G. In this case we define the *degree* of the representation as the rank of the R-module V. More appropriately, it is often used *representation module* or *representation space* (if R is a field).

2.1.2. Examples:

(i) Let G be group and R be a commutative ring with identity element. If we take $V = R$ and $\rho(g) = 1$ for all $g \in G$, where 1 denotes the identity map $R \to R$.

This representation is called the *trivial representation*.

(ii) Let $R = \mathbb{R}$, $V = \mathbb{R}^2$ and $G = S_3$. This group G is isomorphic to the group of symmetries of an equilateral triangle. To be more precise, the symmetries are the three reflections in the angle bisectors, together with three rotations. Now let the center of the triangle place at the origin of V and label the three vertices of the triangle as 1, 2 and 3. Then we have the following representation:

$$() \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(1,2) \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$(1,3) \mapsto \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$(2,3) \mapsto \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$$

$$(1,2,3) \mapsto \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

$$(1,3,2) \mapsto \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$$

where we take basis vectors in the directions of vertices 1 and 2, by making angle of $\frac{2\pi}{3}$ to each other. By definition these matrices define a representation over any ring R. It is easy to observe that these matrices can always multiply together to give a copy of S_3.

2.2. Modules

In this subsection, we briefly give of information about modules. Modules are our main focus object. For the rest the thesis we deeply examine a special kind of modules. Therefore we turn our attention into getting more knowledge on modules.

Definition 2.3. *An abelian group M is called a left R-module if for each $r \in R$ and $m \in M$, a product $rm \in M$ is defined such that*

$$r(m_1 + m_2) = r(m_1) + r(m_2), \tag{2.7}$$

$$(r_1 + r_2)m = (r_1)m + (r_2)m, \tag{2.8}$$

$$(r_1 r_2)m = r_1(r_2 m), \tag{2.9}$$

$$1m = m, \tag{2.10}$$

for all $r, r_1, r_2 \in R$ and $m, m_1, m_2 \in M$.

A subgroup N of M is called a *submodule* if $rn \in N$ for all $r \in R$ and $n \in N$. For left R-modules M, N, a one-to-one mapping f of M onto N is called an *R-isomorphism* if $f(m+n) = f(m) + f(n)$ and $f(rm) = rf(m)$ for all $m \in M$, $r \in R$. Hence if there exists an R-isomorphism between M and N then we say that M and N are *R-isomorphic* and write $M \cong N$.

The *left regular module* $_RR$ of a ring R is the left R-module whose underlying abelian group is the additive group of R, and the module product is given by the ring multiplication rm for $r \in R$ and $m \in {}_RR$

2.2.1. Direct Sum

We are given that K and L are submodules of the R-module M. Then we can define the *sum* of K and L by

$$K + L = \{k + l : k \in \ K, l \in L\}. \tag{2.11}$$

Let $M_1, \cdots, M_r$ be submodules of the R-module M, we say that M is the *(internal) direct sum* of $M_1, \cdots, M_r$ if and only if

$$M = \cup_{i=1}^{r} M_i \tag{2.12}$$

and for all $i \in \{1, 2, \cdots, r\}$, we have

$$M_i \cap \sum_{j \neq i} M_j = \{0\} \tag{2.13}$$

Now in order to get more intuition about modules we give the definition of the $free$ modules. Later we define the $free$ modules differently by constructing relation to other modules.

Definition 2.4. *An R-free set of generators of M is called an R-basis of R-module M. So the set of elements $\{m_I\}$ is an R-basis of M if and only if every element of M can be written uniquely as a finite R-linear combination*

$$\sum r_i m_i, \, where \, r_i \in R, \, m_i \in \{m_I\}. \tag{2.14}$$

Hence we can define a *free (left) R-module* as an R-module which has an R-basis.

Definition 2.5. *An R-module M is called R-torsion-free if $rm = 0$, $r \in R$, $m \in M$, then one of r, m is zero.*

For example,

- A vector space V over a field F is a torsion-free F-module.
- An additive abelian group A is torsion-free as $\mathbb{Z}$-module if and only if A contains no elements of finite order.

Considering representations of the group G as RG-modules has the advantage that many definitions can be adopted from module theory. Therefore we focus on RG-submodules of an RG-module V, and we think them sometimes as *subrepresentations* of the representation conributed by V. After giving some elementary information, we focus completely on modules.

Let R be a ring with identity 1. A non-zero R-module M is said to be *simple* or *irreducible* if M has no R-submodules other than 0 and itself. It is easy to see that a non-zero simple R-module is generated by each of its non-zero elements. An R-module U is called *semisimple* or *completely reducible* if it is a direct sum of simple R-submodules.

When we replace the ring R with a field F, we get that F-modules are F-vector spaces. so the next result comes familiar from linear algebra:

Lemma 2.1. *Let R be a ring with an identity element 1. Assume that $M = S_1 + \cdots + S_k$ is an R-module such that it can be written as a sum of finitely many simple modules $S_1, \cdots, S_k$. If N is any submodule of M then there is a subset I=$\{i_1, \cdots, i_r\}$ of $\{i_1, \cdots, i_k\}$ such that $M = N \oplus_{i_r \in I} S_{i_r}$. In particular,*

(i) N is direct summand of M, and

(ii) M is the direct sum of some subset of S_i by taking N = 0, hence is necessarily semisimple.

We now present a different version of Maschke's theorem. The next result gives us very essential information about representations of a finite group over a field, in which order of group is invertible.

Corollary 2.1. *Let F be a field in which order of G, denoted $|G|$, is invertible. Then every finite dimensional FG-module is semisimple.*

The largest semisimple submodule of a module M is called the *socle* of M, and it is denoted $Soc(M)$. Moreover, we have radical of M, denoted *Rad M* which is defined as the intersection of all the maximal submodules of M and has the property that it is the smallest submodule of M with semisimple quotient. We deeply mention socle and radical in a later chapter.

2.2.2. Schur's Lemma and Wedderburn's Theorem

In this chapter we give Schur's Lemma and Wedderburn's Theorem for semisimple algebras and some consequences. Thanks to Wedderburn's Theorem, we can make the connection between module-theoretic hypothesis of semisimplicity and the ring-theoretic presentation. On the other hand, conceivably one of the important technique to get deep understanding in representation theory is to consider endo-morphism rings.

Theorem 2.1 (Schur's Lemma)**.** *Let R be a ring with an identity element 1 and S_1 and S_2 be simple R-modules. Then $\mathrm{Hom}_R(S_1, S_2) = 0$ unless $S_1 \cong S_2$, in which case the endomorphism ring $\mathrm{End}_R(S_1)$ is a division ring. If R is a finite dimensional algebra over an algebraically closed field F, then every R-module endomorphism of S_1 is multiplication by some scalar. Thus $\mathrm{End}_R(S_1) \cong F$ in this case.*

It has been observed that requiring F to be algebraically closed guarantees that the division rings $\mathrm{End}_R(S)$ are no larger than F, and this is often a significant simplifying condition. In the case $\mathrm{End}_R(S) = F$ for all simple A-modules S, we call F a *splitting field* for the F-algebra R. For now, we assume that algebraically closed fields are always splitting fields.

The next lemma is the fundamental tool in recovering the structure of an algebra from its representations. First we define the *opposite* ring of R, denoted by R^{op}, which is the ring that has the same set and the same addition operation as R , but with a new multiplication given by $a \cdot b = ba$.

Lemma 2.2. *For any ring R with an identity element 1,* $\mathrm{End}_R({}_RR) \cong R^{op}$.

Now we write Artin–Wedderburn's Theorem. Before stating the theorem we give the definition of a semisimple ring. A ring R with an identity element 1 is called *semisimple* if all of its modules are semisimple.

Theorem 2.2 (Artin–Wedderburn)**.** *Let A be a finite dimensional algebra over a field F with the property that every finite dimensional module is semisimple. Then A is a direct sum of matrix algebras over division rings. Specifically, if*

$$ {}_AA \cong S_1^{n_1} \oplus \cdots \oplus S_r^{n_r} \tag{2.15} $$

where the set $S_1, \cdots, S_1$ are non-isomorphic simple modules with the multiplicities $n_1, \cdots, n_r$, in the regular representation, then

$$ A \cong M_{n_1}(D_1) \oplus \cdots \oplus M_{n_r}(D_r) \tag{2.16} $$

where $D_i = \mathrm{End}_A(S_i)^{op}$. Furthermore, if F is algebraically closed then $D_i = F$ for all i.

Moreover, we interpret that every such direct sum of matrix algebras is a semisimple algebra. Each matrix algebra over a division ring D is a *simple* algebra and up to isomorphism it has a unique simple module.

3. BRIEF INTRODUCTION TO ENDO-TRIVIAL MODULES

For this chapter we fix some notations. Let p be a prime number, k be an algebraically closed field of characteristic p, G be an arbitrary finite group with order divisible by p, P be a finite p-group, and kG be the group algebra. We also assume that all kG-modules are finitely generated left modules.

Recall that if a Sylow p-subgroup of G is not cyclic, then there are infinitely many indecomposable modules and in [1] E. Dade stated that it is impossible to classify them in most cases. Therefore we are looking for a class of modules which is large enough to be interesting, small enough so that we can classify the modules and connect to other important kinds of modules. Thanks to the lectures of Thevenaz at Mathematics village at Şirince, we learn that there are suitable modules satisfying above conditions. In the view of these, he give a motivation to study $endo - trivial$ modules which we define later.

3.1. Projective Modules

In this section we shall start by studying the general properties of indecomposable modules which do not have non-trivial direct sum decompositions. Our aim for this section is to construct the basic characterization of indecomposable modules in terms of their endomorphism algebras.

Recall that if an algebra A is local, then $A/\text{rad } A$ is isomorphic to k.

Lemma 3.1. *An algebra A is local if and only if every element of A is nilpotent or invertible.*

Theorem 3.1. *The A-module U is indecomposible if and only if* $\text{End}(U)$ *is local.*

In the previous chapter we stated the Krull-Schmidt theorem, it is about the uniqueness of decomposition into indecomposable simple modules. Next lemma is its inference:

Corollary 3.1. *If M, U and V are A-modules with $M \oplus U \cong M \oplus V$ then $U \cong V$.*

3.2. Free Modules

In the previous chapter we mentioned free modules. In this chapter we are mainly interested in free modules. If an A-module U is isomorphic with a direct sum $A \oplus \cdots \oplus A$, then we call a *free module*. Our main concern of this section lies in decomposition of free modules into direct sums of indecomposable modules.

Firstly we begin with the fundamental characterization of summands of free modules.

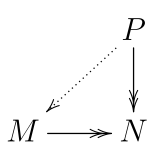
Theorem 3.2. *If P is an A-module then the following are equivalent:*

(i) P is a direct summand of a free module;

(ii) If φ is a homomorphism of the A-module M onto P then the kernel of φ is a direct summand of M;

(iii) If φ is a homomorphism of the A-module M onto the A-module N and Ψ is a homomorphism of P to N then there exists a homomorphism ρ of P to M such that $\varphi\rho = \Psi$.

The modules with these properties are called *projective*. The last property of this theorem can be understood much more in terms of the usual diagram:



Now we introduce the important class of indecomposable projective modules. By Krull-Schmidt Theorem certainly these are the indecomposable summands of the free module A. We are ready to state the theorem:

Theorem 3.3. *There is a one-to-one correspondence between isomorphism classes of indecomposable projective A-modules and isomorphism classes of simple A-modules given by associating $P/\mathrm{rad}P$ to each indecomposable projective A-module P.*

As a consequence, it can be easily seen that in a decomposition of the free module A into a direct sum of indecomposable submodules, each isomorphism type of indecomposable projective module appears the dimension of the corresponding simple module many times.

3.3. Duality

Duality is very important tool to improve our knowledge about projective modules. The notion of duality is used in linear algebra but we extend this idea from k-modules, that is vector spaces, to kG-modules.

We remember from linear algebra that if V is a vector space over k then the dual space of V is denoted by V^*.

It is easy to see that if V is a kG-module then V^* is also a kG-module. If $g \in G$ and $\varphi \in V^*$ then the action on the elements of V^* is defined by $(g\varphi) = \varphi(g^{-1}v)$ for any $v \in V$. Also if V and W are kG-modules and ρ is a kG-homomorphism from V to W then a dual of the map ρ, ρ^*, which is a kG-homomorphism from the kG-module W^* to the kG-module V^*. Indeed, kG-modules have all the usual properties of duality. For instance, If V is a kG-module then $V \cong V^{**}$. Also we have the following equalities. If V and W are kG-modules then

$$(V \oplus W)^* \cong V^* \oplus W^*, \tag{3.1}$$

$$(V \otimes W)^* \cong V^* \otimes W^*. \tag{3.2}$$

We now look at the relation between duality and simplicity of module.

Lemma 3.2. *(i) The kG-module U is simple if and only if U^* is simple.*

(ii) As a consequence of the lemma, we can observe that the kG-module U is semisimple if and only if U^* is semisimple.

(iii) If the kG-module U is free then U^* is free.

(iv) Therefore we can deduce that if a kG-module P is projective then its dual P^* is also projective.

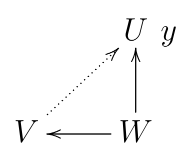
Proposition 3.1. *If U is a kG-module then the following are equivalent:*

(i) U is a direct summand of a free module;

(ii) If φ is a one-to-one homomorphism of U into the kG-module V then $\varphi(U)$ via a direct summand of V;

(iii) If Ψ is a one-to-one homomorphism of the kG-module W into the kG-module V and φ is a homomorphism of W to U then there is a homomorphism ρ of V to U such that $\varphi{=}\rho\Psi$.

To understand (*iii*) better we draw a graph:



A module which satisfies above properties is called *injective*. The followings are equal to the definitions of projective and injective modules respectively.

Let M be a kG-module. Then we have:

(i) M is projective if and only if any surjective homomorphism π maps N to M splits, in other words we can write N as $N \cong M \oplus \ker\pi$

(ii) M is injective if and only if any injective homomorphism ι maps M to N splits, in other words we can write N as $N \cong \mathrm{im}(\iota) \oplus N/\mathrm{im}(\iota)$

In the view of the above proposition we have the following theorem:

Theorem 3.4. *A kG-module P is projective if and only if it is injective.*

Proof. Let P, V and W be kG-modules. If P is projective if and only if P is a summand of free module $(kG \oplus \cdots \oplus kG)$. Then by assumption we have the following diagram

$$\begin{array}{ccc} & & P \\ & & \uparrow{\scriptstyle\alpha} \\ V & \xleftarrow{\beta} & W \end{array} \tag{3.3}$$

When we take take the dual of the diagram, we have

$$\begin{array}{ccc} & & P^* \\ & & \downarrow{\scriptstyle\alpha^*} \\ V^* & \xrightarrow[\beta^*]{} & W^* \end{array} \tag{3.4}$$

where β^* is surjective and by projectivity of P^* there exists $f : P^* \to V^*$ such that $\beta^* f = \alpha^*$. Since $f^*\beta = \alpha$, we see that P is injective.

Conversely, to see that injective modules are projective, we can use the same argument to show that their duals are projective, hence they are injective. $\square$

3.4. Tensor Products

In this section we briefly give information about the properties of tensor product.

