DYNAMIC R&D COMPETITION UNDER INFORMATION ASYMMETRY

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ABSTRACT

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We consider a two-stage dynamic research and development (R&D) game in which two firms compete for a constant prize. It is assumed that the prize is given only to the winner of the competition and in order to get the prize, firms need to complete both stages of the R&D game. We aim to analyze how firms alter their investment rates as the technological states of the competition changes over time. We characterize the Nash equilibrium investment rates using backward induction both for the case with complete and incomplete information. Our observations for the complete information case are in line with the R&D literature. In equilibrium, we observe that the leader firm invests more than the follower firm and sensing the victory is close, it intensifies its R&D investments, while the discouraged follower firm reduces its expenditures. In addition, we observe that the rivalry is more intense when both firms reach the second stage of the game. For the incomplete and asymmetric information case, we again observe an intense rivalry in case both firms complete the first stage. However, as there is incompleteness and the speed of the firms' research capabilities are not the same in this case, the dynamics of the game change. When both/either of the firms complete the first stage, it is observed that the firm with the higher R&D capability invests more in innovation than its competitor; independent of it is the leader or the follower.

ÖZET

ASİMETRİK BİLGİ ALTINDA DİNAMİK AR-GE REKABETİ

İki rakip firmanın sabit bir ödül için yarıştığı iki aşamalı dinamik araştırma ve geliştirme (AR-GE) oyunu düşünülmüştür. Ödülü yalnızca oyunun kazananı alır ve ödülü alabilmek için firmaların AR-GE oyununun iki aşamasını da tamamlaması gerekir. Bu çalışma kapsamında, firmaların zaman içerisinde teknolojik konumları değiştikçe, yatırım oranlarını nasıl değiştirdikleri analiz edilmiştir. Firmaların Nash dengesindeki yatırım oranları tam bilgili ve eksik bilgili durumlar için karakterize edilmiştir. Tam bilgili oyun ile ilgili gözlemlerimiz bu alandaki literatür araştırmalarıyla paralellik göstermektedir. Firmaların dengedeki yatırım oranları incelendiğinde, lider firmanın takipçi firmaya göre daha fazla AR-GE yatırımı yaptığı ve ödüle yaklaştıkça AR-GE yatırımını arttırdığı gözlenmiştir. Lider firmanın tersine, takipçi firma ise yatırımlarını azaltmaktadır. Ayrıca, iki firmanın da ikinci aşamaya geçtiği durumda rekabetin daha da yoğunlaştığı görülmektedir. Eksik bilgili ve asimetrik oyunda, bir önceki durumdakiyle benzer şekilde, iki firma da ilk aşamayı bitirdikten sonra rekabetin yoğunlaştığı görülmektedir. Fakat bu oyunda eksik bilgi söz konusu olduğundan ve firmaların AR-GE hızları aynı olmak zorunda olmadığından, oyunun dinamikleri değişmektedir. İki firmanın ya da firmalardan sadece birinin ilk aşamayı bitirdiği durumda, AR-GE hızı yüksek olan firmanın -lider ya da takipçi olmasının bir önemi olmaksızın- rakibinden daha fazla yatırım yaptığı gözlemlenmiştir.

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a_{ij}	i^{th} player's investment rate where both players are at technology level 0, $i = 1, 2$, $j = 1, 2$, $i \neq j$.
a_{ij}^{H}	i^{th} player's investment rate where both players are at technology level 0 and Firm 2 has a high type hazard rate, $i = 2$, $j = 1$.
a_{ij}^{L}	i^{th} player's investment rate where both players are at technology level 0 and Firm 2 has a low type hazard rate, $i = 2$, $j = 1$.
a_{ij^*}	i^{th} player's investment rate where only the j^{th} player has completed the first stage, $i = 1, 2$, $j = 1, 2$, $i \neq j$.
$a_{ij^*}^{H}$	i^{th} player's investment rate where only the j^{th} player has completed the first stage and Firm 2 has a high type hazard rate, $i = 1, 2$, $j = 1, 2$, $i \neq j$.
$a_{ij^*}^{L}$	i^{th} player's investment rate where only the j^{th} player has completed the first stage and Firm 2 has a low type hazard rate, $i = 1, 2$, $j = 1, 2$, $i \neq j$.
a_{i^*j}	i^{th} player's investment rate where only the i^{th} player has completed the first stage, $i = 1, 2$, $j = 1, 2$, $i \neq j$.
$a_{i^*j}^{H}$	i^{th} player's investment rate where only the i^{th} player has completed the first stage and Firm 2 has a high type hazard rate, $i = 1, 2$, $j = 1, 2$, $i \neq j$.
$a_{i^*j}^{L}$	i^{th} player's investment rate where only the i^{th} player has completed the first stage and Firm 2 has a low type hazard rate, $i = 1, 2$, $j = 1, 2$, $i \neq j$.
$a_{i^*j^*}$	i^{th} player's rate of investment where both players have completed the first stage, $i = 1, 2$, $j = 1, 2$, $i \neq j$.
$a_{i^*j^*}^{H}$	i^{th} player's rate of investment where both players have completed the first stage and Firm 2 has a high type hazard rate, $i = 1, 2$, $j = 1, 2$, $i \neq j$.