Lemma 3.3. *If U, V, and W are kG-modules then*

$$\mathrm{Hom}_{kG}(U \otimes V, W) \cong \mathrm{Hom}_{kG}(U, V^* \otimes W). \tag{3.5}$$

As before, we know the action of the elements g in G on the elements φ of $\mathrm{Hom}_{kG}(U, V)$. For special case $V = k$, then $\mathrm{Hom}_k(U, k) \cong U^*$ as kG-modules.

Lemma 3.4. *If V is a kG-module and P is a projective kG-module then $V \otimes P$ is also projective.*

Before we complete this section, we want to state a well-known useful fact:

Lemma 3.5. *Let M and N be kG-module. Then there is a natural isomorphism ϕ: $M^* \otimes_k N \to Hom_k(M, N)$ which is defined by $\phi(f \otimes n)(m) := f(m) \cdot n$. Specifically* $\mathrm{End}_k(M) \cong M^* \otimes M$.

3.5. Useful Operations

For the sake of this section, we fix some notations.

Let M and N be kG-modules. We define the tensor product of M and N over k as $M \otimes N := M \otimes_k N$. Consider the action of G on $M \otimes N$ which is defined by $g(m \otimes n) := gm \otimes gn$. With this action of G, $M \otimes N$ is a kG-module.

3.5.1. Operations on modules

Restriction: Let G be a group and H be a subgroup of G. Then kH is a subring of kG. We can say that any kG-module M can be restricted to a kH-module, which we write as $M \downarrow_H^G := \mathrm{Res}_H^G M$.

Theorem 3.5. *Let P be a projective kG-module and H is a subgroup of G then* $\mathrm{Res}_H^G P$ *is a projective kH-module where* $\mathrm{Res}_H^G P$.

It is sufficient to show that the free kG-module kG is free as a kH-module, since every projective module is a direct summand of modules each isomorphic with kG.

Induction: If N is a kH-module, the induced module is defined as the extension of scalars $N\uparrow_H^G := kG \otimes_{kH} N$ which is denoted as $\mathrm{Ind}_H^G N$. Notice that kG is a free right kH-module with rank $|G : H|$ so there is an isomorphism of k-vector spaces $N\uparrow_H^G := \bigoplus_{x\in[G/H]} x \otimes N$, where $[G/H]$ represents a set of all left cosets of H in G, and where $x \otimes N$ denotes the conjugate module of N by x.

Special case: Let take $N = k$ which is the trivial kH-module. Then $k\uparrow_H^G$ is the permutation module with basis G/H.

Lemma 3.6. *Let N be a kH-module and M be a kG-module. Then we have*

$$N\uparrow_H^G \otimes M \cong (N \otimes M\downarrow_H^G)\uparrow_H^G . \tag{3.6}$$

Proof. We assume that N is a kH-module and M is a kG-module. By definition, $N\uparrow_H^G \otimes M = (kG \otimes_{kH} N) \otimes M$ and $(N \otimes M\downarrow_H^G)\uparrow_H^G = kG \oplus_{kH} (N \otimes M\downarrow_H^G)$. Let φ be a map from $(kG \otimes_{kH} N) \otimes M$ to $kG \oplus_{kH} (N \otimes M\downarrow_H^G)$ such that

$$\begin{aligned} \varphi : (kG \otimes_{kH} N) \otimes M &\leftrightarrow kG \oplus_{kH} (N \otimes M\downarrow_H^G) : \psi \\ (g \otimes n) \otimes m &\mapsto g \otimes (n \otimes g^{-1}m) \\ (g \otimes n) \otimes gm &\leftarrow g \otimes (n \otimes m). \end{aligned}$$

It is easy to show that the maps φ and ψ are kG-homomorphisms. Then, we have that $\varphi \circ \psi$ is the identity map and also $\psi \circ \varphi$ is the identity map. Hence, we get

$$N \uparrow_H^G \otimes M \cong (N \otimes M \downarrow_H^G) \uparrow_H^G . \tag{3.7}$$

□

Special cases of the lemma :

(i) Let N be the trivial module k, then we have $k \uparrow_H^G \otimes M \cong M \downarrow_H^G \uparrow_H^G$.

(ii) Let N be the trivial module k and $H = \{1\}$. Then we know that $k \uparrow_{\{1\}}^G = kG$. By the lemma, $kG \otimes M = M_0 \uparrow_{\{1\}}^G$ where $M_0 = M$ is viewed as a k-vector space. Let $\{m_1, \cdots, m_r\}$ be basis of M_0. Then we can write

$$M_0 = km_1 \oplus \cdots \oplus km_r \tag{3.8}$$

hence we have that $M_0 \uparrow_{\{1\}}^G = \bigoplus m_i \uparrow_{\{1\}}^G \cong kG \oplus \cdots \oplus kG =$ free kG-module. Therefore, we can deduce that $free \otimes M \cong free$. So we get that $projective \otimes M \cong projective$.

Frobenius Relations: Let M and N be kG-modules and G be group with subgroup H. Then there are natural isomorphisms of abelian groups.

$$\mathrm{Hom}_{kG}(N \uparrow_H^G, M) \cong \mathrm{Hom}_{kH}(N, M \downarrow_H^G) \tag{3.9}$$

$$\mathrm{Hom}_{kG}(M, N \uparrow_H^G) \cong \mathrm{Hom}_{kH}(M \downarrow_H^G, N). \tag{3.10}$$

Corollary 3.2. *Let L,K be submodules of a kG-module N with $L \subseteq K \subseteq N$. Let Q be a projective submodule of K/L. Then Q lifts to a submodule $\tilde{Q}$ of K, (that is $\tilde{Q} \cong Q$ via the map $\tilde{Q} \xrightarrow{\cong} Q$ which is a restriction of the map $K \twoheadrightarrow K/L$), and $\tilde{Q}$ is a direct summand of N (that is $N = \tilde{Q} \oplus X$).*

Proof. Q is projective then it is also injective and the map $\rho : Q \hookrightarrow K/L$ splits. Therefore we have $K/L = Q \oplus R$. Let $\widehat{Q}$ be the inverse image of Q in K. By correspondence theorem we have $L \subseteq \widehat{Q} \subseteq K$. Also the map π: $\widehat{Q} \twoheadrightarrow \widehat{Q}/\mathrm{L} \cong Q$ splits since Q is projective. Hence $\widehat{Q} \cong \tilde{Q} \oplus L$ with $\tilde{Q} \cong Q$. Now $\tilde{Q}$ is an injective submodule of N. Therefore, $N = \tilde{Q} \oplus X$ for some kG-module X.

□

Now, we define the *augmentation* map for a group P with p-power order where p is characteristic of k.

$$\varepsilon = \varepsilon_P : kP \to k$$
$$u \mapsto 1$$

for all u in P. Note that $\mathrm{Ker}(\varepsilon)$ is a two sided ideal which is generated by all elements of the form $u - 1$ for u in $P - \{1\}$.

Proposition 3.2. *Let P be a p-group.*

(i) The trivial module is the only simple kP-module, up to isomorphism.
(ii) Ker*(ε) is equal to the Jacobson radical J(kP), hence* Ker*(ε) is a nilpotent ideal.*
(iii) There is a unique trivial submodule of kP, namely $\{\lambda(\sum_{u\in P} u)|\lambda \in k\}$.
(iv) kP is indecomposable. In particular every projective kP-module is free.

Proof. (i) Let S be a simple kP-module and let s be a non-zero element of S. The field $\mathbb{F}_p = \{\overline{1}, \cdots, \overline{p-1}\}$ is a subset of k. Let A be $\mathbb{F}_pP$- submodule generated by s, namely it is finite algebra. Then it is easy to observe that A is finite. P acts on A and we can decompose into orbits. Hence we have

$$A = \bigsqcup \left[P - orbits\right] = \left[A^p\right] \sqcup \left[non - trivial\, orbits\right] \tag{3.11}$$

where A is $\mathbb{F}_p$-vector space with dimension power of p and A^p is the orbits of cardinality 1. Notice that cardinality of non-trivial orbits is also a p-power. Therefore p divides $|A^p|$. Since A^p contains 0, A^p contains some non-zero element, say t. Then kt is a trivial submodule of the simple module S, hence $S = kt$. This gives us that S is trivial.

(ii) Firstly notice that $\left(kP/\ker\varepsilon\right) \cong k$ which is the only simple module as a quotient of kP. Let $u \in P$. Since order of P is some power of p, $u^{p^k} = 1$. Then we have $(u-1)^{p^k} = u^{p^k}$ - 1 = 1 - 1 = 0. This means $u - 1$ is nilpotent so this gives the result.

(iii) Let a be a non-zero element of kP such that a generates a trivial submodule of kP. Then we can write

$$a = \sum \lambda_u u \tag{3.12}$$

for some λ_u in k.

For any $w, v \in P$, we have

$$a = wv^{-1}a = \sum_{u \in P} \lambda_u wv^{-1}u. \tag{3.13}$$

Since P is a k-basis for P, the decomposition of a is unique. When we compare coefficients in (3.12) and (3.13), we see that coefficient of w in (3.12) is λ_w and coefficient of w in (3.13) since λ_v in order to get w; we should put $u = v$. Thus we get $\lambda_w = \lambda_v$. Thus

$$a = \lambda \left(\sum_{u \in P} u \right) \tag{3.14}$$

for some $\lambda \in k$ and kP has a unique simple submodule.

(iv) If $kP = M \oplus N$ for some non-zero submodules M and N, then there are infinitely many simple submodules of kP. But by part (iii), this is not possible. So kP is indecomposable. In particular, every projective kP-module is free.

□

Now we introduce the notion of elementary abelian group to give a very important result about understanding projective module.

Definition 3.1. *An elementary abelian p-group is a group isomorphic to* $C_p \times \cdots \times C_p$ *where* C_p *is a cyclic group of order p. The number of factors is called the rank of the group.*

Now, we are ready to state the theorem:

Theorem 3.6 (Chouinard's Theorem)**.** *Let G be a finite group. A kG-module M is projective if and only if the restriction of M to every elementary abelian p-subgroup of G is projective.*

3.6. Category Theory

Now we introduce the notion of a stable category. From the categorical point of view, we can understand endo-trivial modules better.

Let $\mathrm{mod}(kG)$ denote the category of finitely generated left kG-modules and $\mathrm{Mod}(kG)$ the category of all kG-modules. A useful property of $\mathrm{mod}(kG)$ is the next lemma.

Lemma 3.7. *A short exact sequence* $0 \to A \to B \to C \to 0$ *in* $\mathrm{mod}(kG)$ *splits if and only if* $B \cong A \oplus C$*.*

Now we are ready to define *stable category* $\mathrm{stmod}(kG)$ which is the category of finitely generated kG-modules modulo projectives. The objects in this category are the same objects in $\mathrm{mod}(kG)$ and the morphism from a module M to a module N are given by

$$\underline{\mathrm{Hom}}_{kG}(M,N) = \mathrm{Hom}_{kG}(M,N)/\mathrm{PHom}_{kG}(M,N),$$

where $\mathrm{PHom}_{kG}(M, N)$ denotes the set of all morphisms from M to N that factor through a projective module P.

Definition 3.2. *A morphism $\phi : M \to N$ is said to factor through a projective module if there exists $P \in \mathrm{mod}(kG)$ and two morphisms $\alpha \in \mathrm{Hom}_{kG}(M, P)$ and $\beta \in \mathrm{Hom}_{kG}(P, N)$ such that $\phi = \beta\alpha$.*

There is a usual fact for the stable category. Let M and N be kG-modules. Then $M \cong N$ in $\mathrm{stmod}(kG)$ if and only if there are projective modules Q_1 and Q_2 such that as kG-modules

$$M \oplus Q_1 \cong N \oplus Q_2. \tag{3.15}$$

3.7. General Aspect of Endo-permutation Modules

Our main goal is the study of the class of endo-trivial modules which are special case of endo-permutation modules. We give some definitions and properties of the endo-permutation and class of endo-trivial modules by using the definitions and some useful facts in [19].

Definition 3.3. *Let G be a finite group. A kG-module is called a* permutation *module if and only of it has a G-invariant k-basis.*

Definition 3.4. *Let P be a finite p-group. A kP-module M is called endo-permutation if and only if its endomorphism algebra $\mathrm{End}_k(M)$ is a permutation kP-module.*

Definition 3.5. *A kG-module V is called endo-trivial if $\mathrm{End}_{kG}(V) \cong k \oplus P$ where P is a projective kG-module.*

We also give another definition which is stated by Dade.

Definition 3.6. *A kG-module V is called $endo-trivial$ if it is invertible with respect to $\otimes$ in $\mathrm{stmod}(kG)$. In other words, there is a kG-module W such that $V \otimes W \cong k$ in $\mathrm{stmod}(kG)$, that is $V \otimes W \cong k \oplus P$ where k is trivial kG-module and P is a projective.*

Also we want to underline basic results that comes from definition of endo-trivial modules:

(i) If the characteristic p of the field k divides the order of the finite group G, then k is not projective. In fact in the other case the theory does not work and everything collapses.

(ii) Let P be a Sylow p-subgroup of G. If we restrict $V \otimes W \cong k \oplus (projective)$ to P we get that $V \downarrow_P^G \otimes W \downarrow_P^G \cong k \oplus (free)$. Then by the property of tensor product we have $\dim(V){\cdot}\dim(W) = 1 +$ multiple of p. Since the right-hand side is not a multiple of p, this implies that $\dim(V)$ is prime to p.

Lemma 3.8. *Let V be a kG-module. Then we have*

(i) If $\dim$*(V) is prime to p, then there is a split short exact sequence 0* $\rightarrow \mathrm{Ker}(tr) \rightarrow \mathrm{End}_k(V) \rightarrow k \rightarrow 0$ *so that* $V \otimes V^* \cong k \oplus \ker(tr)$

(ii) $V \mid V \otimes V^* \otimes V$.

Proposition 3.3. *Let V be an endo-trivial kG-module. Then the inverse of V in* stmod*(kG) is the dual V^* of V.*

Proof. Assume that V is an endo-trivial kG-module. Let W be the inverse of V in stmod(kG). Then by definition of having inverse in stmod(kG), we have that

$$V \otimes W \cong k \oplus Q \tag{3.16}$$

where Q is a projective module. Also by the previous lemma, we have

$$V \otimes V^* \cong k \oplus L \tag{3.17}$$

and

$$W \otimes W^* \cong k \oplus M. \tag{3.18}$$

On the other hand, we know that

$$
\begin{aligned}
W^* \cong k \otimes W^* &\cong (k \oplus Q) \otimes W * && (3.19)\\
&\cong V \otimes W \otimes W^* && (3.20)\\
&\cong V \otimes (k \oplus M) && (3.21)\\
&\cong V \oplus (V \otimes M) && (3.22)
\end{aligned}
$$

Since W is the inverse of V, we have $k \cong V \otimes W$

$$
\begin{aligned}
k \cong k^* &\cong V^* \otimes W^* && (3.23)\\
&\cong V^* \otimes (V \oplus (V \otimes M)) && (3.24)\\
&\cong (V^* \otimes V) \oplus (V^* \otimes V \otimes M) && (3.25)\\
&\cong k \oplus L \oplus (V^* \oplus V \otimes M) \text{ in stmod(kG)} && (3.26)
\end{aligned}
$$

Then by Krull-Schmidt theorem, L is projective module. This implies $W \cong V^*$ in stmod(kG). □

Lastly, we give one more lemma about endo-trivial modules in order to complete this section.

Lemma 3.9. *Let V be an endo-trivial kG-module.*

(i) If $V = V_1 \oplus V_2$, then one of them, say V_1, is an endo-trivial and the other module V_2 is a projective module.

(ii) $V = \tilde{V} \oplus$ (projective) with $\tilde{V}$ is an endo-trivial and indecomposable module .

Proof. If p does not divide $\dim(V)$, then p does not divide $\dim(V_1)$. This implies that

$$\mathrm{End}_k(V_1) \cong k \oplus L \tag{3.27}$$

for some projective module L. Since V is an endo-trivial module we have

$$\begin{aligned} k \oplus (proj) &\cong \mathrm{End}_k(V) \cong \mathrm{End}_k(V_1 \oplus V_2) &(3.28)\\ &\cong \mathrm{End}_k(V_1) \oplus \mathrm{Hom}_k(V_1, V_2) \oplus \mathrm{End}_k(V_2, V_1) \oplus \mathrm{End}_k(V_2). &(3.29)\end{aligned}$$

Then we get that L is a projective module and this implies that V_1 is an endo-trivial module. Also, from a previous lemma,

$$V_2 | V_2 \otimes V_2^* \otimes V_2 \cong \mathrm{End}_k(V_2) \otimes V_2. \quad (3.30)$$

Since $\mathrm{End}_k(V_2)$ is a projective module then tensor product of $\mathrm{End}_k(V_2)$ with V_2 is also a projective module. Hence we get V_2 is projective.

For the second part of the lemma, by deleting projective summands, we can always assume that an endo-trivial module is indecomposable. Therefore we get the result of the lemma. □

3.7.1. The Group T(G)

Our main problem is to classify all endo-trivial modules. For this purpose, we describe a new group and examine this group.

The group $\mathrm{T}(G)$ is a group that consists of the isomorphism classes of all invertible objects in the category $\mathrm{stmod}(kG)$. In other words, it is the group of endo-trivial modules which can be assumed indecomposable. The identity element of $\mathrm{T}(G)$ is the trivial module $[k]$ and for any element $[V]$ in $\mathrm{T}(G)$, the inverse of $[V]$ is $[V^*]$. Moreover $\mathrm{T}(G)$ is an abelian group for tensor product $(\otimes)$.

Proposition 3.4. *Let V be a kG-module. V is an endo-trivial module if and only if the restriction of V to E, namely $V \downarrow_E^G$ is an endo-trivial module for all elementary abelian p-subgroups E of G.*

Proof. We can assume that $\dim(V)$ is prime to p. Therefore, we can write $V \otimes V^* \cong k \oplus L$, for some kG-module L.

V is an endo-trivial module if and only if L is projective module. By *Chouinard Theorem*, we have $L \downarrow_E^G$ is projective module for all subgroups E of G. When we apply restriction on $V \otimes V^*$, we get $V \downarrow_E^G \otimes V^* \downarrow_E^G \cong \oplus(proj)$ for all $E \leq G$. Therefore we can conclude that $V \downarrow_E^G$ is an endo-trivial module for all E. □

Remark 3.1. *It is immediate from the above proposition that if V is an endo-trivial module and P is a Sylow p-subgroup of G, then*

$$\dim(V) = \begin{cases} \pm 1 \mod |P| & \text{if } p \text{ is odd} \\ \pm 1 \mod \frac{|P|}{2} & \text{if } p = 2 \end{cases}$$

Indeed, we have $V \downarrow_E^G \otimes V^* \downarrow_E^G \cong k \oplus (proj)$ and $\dim(V)= \dim(V \downarrow_E^G) = \dim(V^* \downarrow_E^G)$ and $\dim(proj)$ is multiple of p. If we say $\dim(V)= n$, then we get $n^2 = 1 \mod |P|$ so this gives the assertion.

3.7.2. Example

If $\dim(V)$=1, then V is an endo-trivial module. This does not give anything for a p-group since k is the only one-dimensional module up to isomorphism.

3.7.3. Heller translator

The Heller or *syzygy* operator Ω enables us a way to generate new indecomposable modules from old ones. This operator is very important for homological algebra but we do not discuss this in here.

Definition 3.7. *A submodule S of a module is* superfluous *if $L \subseteq M$ is a submodule with $L + S = M$, then $L = M$.*

Definition 3.8. *A* projective cover *of a module B is an ordered pair* (P, φ) *where P is projective module and* $\varphi : P \to B$ *is a surjective map with* ker φ *is a superfluous submodule of P.*

We are ready to state the definition of Heller operator.

Definition 3.9. *Let M be a kG-module and P be a projective kG-module. We define* $\Omega(M)$ *to be the kernel of a surjective map* $\alpha : M \to P$ *from P to M so that there is a short exact sequence* $0 \to \Omega M \to P \to M \to 0$.

We know that projective covers are unique up to isomorphism of the diagram, ΩM is well-defined up to isomorphism. It can be deduced that $\Omega M = 0$ if and only if M is projective. Therefore we can say that the operator is not invertible for all modules. Now we define the inverse Heller translate.

Definition 3.10. *Let M be a kG-module. We may define* $\Omega^{-1}M$ *as the cokernel of the injective hull of M so that there is a short exact sequence*

$$0 \to M \to I_M \to \Omega^{-1}M \to 0.$$

Before we write the next lemma, we want to state *Schanuel's Lemma.*

Theorem 3.7 (Schanuel's Lemma)**.** *Let R be a ring. Suppose that we have two exact sequences of R-modules*

$$0 \to K_1 \to P_1 \to M \to 0 \quad (3.31)$$

$$0 \to K_2 \to P_2 \to M \to 0 \quad (3.32)$$

where P_1 *and* P_2 *are projective modules and M is a kG-module. Then*

$$K_1 \oplus P_2 \cong K_2 \oplus P_1. \quad (3.33)$$

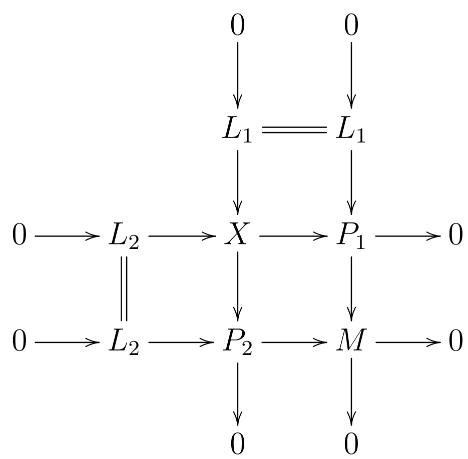
Lemma 3.10. *The Heller operator ΩM is unique in* $\text{stmod}(kG)$*.*

Proof. Suppose that we have two exact sequences with P_1 and P_2 are projective modules and $L_1 = \ker \pi_1$ and $L_2 = \ker \pi_2$

$$0 \to L_1 \to P_1 \to M \to 0 \tag{3.34}$$

$$0 \to L_2 \to P_2 \to M \to 0 \tag{3.35}$$

Then we have a diagram:



Since P_1 is a projective module, the first row splits so we can write $X \cong P_1 \oplus L_2$. Similarly, P_2 is a projective module and first column splits so we can write $X \cong P_2 \oplus L_1$. Therefore we get that $P_2 \oplus L_1 \cong P_1 \oplus L_2$. By Schanuel's Lemma, hence we have $L_1 \cong L_2$ in $\text{stmod}(kG)$.

□

Proposition 3.5. *Let F be a field or a complete discrete valuation ring and G be a finite group.*

(i) *For any FG-lattice M we have* $(\Omega^{-1}M)^* \cong \Omega(M^*)$.

(ii) *If M is an FG-lattice with non-zero projective summands then* $\Omega^{-1}\Omega(M) \cong M \cong \Omega\Omega^{-1}(M)$

(iii) *For given FG-modules* M_1 *and* M_2, *we have* $\Omega(M_1 + M_2) \cong \Omega M_1 \oplus \Omega M_2$ *and* $\Omega^{-1}(M_1 + M_2) \cong \Omega^{-1}M_1 \oplus \Omega^{-1}M_2$

(iv) *Let M be an FG-lattice with no non-zero projective summands. Then M is an indecomposable module if and only if* ΩM *is an idecomposable module if and only if* $\Omega^{-1}M$ *is an indecomposable module.*

Proof. (i) Firstly we take the dual of the sequence $0 \to \Omega(M^*) \to P_{M^*} \to M^* \to 0$ since it computes $\Omega(M^*)$. Then we have $0 \to M \to (P_{M^*})^* \to \Omega(M^*)^* \to 0$ and it computes $\Omega^{-1}(M)$. Therefore $\Omega^{-1}(M) \cong \Omega(M^*)^*$ and when we take the dual of map again then it gives the result.

(ii) To prove this, we first give a useful result. Let $0 \to M \to N \to L \to 0$ be a short exact sequence of FG-latices. If L has no non-zero projective summand and $N \to L$ is a projective cover then $M \to N$ is an injective hull. If M has no no-zero projective summand and $M \to N$ is injective hull then $N \to L$ is a projective cover.

Now we go back to proof. These isomorphisms come from the above result since when we look at the sequence it is clear that the sequence constructs $\Omega^{-1}\Omega(M)$ whenever it constructs ΩM. Similarly, it is obvious that it constructs $\Omega\Omega^{-1}(M)$.

(iii) This comes from the fact that the projective cover of $M_1 \oplus M_2$ is the direct sum of the projective covers of M_1 and M_2, and similarly with injective hulls.

(iv) If $\Omega(M)$ were to decompose then so would $M \cong \Omega^{-1}\Omega(M)$ by (3). Thus the indecomposability of M implies the indecomposability of ΩM. The reverse implication and the equivalence with the the indecomposability of $\Omega^{-1}M$.

□

As a consequences of the proposition we can state that

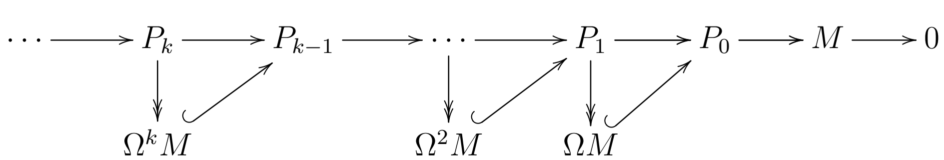
- M is an indecomposable non-projective FG-module if and only if ΩM is an indecomposable non-projective FG-module.
- Inductively, $\Omega^k M = \Omega(\Omega^{k-1} M)$. Also we have a similar result for the inverse Heller translate.

This proposition gives us that Ω permutes the isomorphism types of indecomposable FG-lattices with the inverse permutation Ω^{-1}. As a convention $\Omega^0 M = M$.

The Heller operator has an important role in homological algebra. We mention this a little bit. An essential idea in homological algebra is that of a *projective resolution* of an FG-module M. *Projective resolution* is a sequence of projective modules

$$\cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} 0 \xrightarrow{d_{-1}} 0 \cdots$$

This sequence is exact everywhere except at P_0, where its homology is M. To be more precise, we have also the following sequence



Now we state our last proposition about the Heller operator.

Proposition 3.6. *Let F be a field or a complete discrete valuation ring and G a finite group. For any FG-lattice M, we have $M \otimes_F \Omega^k F \cong \Omega^k M \oplus Q_k$ for each k, where Q_i is a projective FG-module.*

From this proposition, we can deduce that the modules Ω^k where $k \in \mathbb{Z}$, are closed under the operations of taking indecomposable summands of the tensor products.

4. CLASSIFICATION OF ENDO-TRIVIAL MODULES

Our main goal in this chapter is to classify the endo-trivial modules over p-groups. We follow t [1] "*Endo-permutation modules over p-groups 2*" written by Everett C. Dade. Sometimes we refer the first article of Dade about endo-permutation modules to recall the definition and some useful result. To be more clear, we fix some notations in this section. Let D be a commutative valuation ring, P a finite p-group and DP the group algebra of P. We denote the residue class field F which is defined as $D/J(D)$ where $J(D)$ is the Jacobson radical of D. Let Q be any subgroup of P, L be any indecomposable DQ-lattice and Λ be a fixed family of representatives for the conjugacy classes of subgroups of P . Let $\mathrm{End}_{DQ}(L)$ be the endomorphism ring of L which is a non-commutative local ring. We say that a DP-lattice is *projective-free* if it has no non-zero projective DP-direct summand.

4.1. Endo-trivial modules

Previously we defined the $endo - trivial$ modules and gave some properties of these modules and also stated some useful result about these modules. Now in the view of Dade's article [1], we define it again.

Let L be any DP-module. We call L an *endo-trivial* module if it satisfies

$$\mathrm{End}_D(L) \cong D_P \oplus (DP)^n \tag{4.1}$$

for some integer $n \geq 0$. If P is not a trivial group, then it is easy to notice that only the projective-free part of the endomorphism ring is a trivial DP-module D_P in above equation. When P is trivial, the endo-trivial DP-modules are clearly the non-zero DP-lattices.

Now, in [1] , we have definitions which are useful for now and the latter sections. Let E be the endomorphism ring $\mathrm{End}_D(\mathrm{L})$ of an endo-permutation DP-module L.

Then E is also a permutation DP-module so that it has a non-unique direct sum decomposition

$$E = \bigoplus_{Q \in \Lambda} E[Q] \tag{4.2}$$

where each $E[Q]$ is a DP-sublattice of E isomorphic to a direct sum of copies of $(D_Q)^P$. Now we define the natural action of P on E. This action sends $g \in P$, $\phi \in E$ into $g * \phi \in E$ which can be stated as

$$g * \phi = g\phi g^{-1} : l \mapsto g(\phi(g^{-1}l)), \quad \text{for all} \quad l \in L. \tag{4.3}$$

It is easy to check that for fixed $g \in P$, the above map is an automorphism of the D-algebra E.
Let Q be any subgroup of P. The trace map $\mathrm{tr_{Q \mapsto P}}$ from centralizer $C(Q)$ of Q in E to centralizer $C(P)$ of P in E is defined as

$$\mathrm{tr_{Q\mapsto P}}(u) = \sum_{g \in [P/Q]} g * \phi, \quad \text{for all} \quad \phi \in C(Q), \tag{4.4}$$

where [P/Q] is any representatives g for the left coset gQ of Q in P.

In [1], we have that
If Q is any subgroup of P and D_Q is the trivial rank-one D_Q-lattice, then the induced DP-lattice $(D_Q)^P$ is indecomposable with vertex Q.

It is easy to deduce from above result and Krull-Schmidt Theorem that

Remark 4.1. *Assume that $P > 1$. A DP-lattice L is an endo-trivial DP-module if and only if it is an endo-permutation DP-module such that* $\mathrm{rank}_D(E[P]) = 1$ *and* $E[Q] = 1$ *whenever $P > Q > 1$ in the decomposition (4.2) for* $E = \mathrm{End}_Q(L)$

Above result we mentioned about vertex of a group. Now we define the vertex and source of a group and also the capped endo-permutation module in the view of [1].

After then we give to the reader a proposition and the definition of vertex in the view of [4].

Definition 4.1. *Let Q be a subgroup of a finite group of G and U an indecomposable module. We say that Q is a vertex of U if Q is a minimal subgroup of G relative to which the indecomposable module U is projective. The vertex of U is defined up to conjugacy in G. Usually we write vtx(U) to denote a subgroup Q that is a vertex of U.*

Now we give some properties of a vertex of a module without proof.

Proposition 4.1. *Let D be a field of characteristic p or a complete discreate valuation ring with residue field of characteristic p.*

(i) The vertex of every indecomposable DG-module is a p-group.
(ii) An indecomposable DG-module is projective if and only if it is free as a D-module and its vertex is 1.
(iii) The vertex of the trivial DG-module D is a Sylow p-subgroup of G.

Definition 4.2. *Let P be a finite p-group. An endo-permutation kP-module M is said to be capped if it has at least one indecomposable direct summand with vertex P.*

Remark 4.2. *In particular, if M is indecomposable, then M is capped if and only if it has vertex P, or equivalently, M is not induced from a proper subgroup.*

Now we are ready to write the proposition.

Proposition 4.2. *If $P > 1$, then a DP-lattice L is an endo-trivial DP-module if and only if its projective-free part L_P is an endo-trivial module. In this case, L is also a capped endo-permutation DP-module and $L_P \cong cap(L)$ is DP-indecomposable. Therefore, any direct summand R of L is either projective or endo-trivial with R_f isomorphic to L_f.*

Proof. As we know L is isomorphic to the direct sum $L_f \oplus R$ as a DP-module where R is some projective DP-module. Also we know that the endomorphism ring $\mathrm{End}_D(L)$

of L is isomorphic as a DP-module to

$$\mathrm{End}_D(L_f) \oplus \mathrm{Hom}_D(L_f, R) \oplus \mathrm{Hom}_D(R, L_f) \oplus \mathrm{End}_D(R). \tag{4.5}$$

Since R is projective it is immediate that $\mathrm{Hom}_D(L_f, R)$, $\mathrm{Hom}_D(R, L_f)$ and $\mathrm{End}_D(R)$ are also projective. Thus Krull-Schmidt Theorem gives us that $\mathrm{End}_D(L_f)$ has the form $\mathrm{End}_D(L) \cong D_P \oplus (DP)^n$ if and only if $\mathrm{End}_D(L_f)$ does. Hence we get the first statement of the proposition.

If L is an endo-trivial module, then by Proposition 3.9 in [1] which is stated as the endo-permutation DP-module L is capped if and only if $\mathrm{End_D}(L)$ has a DP-direct summand isomorphic to D_P and by the above result it is capped endo-permutation DP-module. If L_f is the direct sum of two non-zero DP-modules R_1 and R_2, then we can easily say that direct sum of endomorphism rings of these modules is isomorphic to an DP-direct summand of $\mathrm{End}_D(L)$. By using Equation 4.1 and Krull-Schmidt Theorem, one of the DP-modules $\mathrm{End}_D(R_1)$ and $\mathrm{End}_D(R_2)$ must be projective. Let us say that $\mathrm{End}_D(R_1)$ is projective. But then R_1 is a projective DP-direct summand of L_f by Proposition XII.1.1 in [7]. But this contradicts with the definition of the projective-free part of L. Since L_f is an endo-trivial module, it cannot be zero. Then this gives that L_f is DP-indecomposable. Clearly it must be isomorphic to $\mathrm{cap}(L)$. Thus we get the second statement of the proposition. The rest of the proposition follows from Krull-Schmidt Theorem.

□

The family of endo-trivial modules are closed under some operations but this is not true for the operations induction and direct sums.

Proposition 4.3. *If L and R are endo-trivial DP-modules, then $L \otimes R$, L^* and $\mathrm{Hom}_D(L, R)$ are also endo-trivial modules. Moreover, the restriction L_Q is an endo-trivial DQ-module for any subgroup Q of P.*

Proof. By Equation 4.1, we have a DP-isomorphism for R

$$\mathrm{End}_D(R) \cong D_P \oplus (DP)^m, \tag{4.6}$$

for some nonnegative integer m. Then this gives that there are DP-isomorphisms

$$\begin{aligned}
\mathrm{End}_D(L \otimes R) &\cong \mathrm{End}_D(L) \otimes \mathrm{End}_D(R) \\
&\cong [D_P \otimes D_P] \oplus [(DP)^n \otimes D_P] \oplus [(D_P)^m \otimes D_P] \oplus [(D_P)^{mn} \otimes D_P] \\
&\cong D_P \oplus [(DP)^{(n+m+nm|P|)}].
\end{aligned}$$

Thus the DP-lattice $L \otimes R$ is endo-trivial.

The natural isomorphism of $\mathrm{End}_D(L^*)$ onto $\mathrm{End}_D(L)$ preserve the DP-module structure of the rings. Thus $\mathrm{End}_D(L^*)$ has the form Equation 4.1, in other words L^* is an endo-trivial module. Moreover, this and the previous result give us that $\mathrm{Hom}_D(L, R)$ $\cong L^* \otimes R$ is an endo-trivial DP-module.

Lastly, we look at the restriction of $\mathrm{End}_D(L)$ on Q. By Equation 4.1, the restriction $\mathrm{End}_D(L_Q)$ provides the following

$$\mathrm{End}_D(L_Q) \cong D_Q \oplus [DQ^{n[P:Q]}]. \tag{4.7}$$

Thus L_Q is an endo-trivial DQ-module. Hence we completely proved the proposition.

□

Proposition 4.4. *If P >1 and L is a projective-free (indecomposable) endo-trivial DP-module, then for any integer k,* $\Omega^k L$ *is also endo-trivial module.*

Proof. Since P>1 we can deduce from Proposition 4.2 that an endo-trivial DP-module L is projective free then it is indecomposable. We assume from [1] that given L and R

are two DP-lattices then there exists a short exact sequence of DP-modules

$$0 \to R \to B \to L \to 0 \tag{4.8}$$

in which B is a projective DP-lattice. Then in [1] Feit's Lemma (2.12) (which says that if (2.7) holds, then the projective-free parts of the DP-lattices $\mathrm{End}_D(R)$ and $\mathrm{End}_D(L)$ are isomorphic.) and Equation 4.1 give us that $\mathrm{End}_D(R)$ is of the form Equation 4.1 so R is endo-trivial. We know from Proposition 4.2 the projective-free part ΩL of R is an endo-trivial DP-module. Similarly, $\Omega^{-1}L$ is an endo-trivial DP-module. Hence we complete the proof of the proposition from this and the properties of the Heller operator which wrote before. □

Corollary 4.1. *If $P > 1$, then $\Omega^k D_P$ is an indecomposable endo-trivial DP-module, for any integer k.*

Proof. To get the result, we apply the proposition to $L = D_P$. □

Also the composition of $\mathrm{End}_D(L)$ in Equation 4.1 can be specified closely. Next lemma provide us this.

Lemma 4.1. *If $P > 1$ and L is an endo-trivial DP-module, then $\mathrm{End}_D(L)$ is the direct sum of its trivial rank-one DP-submodule D1 and the kernel $\mathrm{Ker}(\mathrm{Tr})_L$ of the map Tr_L sending each D-linear transformation ϕ of L into its trace $\mathrm{Tr}_L(\phi)$ in D. This kernel $\mathrm{Ker}(\mathrm{Tr})_{\mathrm{L}}$ is a free DP-submodule of $\mathrm{End}_D(L)$.*

Proof. By using Equation 4.1 we have that

$$[\mathrm{rank}(L)]^2 = \mathrm{rank}(\mathrm{End}_D L) \equiv 1 \quad (mod\ |P|) \tag{4.9}$$

Since P >1, we get that

$$\mathrm{Tr}_L(1) = \mathrm{rank}_D(L) \not\equiv 0 \pmod p. \tag{4.10}$$

Thus we have that $\mathrm{Tr}_L(1)$ is a unit in D by (1.1a) in [1]. Hence this gives us the first conclusion of the lemma.

It is known from (3.2) in [1] that $\mathrm{Ker}(Tr_L)$ is an $DP-$sublattice of $\mathrm{End}_D(L)$. Then by using (4.1) and The Krull-Schmidt therorem we can conclude the proof of the lemma. □

Now we give another characterization of the endo-trivial modules by using the above lemma.

Proposition 4.5. *Let L be a non-zero DP-lattice. Then L is endo-trivial if and only if it satisfies the followings*

$$\mathrm{rank}_D(\mathrm{End}_{DP}(L)) = 1 + \frac{[\mathrm{rank}_D(L)]^2 - 1}{|P|} \tag{4.11}$$

Corollary 4.2. *Any DP-endomorphism ϕ of L satisfying $\mathrm{Tr}_L(\phi)$=0 is of the form $\phi = \mathrm{tr}_{1\to P}(\psi)$, for some D-endomorphism ψ of L.*

Proof. When we take P =1, the proposition is clearly true. Therefore, we look at the case $P > 1$.

Assume that L is endo-trivial, then by Equation 4.1 the isomorphism implies that

$$[\mathrm{rank}_D(L)]^2 = \mathrm{rank}_D(\mathrm{End}_D(L)) = 1 + n|P|, \tag{4.12}$$

$$\mathrm{rank}_D(\mathrm{End}_D(L)) = 1 + n \tag{4.13}$$

After some calculation Equation 4.11 comes directly from this. By using Lemma 4.1, we say that the $\mathrm{Ker}(\mathrm{Tr_L})$ is free. So it provides

$$\mathrm{Ker}(\mathrm{Tr}_L) \cap \mathrm{End}_{DP}(L) = \mathrm{C}(P \ in \ \mathrm{Ker}(\mathrm{Tr}_L)) = \mathrm{tr}_{1\to P}(\mathrm{Ker}(\mathrm{Tr}_L)) \qquad (4.14)$$

where as we defined before $\mathrm{C}(P \ in \ \mathrm{Ker}(\mathrm{Tr_L}))$ is the centralizer of P in $\mathrm{Ker}(\mathrm{Tr_L})$. Hence we get the second assertion of the proposition.

For the other direction, suppose the above assertions hold for some DP-lattice L. Then it follows from Equation 4.11 that

$$[\mathrm{rank}_D(L)]^2 \equiv 1 \ (mod \ \ |P|). \qquad (4.15)$$

Therefore we get $\mathrm{rank}_D(L)$ is not divisible by p. Then since the residue class field has characteristic p, this implies that $\mathrm{End}_D(L)$ is the direct sum of its DP-submodules D with rank 1 and $\mathrm{Ker}(\mathrm{Tr}_L)$ with rank $[\mathrm{rank}_D(L)]^2$ - 1. Hence we can choose the endomorphism ψ of the assertion (ii) in $\mathrm{Ker}(\mathrm{Tr}_L)$.

Assume that $\{\phi_1, \cdots, \phi_n\}$ is a D-basis for $\mathrm{C}(P \ in \ \mathrm{Ker}(\mathrm{Tr}_L))$ as above and $\{\psi_1, \cdots \psi_n\}$ are elements of $\mathrm{Ker}(\mathrm{Tr}_L)$ given by (ii) so that

$$\phi_i = \mathrm{tr}_{1\to P}(\psi_i) \qquad (4.16)$$

for all i. We know that $\mathrm{C}(P \ in \ \mathrm{Ker}(\mathrm{Tr}_L))$ is a D-sublattice of $\mathrm{Ker}(\mathrm{Tr}_L)$ so the images $\{1 \otimes \phi_1, \cdots, 1 \otimes \phi_n\}$ are F-linearly independent in $F \otimes \mathrm{Ker}(\mathrm{Tr}_{\underline{L}})$. As we know before, P is a p-group and F is a field of characteristic p so we have

$$\mathrm{tr}_{1\to P}(1 \otimes \psi_1) = 1 \otimes \phi_1, \cdots, \mathrm{tr}_{1\to P}(1 \otimes \psi_n) = 1 \otimes \phi_n \qquad (4.17)$$

are linearly independent. So $1\otimes\psi_1, \cdots, 1\otimes\psi_n$ are an FP-basis for a free FP-submodule of $F\otimes\mathrm{Ker}(\mathrm{Tr}_L)$. This implies that $\psi_1, \cdots \psi_n$ are an DP-basis for a free DP-submodule B which is also a D-sublattice of $\mathrm{Ker}(\mathrm{Tr}_L)$. Furhermore, Equation 4.11 gives that

$$n = ([\mathrm{rank}_D(L)]^2 - 1)/|P|. \tag{4.18}$$

Thus we have that

$$\mathrm{rank}_D(B) = n|P| = (\mathrm{rank}_D\mathrm{Ker}(\mathrm{Tr}_L)L), \tag{4.19}$$

which constrain the D-sublattice B to equal $\mathrm{Ker}(\mathrm{Tr}_L)$. Hence $\mathrm{Ker}(\mathrm{Tr}_L)$ is a free DP-module and $\mathrm{End}_{DP}(L) = D \dotplus \mathrm{Ker}(\mathrm{Tr}_L)$ providing Equation 4.1. So this proves the proposition.

□

In the view of Proposition 4.5, we can say that the characterization indicates that the definition of an endo-trivial DP-module does not depend on the embedding of P in this ring but depend only on the D-algebra DP.

Proposition 4.6. *Let R be another subgroup of the unit group of the D-algebra DP such that DP is also the group algebra DR. Then a DP-lattice L is an endo-trivial DP-module if and only if it is an endo-trivial DR-module.*

Proof. We saw that Equation 4.11 depends only on the structure of L as a module over the D-algebra $DP = DR$. We know that a DP-endomorphism ϕ of L is of the form $\mathrm{tr}_{1\to P}(\psi)$, for some ψ in $\mathrm{End}_D(L)$, if and only if it can be factored through a projective DP-module B, in other words, ϕ is the composition of some DP-homomorphism ε from L to B and some DP-homomorphism ϑ from B to L (to see more detail look at Proposition XII.1.2 of [7]). Moreover it is clear that the second assertion of the previous proposition depends only on the structure of L as a module over $DP = DR$. Considering Proposition 4.5, we complete the proof of the proposition. □

When we take D as a field F with characteristic p, Proposition 4.6 has its greatest interest. Because as we shall see later, the algebra FP has the property of preserving the structure of the group algebra, there are many ways to embed P in FP.

Proposition 4.7. *Suppose that P is an abelian group. An indecomposable FP-module L is endo-trivial if and only if it is an indecomposable endo-permutation FP-module with vertex P whose isomorphism class* $[\tilde{L}]$ *lies in* $\text{Res Ind}_{\text{P}}(FP)$.

Proof. If $P = 1$ then the proposition is clearly true. So we suppose that $P > 1$.
If L is endo-trivial, then Proposition 4.2 gives us that $L \cong \text{cap}(L)$ is an indecomposable endo-permutation DP-module with vertex P. By using Lemma (5.19) in the [1] and 4.1 we get that $[\tilde{L}]$ lies in $\text{Res Ind}_P(FP)$.

On the other hand, if L is an indecomposable endo-permutation FP-module with vertex P, then by Lemma 6.4 in [1] we say that $\dim E[P]_F = 1$. It is well-known fact that for the abelian group P, every subgroup Q is normal, so $\text{core}(Q) = Q$. Thus $[\tilde{L}]$ in $\text{Res Ind}_P(FP)$ gives that $E[Q] = 0$ for all proper subgroups Q of P by Lemma 5.19 in [1]. When we consider 4.1, it gives the result.

Note that core(Q) is the intersection of all the P-conjugates of Q where Q is not equal to 1 and P. □

If we take $D = F$, then Proposition 4.5 gives a useful result.

Proposition 4.8. *If L is an endo-trivial FP-module and g lies in the center Z(P) of P, then*

$$(g - 1 \textit{ on } L) = \text{tr}_{1 \to P}(\psi) \tag{4.20}$$

for some $\psi \in \text{End}_F(L)$.

Proof. It is easy to see that $(g-1 \ on \ L)$ is in $\mathrm{End}_F(L)$ since g is an element of the center $Z(P)$ of P. Also F is a field of characteristic p and g lies in the p-group P, the eigenvalues of the linear transformation $(g-1 \ on \ L)$ are all 1.Hence $\mathrm{Tr}_L\big((g \ on \ L)\big) = \mathrm{Tr}_L(1)$ and $\mathrm{Tr}_L\big((g-1) \ on \ L\big) = 0$. Thus Equation 4.20 holds for some ψ in $\mathrm{End}_F(L)$ by Proposition 4.5. Hence this completes the proof of the proposition. □

4.2. Cyclic P

In this section we deal with a cyclic p-group P generated by g, that is $P = \langle g \rangle$ where the order of the element g is $q = p^a > 1$. Since F is a field of characteristic p, the group algebra $FP = F\langle g \rangle$ can be regarded as the truncated polynomial ring generated by its element $x = g-1$, since $x^0 = 1$, $x, x^2, \cdots, x^{q-1}$ are an F-basis for $F\langle g \rangle$ and $x^q = 0$.

When we consider Proposition 4.8, we can deduce that any endo-trivial $F\langle g \rangle$-module L satisfies

$$(x \text{ on } L) = \mathrm{tr}_{1\to P}(\psi), \tag{4.21}$$

for some $\psi \in \mathrm{End}_F(L)$. In order to use this relation we must compute $\mathrm{tr}_{1\to \mathrm{P}}(\psi)$ in terms of x.

Lemma 4.2. *Assume that L is any $F\langle g \rangle$-module and $\psi \in \mathrm{End}_F(L)$. Then*

$$\mathrm{tr}_{1\to\langle g \rangle}(\psi) = \sum_{i=0}^{q-1} x^i g \psi x^{q-1-i}. \tag{4.22}$$

Proof. As we know from Equation 3.2 in [1], the expression $w\psi z$, for any w, z in $F\langle g \rangle$ denotes the linear transformation which sends any $l \in L$ into $w(\psi(zl)) \in L$. Therefore Equation (3.2) is

$$(g^{-1}-1) * \psi = g^{-1}\psi g - \psi = g^{-1}\psi(g-1) - (g-1)g^{-1}\psi. \tag{4.23}$$

A simple induction on this formula tells us that:

$$(g^{-1}-1)^j * \psi \equiv \sum_{i=0}^{j}(-1)^i \binom{j}{i}(g-1)^i g^{-j}\psi(g-1)^{j-i}, \tag{4.24}$$

for any integer $j \geq 0$, where $\binom{j}{i}$ is the usual binomial coefficient.

Then since $\binom{q-1}{i} \equiv (-1)^i (\text{mod } p)$, for $i = 0, 1, \cdots, q-1$ as q is a positive power of the characteristic p of F, we have

$$(g-1)^{q-1} = 1 + g^{-1} + \cdots + g^{-q+1} \tag{4.25}$$

is the sum of the elements of $\langle g \rangle$ and so we get

$$\text{tr}_{1\to\langle g\rangle}(\psi) = (g-1)^{q-1} * \psi. \tag{4.26}$$

Therefore Equation 4.22 is Equation 4.24 with $j = q-1$. Hence we have proved the lemma. □

Theorem 4.1. *An $F\langle g\rangle$-lattice L satisfies Equation 4.21 if and only if it is isomorphic to the $F\langle g\rangle$-module*

$$[(F\langle g\rangle/xF\langle g\rangle)^r] \oplus [(F\langle g\rangle/x^{q-1}F\langle g\rangle)^s] \oplus [(F\langle g\rangle)^t], \tag{4.27}$$

for some non-negative integers r, s and t.

Proof. Firstly, we know from Corollary 4.1 that $F\langle g\rangle$-modules
$F\langle g\rangle/xF\langle g\rangle \cong F_{\langle g\rangle}$ and $F\langle g\rangle/x^{q-1}F\langle g\rangle \cong \Omega F_{\langle g\rangle}$ are both endo-trivial. Therefore they satisfies Equation 4.21 by Proposition 4.8. As $F\langle g\rangle$ is a free $F\langle g\rangle$-module, $\text{End}_F(F\langle g\rangle)$ is also endo-trivial. Thus $\text{End}_{F\langle g\rangle}(F\langle g\rangle) \cong \text{tr}_{1\to\langle g\rangle}(\text{End}_F(F\langle g\rangle))$. This implies that $F\langle g\rangle$ also provides Equation 4.21. So every $F\langle g\rangle$-lattice of the form Equation 4.27 satisfies Equation 4.21.

Now assume that L is an $F\langle g\rangle$-lattice satisfying 4.21. As the view of the structure theorem for $F\langle g\rangle$-lattices (look at Lemma (64.2) of [8]), we have that L is a direct sum of indecomposable $F\langle g\rangle$-submodules R, and each of them is isomorphic to $F\langle g\rangle/x^iF\langle g\rangle$ for some $j = 1, 2, \cdots, q$. In order to show that L is of the form in Equation 4.27 we need show that j is $1, q-1$ or q, for any such R.

Assume for a contradiction that it is false. Then R=$F\langle g\rangle r$ is generated by an element r satisfying

$$x^{j-1} \neq 0, \quad x^j r = 0, \tag{4.28}$$

for some $j = 2, \cdots, q-2$. Then from Equation 4.21 and Equation 4.22 we have that

$$xr = \sum_{i=0}^{q-1} x^i g\psi(x^{q-1-i}r) \tag{4.29}$$

for some $\psi \in \mathrm{End}_F(L)$. If we take i= 0 or 1, then $q-1-i \ \geq q-2 \ \geq j$. Therefore $x^{q-1-i}r$= 0 by Equation 4.28. If $i \geq 2$ then $x^i g\psi(x^{q-1-i}r) \in x^iL \subseteq x^2L$. Then the above expression for xr gives us that this element lies in x^2L. Also we know that it lies in the $F\langle g\rangle$-direct summand R of L, so this implies that it lies in x^2R. However this is not impossible since r generates $R \cong F\langle g\rangle/x^jF\langle g\rangle$ and $j \geq 2$. Hence we get the contradiction and it completes the poof. □

Corollary 4.3. *The indecomposable endo-trivial $F\langle g\rangle$-modules are precisely those $F\langle g\rangle$-lattices isomorphic to one of the following two modules*

$$F\langle g\rangle/xF\langle g\rangle \cong F_{\langle g\rangle} \text{ and } F\langle g\rangle/x^{q-1}F\langle g\rangle \cong \Omega F_{\langle g\rangle} \tag{4.30}$$

Proof. We know from Corollary 4.1 that the above indecomposable $F\langle g\rangle$-modules in Equation 4.30 are both endo-trivial. Then by using the theorem and Proposition 4.8 we get that any indecomposable endo-trivial $F\langle g\rangle$-module L is isomorphic either to the modules in Equation 4.30 or to $F\langle g\rangle$. We eliminate the option $F\langle g\rangle$ since $F\langle g\rangle$ has vertex $1 \neq \langle g\rangle$ and L is capped by Proposition 4.2. Hence we prove the corollary. □

Later we shall see that Equation 4.21 holds for some ψ leaving invariant some one-dimensional $F\langle g\rangle$-submodule R of L. The next result is the consequences of this additional condition.

Before we state the proposition, we give the definition of the regular module: The D-module M is called *regular* if for each element $m \in M$ there is $\alpha \in \mathrm{Hom}_D(M, D)$ satisfying $(m\alpha)m = m$.

Now we are ready to state the proposition.

Proposition 4.9. *Assume that R is a one-dimensional $F\langle g\rangle$-submodule of an $F\langle g\rangle$-lattice L. If Equation 4.21 is valid for some ψ which sends R into itself, then either R is an $F\langle g\rangle$-direct summand of L or there is a regular $F\langle g\rangle$-direct summand of L such that $R = x^{q-1}L_1$.*

Proof. Let r be a basis for $R = Fr$. It is clear that if R is not an $F\langle g\rangle$-direct summand of L, then $R \subseteq xL$. Therefore, we have that there is an element l of L such that $r = xl$.

If we take $q = 2$, then it is clear that $xl = r \neq 0$. So this implies that l generates a regular $F\langle g\rangle$-submodule $F\langle g\rangle l \cong F\langle g\rangle$, which must be an $F\langle g\rangle$-direct summand of L. Therefore the proposition possess for this situation.

Now we assume that $q > 2$. From Equation 4.21 and 4.22 we get

$$r = xl = \sum_{i=0}^{q-1} x^i g\psi(x^{q-1-i}l). \tag{4.31}$$

Notice that the element x annihilates the one-dimensional $F\langle g\rangle$-submodule R. Then this gives that $x^2l = xr = 0$. Thus it implies that $x^{q-1-i} = 0$ for $i = 0, 1, \cdots, q-3$. It is easy to check that when we take $i = q-2$, we have that $x^2l = xl = r \in R$. By hypothesis we have

$$g\psi(x^{q-1-i}l) \in g\psi(R) \subseteq R \tag{4.32}$$

and

$$x^i g\psi(x^{q-1-i}l) \in x^{q-2}R \subseteq xR = 0, \tag{4.33}$$

since $q > 2$. Thus we have that all terms in the above sum with $i \leq q-2$ are zero. In other words, we get

$$0 \neq r = x^{q-1}g\psi(l). \tag{4.34}$$

Hence we can conclude that $g\psi(l)$ generates a regular $F\langle g\rangle$-submodule L_1 satisfying the conditions of the proposition. □

4.3. Two-generator abelian P

In this section we take P as a direct product

$$P = \langle g\rangle \times \langle t\rangle \tag{4.35}$$

of cyclic subgroups $\langle g\rangle$ and $\langle t\rangle$ generated by elements g and t where order of these elements q and r satisfying

$$q = p^a \geq r = p^b \geq 1 \tag{4.36}$$

for some integers a and b. Now we define two elements in FP by

$$y = g - 1 \quad z = t - 1. \tag{4.37}$$

In this section we devote ourself to study a projective-free endo-trivial FP-module L. Let (z^{r-1} *on* L) be a linear transformation. Then the kernel L^1 is an FP-submodule of L and it is defined as

$$L^1 = \mathrm{Ker}(z^{r-1} \text{ on } L). \tag{4.38}$$

It is clear that z is an annihilator the factor module L/L^1. So the minimal number $m(L/L^1)$ of generators of L/L^1 as an FP-module is the dimension of the factor space $L/(yL + L^1)$. We denote this constant $m(L/L^1)$ of the FP-module L by $m_1(L)$.

In the view of Proposition 4.4, we use the fact that each $\Omega^n L$ is a projective-free endo-trivial FP-module. In order to minimize $m_1(L)$ we assume that L has been chosen among $\Omega^n L$. In other words

$$m_1(\Omega^n L) \geq m_1(L) \tag{4.39}$$

for all integers n.

Moreover, Proposition 4.8 gives us an F-endomorphism ψ of L such that

$$(y \text{ on } L) = \mathrm{tr}_{1\to \mathrm{P}}(\psi). \tag{4.40}$$

Considering Equation 4.35, this gives that $\phi = \mathrm{tr}_{1\to\langle \mathrm{t}\rangle}(\psi)$ satisfies

$$\phi \in \mathrm{End}_{F\langle t\rangle}(L), \tag{4.41}$$

$$(y \text{ on } L) = \mathrm{tr}_{1\to\langle g\rangle}(\phi). \tag{4.42}$$

By using Proposition 4.3 we can deduce that the restriction $L_{\langle t\rangle}$ is an endo-trivial $F\langle t\rangle$-module. Therefore Proposition 4.2 and Corollary 4.3 imply that $L_{\langle t\rangle}$ is one of the two-forms

$$L_{\langle t\rangle} \simeq [F\langle t\rangle/zF\langle t\rangle] \oplus [F\langle t\rangle]^u, \tag{4.43}$$

$$L_{\langle t\rangle} \simeq [F\langle t\rangle/z^{r-1}F\langle t\rangle] \oplus [F\langle t\rangle]^u \tag{4.44}$$

for some non-negative integer u. Therefore we have that L^1/zL is a one-dimensional FP-submodule of L/zL.

Our minimality condition is very important for proving the next results.

Lemma 4.3. *Assume that Equation 4.39 holds. Then L^1/zL is an FP-direct summand of L/zL.*

Proof. It is easy to check that t centralizes L/zL. Indeed, let w be an arbitrary element of L/zL. Then we can write it as $w= l + zL$. When we apply t on w we get that $t.w$=$t.(l + zL)$ in L/zL. So we need only to show that L^1/zL is an $F\langle g\rangle$-direct summand of L/zL.

The $F\langle t\rangle$-endomorphism ϕ of Equation 4.41 and Equation 4.42 must fix both zL and L^1. Therefore we can obtain an F-endomorphism $\tilde{\phi}$ of L/zL which fixes L^1/zL. Then we can easily see that

$$(y \text{ } on \text{ } L/zL) = \mathrm{tr}_{1\to\langle g\rangle}. \tag{4.45}$$

Thus, the $F\langle g\rangle$-lattice L/zL and L^1/zL which is a one-dimensional $F\langle g\rangle$-submodule provide the hypothesis of Proposition 4.9.

Suppose that L^1/zL is not an $L\langle g\rangle$-direct summand of L/zL. Then from Proposition 4.9 we get that L/zL can be expressed as a direct sum $L_1 \dotplus \cdots \dotplus L_s$ where $s \geq 1$. In this expression for each i, L_i is an indecomposable $F\langle g\rangle$-submodule. Note that L_1

is $F\langle g\rangle$-isomorphic to $F\langle g\rangle$. Also we have that $L^1/zL= y^{q-1}L_1$. It is well-known fact that each indecomposable $F\langle g\rangle$-module is generated by one element (look at Lemma (64.2) of [8]). This gives us non-zero elements $\tilde{l_1}, \cdots, \tilde{l_s}$ of L/zL satisfying:

$$L/zL = F\langle g\rangle\tilde{l_1} \dotplus \cdots \dotplus F\langle g\rangle\tilde{l_s}, \tag{4.46}$$

$$F\langle g\rangle\tilde{l_1} \simeq F\langle g\rangle, \tag{4.47}$$

$$L^1/zL = Fy^{q-1}\tilde{l_1}. \tag{4.48}$$

It clear from Equation 4.46 and Equation 4.48 that the number s of the $\tilde{l_i}$ is the number $m_1(L) = m(L/L^1)$ of generators for the $F\langle g\rangle$-module L/L^1. Assume that $l_1, \cdots, l_s$ are arbitrary elements of L with the property that $\tilde{l_1} \cdots, \tilde{l_s}$ are their corresponding images in L/zL. Therefore by Equation 4.46 and *Nakayama Lemma* we get that $l_1, \cdots, l_s$ construct a minimal set of generators for the FP-lattice L. On the other hand, suppose that B is a free FP-module on s generators $p_1, \cdots, p_s$ and ν is the FP-epimorphism of B onto L which maps p_i into l_i, for $i = 1, \cdots, s$. Then it can be said that B and ν produce a projective cover for L in order to get that kernel $R =$ Ker (ν) is FP-isomorphic to ΩL (for detail information look at [15]).

The images $\tilde{p_1}, \cdots, \tilde{p_s}$ of $p_1, \cdots, p_s$ respectively, as an FP-basis construct the free FP-module B/zB. The epimorphism ν: $B \to L$ induces an $F\langle g\rangle$-epimorphism $\tilde{\nu}$

$$\tilde{\nu} : B \to L$$
$$\tilde{p_i} \mapsto \tilde{l_i}$$

for $i = 1, \cdots, s$. From the above equalities, we deduce that the kernel of $\tilde{\nu}$ is an $F\langle g\rangle$-submodule of $F\langle g\rangle\tilde{p_2} \dotplus \cdots F\langle g\rangle\tilde{p_s}$. Hence we find that the minimal number of generators $m(\mathrm{Ker}(\nu))$ is at most $m-1$.

It is clear that $\mathrm{Ker}(\nu)$ is $(R+zB)/zB$ which is $F\langle g\rangle$-isomorphic to $R/(R\bigcap zB)$. We know that B is a free $F\langle g\rangle$-module so $\mathrm{Ker}(\nu) = zB$. Thus $R/(R\bigcap zB) = (z^{r-1}$ on $R) = R^1$. Hence we reach that

$$m_1(R) = m(R/R^1) = m(\mathrm{Ker}(\nu)) < m = m_1(L). \tag{4.49}$$

Recall that $m(\Omega^n L) > m_1(L)$ for all n but we get $R = \Omega L$. Therefore we have a contradiction. This proves the lemma. □

When we apply the duality to the previous lemma, we get

Lemma 4.4. *If Equation 4.39 holds, then the submodule $z^{r-1}L$ of* Ker(z on L) *is an FP-direct summand.*

Proof. We know that the dual L^* is an FP-module and by Proposition 4.3 we have that $L^* = \mathrm{Hom_F}(L, F)$ is projective-free and endo-trivial. When we multiply by z^{r-1} in L it induces an $F\langle g\rangle$-isomorphism of L/L^1 onto $z^{r-1}L$. Therefore we get that

$$m_1(L) = m(L/L^1) = m(z^{r-1}L). \tag{4.50}$$

We know that $L^* = \mathrm{Ker}(z^{r-1}$ *on* $L^*)$ is the perpendicular subspace to $z^{r-1}L$. Therefore the factor module L^*/L^{1*} is $F\langle g\rangle$-isomorphic to the dual of $z^{r-1}L$. Then we have

$$m_1(L^*) = m(L^*/(L^1)^*) = m(z^{r-1}L). \tag{4.51}$$

Hence we get

$$m_1(L^*) = m_1(L). \tag{4.52}$$

It is easy to deduce from the definition of the Heller operator that $\Omega^n L^*$ is FP-isomorphic to the dual of $\Omega^{-n}L$, for all integers n (see (2.10) and (2.11) in [1]). Also this statement and Equation 4.52 give us that L^* holds the minimality condition in Equation 4.39. When we apply this to Lemma 4.3 we have $(L^1)^*/zL^*$ is a direct FP-summand of L^*/zL^*. Obviously, zL^* is perpendicular to Ker(z on L). Then the factor module L^*/zL^* is FP-isomorphic to the dual of Ker(z *on* L) with $(L^1)^*/zL^*$ moving into the perpendicular space to $z^{r-1}L$. Thus we get $z^{r-1}L$ is an FP-direct summand of Ker(z *on* L). Hence we have proved the lemma. □

By using Lemma 4.3 and Lemma 4.4, we limit the structure of $L_{\langle t\rangle}$.

Lemma 4.5. *If Equation 4.39 L satisfies, then $L_{\langle t\rangle}$ is of the form Equation 4.44.*

Proof. For a contradiction assume that it is not the case. Then we have that $L_{\langle t\rangle}$ is of the form Equation 4.44 and $r \geq 3$ which is necessary condition since otherwise Equation 4.43 and Equation 4.44 are the same. Next it is easy to follow that the FP-submodules $L^1 = \text{Ker}(z^{r-1}$ *on* $L)$ and $L^2 = \text{Ker}(z^{r-2}$ *on* $L)$ have the relation

$$z^2L \subset L^2 = zL^1 \subseteq zL \subseteq L^1 \tag{4.53}$$

Firstly, it must be noted that multiplication by z in L gives an FP-isomorphism of L/zL onto zL/z^2L. In other words,

$$zx : L/zL \to zL/z^2L \tag{4.54}$$

$$L^1/zL \mapsto L^2/z^2L \tag{4.55}$$

Therefore Lemma 4.4 gives us an FP-complement R_1 to L^2/z^2L in zL/z^2L. It means we can write L^2/z^2L as a direct sum

$$L^2/z^2L = R_1 \oplus L^2/z^2L. \tag{4.56}$$

Moreover, multiplying by z^{r-2} in L gives an FP-isomorphism of L^1/L^2 onto Ker(z on L) mapping zL/L^2 onto $z^{r-1}L$. That is

$$z^{r-2}x : L^1/L^2 \to \text{Ker}(z \text{ on } L) \tag{4.57}$$

$$zL/L^2 \mapsto z^{r-1}L. \tag{4.58}$$

Therefore from Lemma 4.4 we get an FP-complement $\tilde{R_2}$ to zL/L^2 in L^1/L^2. That is

$$L^1/L^2 = \tilde{R^2} \oplus zL/L^2. \tag{4.59}$$

Then by the correspondence theorem, let R_2 be the inverse image of $\tilde{R_2}$ in L^1/z^2L. Then we get that FP-submodules R_1 and R_2 of L^1/z^2L satisfy

$$L^1/z^2L = R_1 \dotplus R_2, \tag{4.60}$$

$$0 \subset L^2/z^2L \subset R_2, \tag{4.61}$$

$$zL/z^2L = R_1 \dotplus (L^2/z^2L). \tag{4.62}$$

Now we claim that

$$g \text{ centralizes } R_2.$$

Assume that our claim is false. Then L^2/z^2L and L^1/zL are one dimensional modules since Equation 4.44 is valid for $r \geq 3$. Also by Equation 4.60 and Equation 4.62 we have that L^1/zL is isomorphic to $R_2/(L^2/z^2L)$. So it is immediate that R_2 is an FP-module with dimension 2. Since g does not centralize R_2, its one-dimensional submodule L^2/z^2L must be true for

$$0 \subset L^2/z^2L = yR_2 \tag{4.63}$$

As we know before from Equation 4.41 and Equation 4.42, L^1, L^2 and z^2L are fixed by the $F\langle t\rangle$-endomorphism ϕ. Therefore it induces an F-endomorphism $\tilde{\phi}$ of L^1/z^2L

$$\tilde{\phi} : L^1/z^2L \to L^1/z^2L \tag{4.64}$$

$$L^2/z^2L \mapsto L^2/z^2L. \tag{4.65}$$

Then Equation 4.42 implies that

$$(y \text{ on } L^1/z^2L) = \mathrm{tr}_{1\mapsto\langle g\rangle}(\tilde{\phi}). \tag{4.66}$$

So we consider L^1/z^2L and its submodule L^2/z^2L where its dimension is 1 as $F\langle g\rangle$-lattices that satisfy the assumptions of Proposition 4.9.

Hence we get that L^2/z^2L is not an $F\langle g\rangle$-direct summand of L^1/z^2L, this comes from Equation 4.63. Therefore Proposition 4.9 gives us that

$$0 \subset (L^2/z^2L) \subseteq y^{q-1}(L^1/z^2L). \tag{4.67}$$

But this gives a contradiction with Equation 4.63 and Equation 4.60 because $q-1 \geq r-1 \geq 2$ by Equation 4.36. Therefore this contradiction proves our claim so g centralizes R_2.

It comes from Equation 4.36 that the elements $\tilde{g} = gt$ and t provide all our assumptions that g and t satisfy. Therefore when we put in place of g by $\tilde{g}$ in this construction, then it is easy to see that this does not change the submodules L^1, L^2 zL and z^2L because we define them just in terms of t. Moreover, it is clear that $\tilde{g}$ and g act on L/L^1 in the same way so we can say that $m_1(L) = m(L/L^1)$ does not change.

Therefore Equation 4.39 is still valid. Therefore we can deduce that from argument (4.41), $\tilde{g}$ also centralizes R_2. From the above we know that $\tilde{g} = gt$, so $t = g^{-1}\tilde{g}$ also centralizes R_2. However this is not possible from Equation 4.60 and Equation 4.61 because we have

$$z(L^1/z^2L) = L^2/z^2L \neq 0. \tag{4.68}$$

Hence we get a contadiction again. Therefore we prove the lemma. □

With the above lemma we do not have much possibilities for L.

Lemma 4.6. *If Equation 4.39 holds, then L is FP-isomorphic to F_P.*

Proof. By the preceding lemma, the restriction $L_{\langle t\rangle}$ is of the form Equation 4.43. Also from Lemma 4.4 we have R is the FP-complement of $z^{r-1}L$ in Ker(z on L). In the view of Equation 4.43 we can say that the submodule R has dimension one and it is complement to zL in Ker(z^{r-1} on L)$= L^1$. From Lemma 4.3 we get that L^1/zL has an FP-complement in L/zL. So by the correspondence theorem the inverse image of this complement is an FP-module L which is indecomposable. In the view of Proposition 4.2 the projective-free endo-trivial FP-module L is indecomposable. So we get $L = R$ has dimension one. Furthermore we know that P is a p-group and F is a field of characteristic p, hence this completes the proof of the lemma. □

For the next theorem we do not consider the minimality condition (in Equation 4.39) for a while.

Theorem 4.2. *Assume that P provides the condition in Equation 4.35 and Equation 4.36. The indecomposable endo-trivial FP-modules are just those FP-lattices that are isomorphic to $\Omega^n F_P$ for some integer n.*

Proof. From Corollary 4.1 we have that each $\Omega^n F_P$ is an indecomposable endo-trivial FP-module. Also given any indecomposable endo-trivial FP-module L, from Proposition 4.2 we have known that L is projective-free. To make minimal the number $m_1(\Omega^{-n}L)$ now we take an integer n. Therefore in the view of Lemma 4.6 we have that $\Omega^{-n}L$ is isomorphic to F_P. Hence we get L is isomorphic to $\Omega^{-n}L$. So we complete the proof of the theorem. □

Notice that except for the case the cyclic p-groups $\langle g\rangle$ in which $\Omega^2 F_{\langle g\rangle} \simeq F_{\langle g\rangle}$, the modules $\Omega^n F_{\langle g\rangle}$ are all distinct. In the general case we have the next result.

Proposition 4.10. *Let R be an arbitrary p-group. If R is neither cyclic, quaternion, nor generalized quaternion, then $\Omega^n F_R \ncong \Omega^m F_R$ for any distinct integers n and m.*

Proof. We use the proof of Theorem XII. 11.6 in [7] in order to say that our chosen R can contain an elementary abelian subgroup Q with order p^2. Also in [16] by Johnson the modules $\Omega^k F_Q$ are constructed for all $k > 0$. When we consider their dimensions it is explicit that none of them is isomorphic to F_Q. Now to get a contradiction we assume that $\Omega^n F_Q \cong \Omega^m F_Q$ but $n \neq m$. with out loss of generality we can take that $n > m$. Since they are isomorphic then by (2.11) in [1] we get that $\Omega^{n-m} F_Q \cong F_Q$ with $n - m > 0$. However it gives a contradiction with our assumption so it is not possible. Hence the proposition is true for Q.

Especially when we consider the projective-free part $(\Omega^F_R)_{Qf}$ of the restriction to Q of $\Omega^n F_R$, it is clearly $\Omega^n F_Q$ for any integer n. Therefore FR-isomorphism $\Omega^n F_Q \cong \Omega^m F_Q$ with $n \neq m$ would not imply an FQ-isomorphism between $\Omega^n F_Q$ and $\Omega^m F_Q$. Hence the proof of the proposition is completed. □

4.4. Arbitrary abelian P

Next theorem is principal theorem of our thesis. Main purpose of all we did so far is to prove next theorem. We devote this and next section to complete the proof of the next theorem.

Theorem 4.3. *Assume that $P > 1$ is an arbitrary abelian group. Then the indecomposable endo-trivial FP-modules are precisely those FP-lattices isomorphic to $\Omega^n F_P$ for some integer n.*

Proof. One side of the implication is trivial since from Corollary 4.1 we know that $\Omega^n F_P$ are indecomposable endo-trivial FP-modules. Therefore it is enough to show the converse. Proposition 4.2 says that we can prove the converse by constructing a contradiction from the following assumptions

4.1. *L is a projective-free endo-trivial FP-module,*

4.2. *$\Omega^n L \ncong F_P$, for all integers n,*

4.3. *Theorem 4.3 holds for all proper subgroups of P.*

Assume that the abelian p-group P is of the form of a direct product

$$P = \langle g_1 \rangle \times \cdots \times \langle g_m \rangle, \tag{4.69}$$

where g_i has p-power order $k_i > 1$ for all i and $m \geq 1$ since we supposed that $P > 1$. Similarly we define

$$y_i = g_i - 1 \in FP \tag{4.70}$$

for $i = 1, \cdots, m$.

Regarding Corollary 4.3 and Theorem 4.2, our assumptions 4.1 and 4.2 give that

$$m \geq 3. \tag{4.71}$$

Then we use this and 4.3 in order to show the following lemma:

Lemma 4.7. *Let Q be a subgroup of P with index p. Then there is a unique integer n such that $(\Omega^n L)_Q$ provides the FQ-isomorphism*

$$(\Omega^n L)_Q \cong F_Q \oplus [FQ]^u, \tag{4.72}$$

for some non-negative integer u. In this case $u > 0$.

Proof. In the view of Proposition 4.3, we have that L_Q is an endo-trivial FQ-module. Therefore projective-free part L_{Qf} of L_Q is an indecomposable endo-trivial FQ-module from Proposition 4.2. When we consider the construction of P from Equation 4.71 then we can say that the subgroup Q of P with index p is not a cyclic subgroup of P. Since index of Q is p we can say that it is a proper subgroup of P. So we can deduce that there is an integer n such that

$$\Omega^n(L_{Qf}) \cong F_Q. \tag{4.73}$$

Since Q is non-cyclic abelian p-subgroup of P, the integer n is unique from Proposition 4.10. Precisely we can say that $\Omega^n(L_{Qf})$ is FQ-isomorphic to the projective-free part $(\Omega^n L)_{Qf}$ of the restriction $(\Omega^n L)_Q$ since it comes from the definition of the Heller operator Ω^n (look at Section 2 in [1]). So the previous equation implies that this isomorphism is equivalent to the isomorphism at Equation 4.72. Hence we get the first statement of the lemma.

If we take $u = 0$ in Equation 4.72 then it is easily seen that $\Omega^n L$ is a one-dimensional FP-module. Therefore $\Omega^n F \cong F_P$ because P is a p-group and F is a field of characteristic p. But this contradicts to Hypothesis 4.2. Then we get that u must be greater than 0. Hence we prove the lemma. □

Now we investigate the case where P is an elemantary abelian group.

Lemma 4.8. *$k_i = p$ for all k_i where $i = 1, 2, \cdots, m$.*

Proof. For a contradiction suppose that above assertion is false. Then for some k_i, say k_1, we can say that it is divisible by p^2. Now when we apply Lemma 4.7 to the subgroup Q

$$Q = \langle g_1^p \rangle \times \langle g_1 \rangle \times \cdots \times \langle g_m \rangle \tag{4.74}$$

with index p in P then we get from Equation 4.72 that an integer $u > 0$ and some integer n. By using Equation 4.71 we can obtain that the element $q = y_2^{k_2-1} \cdots y_m^{k_m-1}$ of FQ is an annihilator of F_Q. Moreover we can deduce that it is annihilated by the ideal $y_2 FQ + \cdots + y_m FQ$ generated by the augmentation ideal of the subgroup $\langle g_2 \rangle \otimes \langle g_3 \rangle \times \cdots \times \langle g_m \rangle$ in FQ. Then this and Equation 4.72 give us that the factor FP-module $\Omega^n L/\mathrm{Ker}(w \text{ on } \Omega^n L)$ provides that

$$[\Omega^n L/\mathrm{Ker}(q \text{ on } \Omega^n L)]_{\langle g_1^p \rangle} \cong [F\langle g_1^p \rangle]^u. \tag{4.75}$$

We know Proposition 4.4 says that $\Omega^n L$ is an endo-trivial FP-module. Therefore we obtain from Proposition 4.8 that there is an F-endomorphism ψ of $\Omega^n L$ such that (y_1 on $\Omega^n L$) $= \mathrm{tr}_{1 \mapsto P}(\psi)$. Therefore using Equation 4.69 we obtain that the map

$$\phi = \mathrm{tr}_{1 \mapsto \langle g_2 \rangle \times \langle g_3 \rangle \times \cdots \times \langle g_m \rangle}(\psi) \tag{4.76}$$

is an $F(\langle g_2 \rangle \times \langle g_3 \rangle \times \cdots \times \langle g_m \rangle)$-endomorphism of $\Omega^n L$ such that

$$(y_1 \text{ on } \Omega^n L) = \mathrm{tr}_{1 \mapsto \langle g_1 \rangle}. \tag{4.77}$$

Therefore as q lies in $F(\langle g_2 \rangle \times \langle g_3 \rangle \times \cdots \times \langle g_m \rangle)$, we can say that the endomorphism ϕ fixes the submodule $\mathrm{Ker}(q \text{ on } \Omega^n L)$. Thus ϕ gives an F-endomorphism $\tilde{\phi}$ of $\Omega^n L/\mathrm{Ker}(q \text{ on } \Omega^n L)$ which satisfies that $\mathrm{tr}_{1 \mapsto \langle g_1 \rangle}(\tilde{\phi})$ is just multiplication by y_1 on the module. Therefore, since $\Omega^n L/\mathrm{Ker}(q \text{ on } \Omega^n L)$ is an $F\langle g_1 \rangle$-module it is valid for Equation 4.21. In the view of Theorem 4.1 it is a direct sum of $F\langle g_1 \rangle$-submodules where each of them is isomorphic to one of the modules

$$F\langle g_1\rangle/y^{k_1-1}F\langle g_1\rangle,\ F\langle g_1\rangle/y_1F\langle g_1\rangle \text{ and } F\langle g_1\rangle. \tag{4.78}$$

We assumed that k_1 is divisible by p^2, so we can eliminate first two of these modules since they have non-projective restrictions to $\langle g_1^p\rangle$.

From Equation 4.75 we can deduce that these modules cannot be isomorphic to $F\langle g_1^p\rangle$-direct summands of $\Omega^n L/\mathrm{Ker}(q \text{ on } \Omega^n L)$ so we say that $\Omega^n L/\mathrm{Ker}(q \text{ on } \Omega^n L)$ is free $F\langle g_1\rangle$-module. Since $u > 0$, it is immediate that it is not zero by Equation 4.75.

Since it is non-zero, we obtain that there is an element $\tilde{l}$ in $\mathrm{Ker}(q \text{ on } \Omega^n L)$ which generates a regular $F\langle g_1\rangle$-submodule. So $y_1^{k_1-1}\tilde{l}$ is nonzero. If we choose l as an element of $\Omega^n L$ with $\tilde{l}$ as its image, then we have that $y_1^{k_1-1}\tilde{l}$ is not an element of $\mathrm{Ker}(q \text{ on } \Omega^n L)$. Therefore

$$0 \neq y_1^{k_1-1}q\tilde{l} = y_1^{k_1-1}y_1^{k_1-1}\cdots y_1^{k_m-1}l. \tag{4.79}$$

So this gives us that l generates a regular FP-submodule and clearly it must be an FP-direct summand of $\Omega^n L$. However we know from the definition $\Omega^n L$ is projective free. Hence we get a contradiction. So the proof of the lemma is completed. □

Now we investigate the independence of n in Lemma 4.7 in terms of the choice of a subgroup Q of P. We use the argument of the field extension.

Lemma 4.9. *Let $\tilde{F}$ be any field extension of F. If we change F with $\tilde{F}$ and L with $\tilde{F}\otimes L$, then our assumptions 4.1, 4.2 and 4.3 hold again.*

Proof. From Equation 4.1 for the endo-trivial FP-module L we have that

$$\mathrm{End}_F(L) \cong F_P \oplus (FP)^t, \tag{4.80}$$

for some non-negative integer t. Also it is immediate that the $\tilde{F} \otimes P$-lattice $\tilde{F} \otimes L$ provides the $\tilde{F}L$-isomorphisms

$$\mathrm{End}_{\tilde{F}}(\tilde{F} \otimes L) \cong \tilde{F} \otimes \mathrm{End}_F(L) \cong \tilde{F}_P \oplus (\tilde{F}P)^t. \tag{4.81}$$

Hence clearly it is an endo-trivial $\tilde{F}P$-module. Also it is well-known fact that $\tilde{F} \otimes L$ is a projective-free module since $L$is a projective module of FP. Therefore we get that $\tilde{F}$ and $\tilde{F} \otimes L$ satisfies the assumption 4.1.

Furthermore, $\tilde{F} \otimes (\Omega^n L)$ is $\tilde{F}P$-isomorphic to $\Omega^(\tilde{F} \otimes L)$, for all integer n. Indeed if we take $\Omega^(\tilde{F} \otimes L) \cong \tilde{F}_P$, then we have to say that $\Omega^n L$ is isomorphic to F_P but this gives us a contradiction to our assumption 4.2. Therefore we get that $\tilde{F}$ and $\tilde{F} \otimes L$ satisfy the assumption 4.2.

Lastly we just see that our assumption 4.3 is independent of the choice of field F, so $\tilde{F}$ and $\tilde{F} \otimes L$ provide 4.3. Hence we conclude the proof of the lemma. □

Lemma 4.10. *If $z_1, \cdots, z_m$ is any basis for the F-subspace of $Fy_1 \dotplus \cdots \dotplus Fy_m$ of FP, then FP is the group algebra FR of elementary abelian subgroup R with order p^m which is generated by $1 + z_1, \cdots, 1 + z_m$ in the unit group FP. Moreover, the assumptions 4.1, 4.2 and 4.3 are satisfied by R when we substitute R for P.*

Proof. Firstly, we note that F is a field of characteristic p, then in the view of Lemma 4.8 and Equation 4.70 by construction, $y_i^p = 0$ for all $i = 1, \cdots, m$. Also we know that FP is a commutative F-algebra, so any linear combination z_i of $y_1 \dotplus \cdots \dotplus y_m$ holds for $z_i^p = 0$. Thus we obtain that $R = \langle 1 + z_1, \cdots, 1 + z_m \rangle$ is an elementary abelian group with order $|R| \leq p^m$.

It is easy to see that the F-algebra FP is also generated by $z_1 \dotplus \cdots \dotplus z_m$ because we know that it is generated by $y_1 \dotplus \cdots \dotplus y_m$. Therefore we get that it is epimorphic image of FR. So we notice that the order of R must be P^m since $\dim_F(FP) = p^m$. Hence we get that $FR = FP$. Therefore we complete the first part of the lemma.

For the second part of the lemma, in the view of Proposition 4.6, we get that L is also endo-trivial as an FR-module. Therefore the hypothesis 4.1 is valid if P is replaced by R.

Also from the definition of $\Omega^n L$ in Section 2 of [1] we can say that $\Omega^n L$ depends on the algebra FP but it does not depend on the embedding of the group P in the algebra.Since we know that $F_P = F_R$, when we substitute R for P, we see that the assumption 4.2 is valid for R.

It is evident that any proper subgroup of R is isomorphic to one of the proper subgroup of P. So when P is replaced by R, the assumption 4.3 still holds. With this we conclude the proof of the lemma. □

Lemma 4.11. *With regarding the hypothesises of Lemma 4.10, let Q be a proper subgroup of R with index p which is generated by $1 + z_2, \cdots, 1 + z_m$. Then the integer n in Lemma 4.7 such that Equation 4.72 holds is independent of the choice of the basis $z_1, \cdots, z_m$ of $Fy_1 \dotplus \cdots \dotplus Fy_m$.*

Proof. In the view of Lemma 4.10 we can apply Lemma 4.7 to R instead of P. So we obtain that in order to satisfy Equation 4.72 there is a unique integer n.

On the other hand, Let $\tilde{F}$ be the extension field which is obtained from F by adding m^2 many independent transcendentals T_{ij} where $i, j = 1, \cdots, m$. Then the elements

$$Z_i = \sum_{j=1}^{m} T_{ij} y_j, \tag{4.82}$$

for $i = 1, \cdots, m$, form an $\tilde{F}$-basis for $\tilde{F}y_1 \dotplus \cdots \dotplus \tilde{F}y_m$ in $\tilde{F}P$.Then Lemma 4.9 and Lemma 4.10 implies that $1 + Z_1, \cdots, 1 + Z_m$ generate an elementary abelian group $\tilde{R}$ which has order p^m such that $\tilde{F}P$ is the group algebra $\tilde{F}\tilde{R}$. Moreover when we apply Lemma 4.7 to the subgroup $\tilde{Q}$ of $\tilde{R}$ which is generated by $1+Z_2, \cdots, 1+Z_m$. Therefore similarly we can say that there is a unique integer $\tilde{n}$ such that the projective-free part

$(\Omega^{\tilde{n}}(\tilde{F}\otimes L))_{\tilde{Q}f}$ of the restriction $(\Omega^{\tilde{n}}(\tilde{F}\otimes L))_{\tilde{Q}}$ is isomorphic to $\tilde{F}_{\tilde{Q}}$. Hence clearly we obtain that $\tilde{n}$ does not depend on the choice of $z_1, \cdots, z_m$. Therefore the proof of the lemma be complete if we show that $\tilde{n} = n$.

It is clear that $.\Omega^n(\tilde{F}\otimes L)$ is $\tilde{F}P$ to $\tilde{F}\otimes\Omega^n L.$ when we think the regular module $\tilde{F}\tilde{Q}$ as an $\tilde{F}\tilde{Q}$-direct summand, we have that the multiplicity of this module is

$$v = \dim_{\tilde{F}}(Z_2^{p-1}\cdots Z_m^{p-1}(\tilde{F}\otimes\Omega^n L)). \tag{4.83}$$

Furthermore, If $l_1, \cdots, l_r$ are the F-basis element for $\Omega^n L$, then we say that $1\otimes l_1, \cdots, 1\otimes l_r$ construct an $\tilde{F}$-basis for $\tilde{F}\otimes\Omega^n L$. In the view of Equation 4.82 it is precise that the elements S_{ab} such that

$$Z_2^{p-1}\cdots Z_m^{p-1}(1\otimes l_a) = \sum_{h=1}^{d} S_{ab}(1\otimes l_b) \tag{4.84}$$

for $a = 1, \cdots, d$ are polynomials in the variables T_{ij} with coefficients in F. Now we change our understanding by specializing the variables T_{ij} to the elements λ_{ij} of F such that

$$z_i = \sum_{j=1}^{m} \lambda_{ij} y_j, \tag{4.85}$$

for $i = 1, \cdots, m$. So we have that $Z_2, \cdots, Z_m$ specialize to $z_2, \cdots, z_m$ and lastly S_{ab} specialize to the unique s_{ab} in F such that

$$z_2^{p-1}\cdots z_m^{p-1}(1\otimes l_a) = \sum_{h=1}^{d} s_{ab} l_b \tag{4.86}$$

for $a = 1\cdots, d$. We know that the multiplicity of the module is d so we obtain that $(v{+}1)\times(v{+}1)$ minor of the $d\times d$ matrix (S_{ab}) vanishes, therefore every $(v{+}1)\times(v{+}1)$ minor of (s_{ab}) vanishes. Thus we have that v is not smaller than $\dim_F(z_2^{p-1}\cdots z_m^{p-1}(\Omega^n L))$. Notice that $\dim_F(z_2^{p-1}\cdots z_m^{p-1}(\Omega^n L))$ is nothing but the multiplicity u in Equation 4.72.

So eventually we get

$$\begin{aligned} 1 + up^{m-1} &= \dim_F(\Omega^n L) && (4.87)\\ &= \dim_{\tilde{F}}\big((\tilde{F} \otimes \Omega^n L)_{\tilde{Q}f}\big) + vp^{m-1} && (4.88)\\ &\geq \dim_{\tilde{F}}\big((\tilde{F} \otimes \Omega^n L)_{\tilde{Q}f}\big) + up^{m-1}. && (4.89)\end{aligned}$$

Therefore there is not much possibility of the dimension of $[\tilde{F} \otimes \Omega^n L]_{\tilde{Q}f}$ under the above equations so $[\tilde{F} \otimes \Omega^n L]_{\tilde{Q}f}$ is one-dimensional. Also we know that $\tilde{Q}$ is a p-group and $\tilde{F}$ is a field of characteristic p. Thus it happens only if

$$[\Omega^n(\tilde{F} \otimes L)]_{\tilde{Q}f} \cong (\tilde{F} \otimes \Omega^n L)_{\tilde{Q}f} \cong \tilde{F}_{\tilde{Q}}. \tag{4.90}$$

Hence we show that $\tilde{n} = n$. This completes the proof of the lemma. □

Let H be the subgroup $\langle g_1 \rangle \times \langle g_2 \rangle$ of P. Now we work on the FH-module

$$R = \Omega^n L/\mathrm{Ker}(y_3^{p-1} \cdots y_m^{p-1} \text{on } \Omega^n L) \tag{4.91}$$

where n is the integer of Lemma 4.11. By considering Lemma 4.9 we may assume that the filed F is algebraically closed. Then we have

Lemma 4.12. *The objects F, H and R satisfy*

4.4. *F is an algebraically closed field of characteristic p,*

4.5. *The group H is the direct product $\langle g_1 \rangle \times \langle g_2 \rangle$ of cyclic groups which is generated by elements g_1 and g_2 where the order of both of these elements is p,*

4.6. *R is a projective-free FH-lattice,*

4.7. *If z is any non-zero F-linear combination of $y_1 = g_1 - 1$ and $y_2 = g_2 - 1$ in FH, then 1+z is an element of order p in the unit group of FH and the restriction $R_{\langle 1+z \rangle}$ is a free $F\langle 1 + z \rangle$-module.*

Proof. It is immediate from consequences of 4.69 and Lemma 4.8 that the assumptions 4.4 and 4.5 hold.

Notice that clearly R is an FH-lattice. If we assume that R is not projective-free, then we can say that there is an element k such that $y_1^{p-1}y_2^{p-1}k \neq 0$. So any element l in $\Omega^n L$ which maps onto k such that $y_1^{p-1}y_2^{p-1}\cdots y_m^{p-1}l \neq 0$. Therefore we obtain that l generates a regular FP-submodule with the property that this regular FP-submodule must be FP-direct summand of $\Omega^n L$. But this is not possible since we know that $\Omega^n L$ is projective-free by definition. Thus R is projective-free and we get that 4.6.

Now if we assume z is non-zero linear combination of y_1 and y_2, then we may adjoin one of the elements y_1 and y_2, choose y_1 to construct a basis y_1 and z for $Fy_1 \dotplus Fy_2$. Therefore $y_1, z, y_3, y_4, \cdots, y_m$ is an F-basis of $Fy_1 \dotplus \cdots \dotplus Fy_m$. From Lemma 4.10 the element $1+z$has order p in the unit group of FH. Then in the view of Lemma 4.11, we obtain that Equation 4.72 holds for the elementary abelian group Q

$$Q = \langle 1+z \rangle \times \langle g_3 \rangle \times \cdots \times \langle g_m \rangle. \tag{4.92}$$

So by 4.71 we have that

$$R_{\langle 1+z \rangle} = [\Omega^n L/\mathrm{Ker}(y_3^{p-1}\cdots y_m^{p-1} \text{ on } \Omega^n L)]_{\langle 1+z \rangle} \cong u \times F\langle 1+z \rangle. \tag{4.93}$$

Hence we see that 4.7 holds. Thus we conclude the proof of the lemma. □

In order to complete the proof of the theorem we need more ingredients. In the next section we show that our assumptions of Lemma 4.12 restrict R to be 0. But it is easy to see that this is not possible since we know that $u > 0$ in Lemma 4.7. This contradiction give the proof of Theorem 4.3. □

4.5. A Lemma

In this section our main goal is to prove the next lemma so that we complete the proof of the Theorem 4.3.

Lemma 4.13. *If we satisfy the hypothesises of Lemma 4.12, then R =0.*

Proof. From the hypothesises 4.4 and 4.5 of Lemma 4.12, the group algebra FH is trancuated polynomial ring $F[y_1, y_2]$ such that $y_1^p = y_2^p = 0$.Then its radical J= $\mathbf{J}(FH)$ is the ideal generated by y_1 and y_2. Therefore we have that

4.8. J^{p-1} *is the ideal generated by all elements* $y_1^i y_2^{p-1-i}$ *, where* $i = 0, 1, \cdots, p-1$

4.9. J^p *is the ideal generated by all* y_1^i, y_2^{p-i}*, where* $i = 1, 2, \cdots, p-1$

Now assume that $z = f_1y_1 + f_2y_2$ in FH, where f_1, f_2 in F. when we take $f_1 \neq 0$, then by 4.8 we obtain that zJ^{p-1} contains $f_1y_1y_2^{p-1} = zy_2^{p-1}$, $f_1y_1^2y_2^{p-2} + f_2y_1y_2^{p-1} = zy_1y_2^{p-2}$, $f_1y_1^2y_2^{p-2} + f_2y_1y_2^{p-1} = zy_1y_2^{p-2}$, $f_1y_1^3y_2^{p-3} + f_2y_1^2y_2^{p-2} = zy_1^2y_2^{p-3}$, $\cdots$, $f_1y_1^{p-1}y_2 + f_2y_1^{p-2}y_2^2 = zy_1^{p-2}y_2$, which forms a basis for J^p by using the assertion 4.9 above. So we get $J^p \subseteq zJ^{p-1}$. On the other hand, It is clear that the product zJ^{p-1} is in $\Re^p$ because $z \in J$. Thus we get $zJ^{p-1} = J^p$. We can do the similar argument for $f_2 \neq 0$ then we obtain the same equality. Therefore we have

4.10. *If we can write z as a non-zero F-linear combination of* y_1 *and* y_2 *in FH then we have that* $zJ^{p-1} = J^p$.

Now we focus on the FH-submodules such as

$$J^pR \subseteq J^{p-1}R \subseteq (J^{p-1}R : J) = \{k \in R | Jk \subseteq J^{p-1}R\} \tag{4.94}$$

of R. It is clear that the factor modules $\Re^{p-1}R/J^pR$ and $(J^{p-1}R : J)/J^{p-1}R$ are annihilated by J. Moreover, we can deduce that multiplication in R by any non-zero

F-linear combination z of y_1 and y_2 in FH gives an F-linear map Z

$$Z : (J^{p-1}R : J)/J^{p-1}R \longrightarrow J^{p-1}R/J^pR \tag{4.95}$$

Now assume that k is an element of $(J^{p-1}R : J)$ such that $k + J^{p-1}R$ is contained in the kernel of Z. Then we know from 4.10 that

$$zk \in J^pR \subseteq zJ^{p-1}R \tag{4.96}$$

Hence there is some element l in $J^{p-1}R$ such that $zk = zl$. So $z(k-l) = 0$. So by using Assumption 4.7 we have that there is an element m in R such that $k-l = z^{p-1}m$. Now we assume that $z^{p-1} \in J^{p-1}$ so that

$$k = l + z^{p-1}m \in J^{p-1}R + J^{p-1}R = J^{p-1}R. \tag{4.97}$$

Thus we get that $k + J^{p-1}R$=0. In other words, we obtain

4.11. *If z is a non-zero F-linear combination of y_1 and y_2 in FH, then multiplication by $z \in R$ gives an F-monomorphism Z of $(J^{p-1}R : J)/j + p - 1R$ into $J^{p-1}R/J^p$.*

Now we claim that

$$y_1(J^{p-1}R : J) = J^{p-1}R \tag{4.98}$$

In order to get a contradiction we assume that the claim is false. Then
$y_1(J^{p-1}R : J) \subset J^{p-1}R$. In the view of 4.1 we have that

$$J^{p-1}R = y_2^{p-1}R + y_1y_2^{p-2}R + \cdots + y_1^{p-1}R = y_2^{p-1}R + y_1J^{p-2}R. \tag{4.99}$$

We know that $J^{p-2}R$ is a subset of $(J^{p-1}R : J)$, then this gives us that there is an element k in R such that

$$y_2^{p-1}k \notin y_1(J^{p-1}R : J). \tag{4.100}$$

Notice that if $y_1^{p-1}y_2^{p-1}k \neq 0$, then k can generates a regular FH-submodule $FHk= FH$ of R. So this regular module must be an FH-direct summand of R. But this gives a contradiction to the hypothesis 4.6. Hence we get $y_1^{p-1}y_2^{p-1}k$ =0. Therefore for $z = y_1$ in the view of the hypothesis 4.7 we have an element $l \in R$ such that

$$y_2^{p-1}k = y_1 l. \tag{4.101}$$

Now we use again the hypothesis 4.7 since we know that $y_1 y_2 l = y_2^p k = 0$ so we get an element $m \in R$ such that

$$y_1^{p-1}k = y_1 l. \tag{4.102}$$

So $y_1 l$ and $y_2 l$ is contained in $j^{p-1}R$. In other hand, $l \in (J^{p-1}R : J)$. Therefore we get

$$y_2^{p-1}k = y_1 l \in y_1(J^{p-1}R : J), \tag{4.103}$$

But we again get a contradiction to Equation 4.100. Thanks to this contradiction we prove the assertion 4.98. Lastly in the view of 4.11 and 4.98 we obtain that multiplication by y_1 in R gives an F-isomorphism I_1

$$I_1 : (J^{p-1}R : J)/J^{p-1}R \longrightarrow J^{p-1}R/J^pR \tag{4.104}$$

Similarly, multiplication by y_2 in R gives an F-isomorphism I_2:

$$I_2 : (J^{p-1}R : J)/J^{p-1}R \longrightarrow J^{p-1}R/J^pR. \tag{4.105}$$

Now if $(J^{p-1}R : J)/J^{p-1}R \neq 0$, then the linear transformation $I_1^{-1}I_2$ has an eigenvalue e in F by 4.4. Therefore it is precise that:

$$I_2 - eI_1 : (J^{p-1}R : J)/J^{p-1}R \longrightarrow J^{p-1}R/J^pR \tag{4.106}$$

has a non-zero kernel. However it is easy to see from 4.11 that $I_2 - eI_1$ is the map Z which is induced by multiplication by $z = y_2 - ey_1$ in R and $\mathrm{Ker}(Z) = 0$. Therefore with this contradiction we obtain that:

$$(J^{p-1}R : J)/J^{p-1}R = 0. \tag{4.107}$$

If we assume that $R \neq 0$, then we have $J^{p-1}R \subseteq JR \subset R$ because J is a nilpotent ideal in JH. Also again using the nilpotency of F we obtain that $(J^{p-1}R : J) \supset J^{p-1}R$. But this contradicts to Equation 4.107. Thus we get $R = 0$ and this completes the proof of the lemma. □

The last lemma gives a criterion for the projectivity of FH-modules which has had applications (look at [6]). In order to improve its utility we rewrite the lemma in positive form for arbitrary abelian p-groups.

Lemma 4.14. *Assume that F is algebraically closed field and P is an elementary abelian p-group with order $p^m > 1$ which is generated by $g_1, \cdots, g_m$. Then an FP-lattice L is free if and only if its restriction $L_{\langle 1+w\rangle}$ is $F\langle 1+w\rangle$-free for any non-zero f-linear combination w of $y_1 = g_1 - 1, \cdots, y_m = g_m - 1$ in FP.*

Proof. Firstly we assume that L is FP-free, then Lemma 4.10 provides that $L_{\langle 1+w\rangle}$ is $F\langle 1+w\rangle$-free for any non-zero $w \in Fy_1 + Fy_2 + \cdots + Fy_m$.

If we take $m = 1$, then when we select $w = y_1$, it forces $L_{\langle 1+w\rangle} = L_{\langle g_1\rangle} = L_P = L$ to be FP-free.

If we take $m = 2$, then we know from Lemma 4.13 it gives us that $L_f = 0$. So L is FP-projective and also FP-free since F has characteristic p.

Now we suppose that $m \geq 3$. It is easy to see that the result is valid for all smaller values of m.

Let R be the $F\langle g_1, g_2\rangle$-module:

$$R = L/\text{Ker}(y_3^{p-1}, \cdots, y_m^{p-1} \text{ on } L) \tag{4.108}$$

If we take z is any non-zero element of $Fy_1 \dotplus Fy_2$, then there is some $i_1, 2$ such that $y_i, z, y_3, \cdots, y_m$ give an F-basis for $Fy_1 \dotplus \cdots \dotplus Fy_m$. It follows from Lemma 4.10 we get that $FR = FP$, where $R = \langle g_i, 1 + z, g_3, \cdots, g_m$ is also elementary of order p^m. Moreover, when we look at the subspaces $F(g_i - 1) \dotplus Fz \dotplus F(g_3 - 1) \dotplus \cdots \dotplus F(g_m - 1)$ and $Fy_1 \dotplus \cdots Fy_m$ we observe that they coincide with each other. By induction we get that $L_{\langle 1+z, g_3, \cdots, g_m\rangle}$ is a free $F\langle 1 + z, g_3, \cdots, g_m\rangle$-module. Then we get that for any non-zero $z \in Fg_1 + Fg_2$, $R_{\langle 1+z\rangle}$ is a free $F\langle 1 + z\rangle$-module. Therefore by the lemma for $m = 2$ R is a free $F\langle g_1, g_2\rangle$-module. This restricts L to be FP-free. Hence this completes the proof of the lemma. $\square$

5. CONCLUSION

In this thesis, we investigate endo-trivial modules and classification of endo-trivial modules over p-groups, especially abelian groups. In the first two chapter, we give the basic definitions and brief information about representation theory and modules. In the third chapter, we firstly we define projective and free modules and introduce duality notion on this modules. Then we remind tensor product which is a useful tool to get deep understanding about endo-trivial modules and give basic properties of category theory. Also we slightly mention endo-trivial modules and Heller operator. In the last Chapter we exemplify the endo-trivial modules over p-group P. Respectively we take p- group P as a cyclic group with one generator, a non-cyclic group with two generator and arbitrary abelian group to investigate corresponding form of endo-trivial modules.

REFERENCES

1. Alperin, J. L., 1976, “The main problem of block theory”, *Proc. Conf. on Finite Groups*, Academic Press, New York.

2. Alperin, J.L., 1986, *Local Representation Theory*, Cambridge University Press, Cambridge-New York.

3. Benson, D. J., 1998, *Representations and Cohomology. I. Basic Representation Theory of Finite Groups and Associative Algebras*, Cambridge Studies Advanced in Mathematics, vol. 30, Cambridge University Press, Cambridge.

4. Benson, D.J., 2006, *Moduler Representation Theory*, vol. 2, Springer-Verlag, Berling Heidelberg-New York.

5. Brauer, R., 1956, “Zur Darstellungstheorie der Gruppen endlicher Ordnung”, *Math. Z.*, 6, pp. 406-444.

6. Carlson, J. F., 1979, “The dimensions of periodic modules over modular group algebras”, *III. J. Math*, 23, pp. 295-306.

7. Cartan, H., S. Eilenberg, 1956, *Homological Algebra*, Princeton University Press, Princeton.

8. Curtis, C., I. Reiner, 1962, *Representation Theory of Finite Groups and Associative Algebras*, Wiley Interscience, New York.

9. Dade, E., 1972, “Une extension de la théorie de Hall et Higman”, *J. Alg*, 20, pp. 570-609.

10. Dade, E., 1978, “Endo-permutation modules over p-groups I”, *Ann. of Math*, 107, pp. 459-494.

11. Glauberman, G., 1968, “Correspondence of chatacters for relatively prime operator groups”, *Can. J. Math.*, 20, pp. 1465-1488.

12. Green, J., 1959, “On the indecomposable representations of a finite group”, *Math. Z.*, 70, pp. 430-445.

13. Green, J., 1962, “Blocks of modular representations”, *Math. Z.*, 79, pp. 100-115.

14. Green, J., 1964,“A transfer theorem for modular representations”, *J. Alg.*, 1, pp.73-84.

15. Green, J., 1974, “Walking around the Brauer tree”, *J. Austral. Math. Soc.*, 17, pp. 197-213.

16. Johson, D.L., 1969, “Indecomposable representations of the group (p, p) over a field of characteristic p”, *J. London Math. Soc. (2)*, 1, pp. 43-50.

17. Heller, A., 1961, “On group representations over a valuation ring”, *Proc. Nat. Acad. Sci.*, 47, pp. 1194-1197.

18. Huppert, B., 1967, *Endliche Gruppen I*, Springer-Verlag, Berlin-Heidelberg-New York.

19. Lassueur, C., 2011, “Relative Projective and Relative Endotrivial Modules”, *Journal of Algebra*, 337, pp. 285-317

20. Webb, P., 2016, *A Course in Finite Group Representation Theory*, Cambridge University Press.