$a^L_{i^*j^*}$	i^{th} player's rate of investment where both players have completed the first stage and Firm 2 has a low type hazard rate, $i = 1, 2$, $j = 1, 2$, $i \neq j$.
$F_{ij}(t\|a_{ij})$	Probability distribution function of the time needed to jump to the next stage for a given a_{ij}
$f_{ij}(t\|a_{ij})$	Probability density function of the time needed to jump to the next stage for a given a_{ij}
$h(a)$	Technological speed (hazard rate) of Firm 1 as a function of the investment rate
$h^H(a)$	Technological speed (hazard rate) of Firm 2 as a function of the investment rate when its hazard rate type is high
$h^L(a)$	Technological speed (hazard rate) of Firm 2 as a function of the investment rate when its hazard rate type is low
p	Probability of Firm 1's belief in the type of Firm 2's hazard rate
r	Discount rate
V_{ij}	i^{th} player's present expected discounted payoff value where both players are at technology level 0, $i = 1, 2$, $j = 1, 2$, $i \neq j$.
V^H_{ij}	i^{th} player's present expected discounted payoff value where both players are at technology level 0 and Firm 2 has a high type hazard rate, $i = 2$, $j = 1$.
V^L_{ij}	i^{th} player's present expected discounted payoff value where both players are at technology level 0 and Firm 2 has a low type hazard rate, $i = 2$, $j = 1$.
V_{ij^*}	i^{th} player's present expected discounted payoff value where only the j^{th} player has completed the first stage, $i = 1, 2$, $j = 1, 2$, $i \neq j$.
$V^H_{ij^*}$	i^{th} player's present expected discounted payoff value where only the j^{th} player has completed the first stage and Firm 2 has a high type hazard rate, $i = 1, 2$, $j = 1, 2$, $i \neq j$.
$V^L_{ij^*}$	i^{th} player's present expected discounted payoff value where only the j^{th} player has completed the first stage and Firm 2 has a low type hazard rate, $i = 1, 2$, $j = 1, 2$, $i \neq j$.

V_{i^*j}	i^{th} player's present expected discounted payoff value where only the i^{th} player has completed the first stage, $i = 1, 2$, $j = 1, 2$, $i \neq j$.
$V_{i^*j}^H$	i^{th} player's present expected discounted payoff value where only the i^{th} player has completed the first stage and Firm 2 has a high type hazard rate, $i = 1, 2$, $j = 1, 2$, $i \neq j$.
$V_{i^*j}^L$	i^{th} player's present expected discounted payoff value where only the i^{th} player has completed the first stage and Firm 2 has a low type hazard rate, $i = 1, 2$, $j = 1, 2$, $i \neq j$.
$V_{i^*j^*}$	i^{th} player's payoff where both players have completed the first stage, $i = 1, 2$, $j = 1, 2$, $i \neq j$.
$V_{i^*j^*}^H$	i^{th} player's payoff where both players have completed the first stage and Firm 2 has a high type hazard rate, $i = 1, 2$, $j = 1, 2$, $i \neq j$.
$V_{i^*j^*}^L$	i^{th} player's payoff where both players have completed the first stage and Firm 2 has a low type hazard rate, $i = 1, 2$, $j = 1, 2$, $i \neq j$.
W	Reward received at the end of the game
τ	Time needed to jump to the next stage

LIST OF ACRONYMS/ABBREVIATIONS

R&D	Research and Development

1. INTRODUCTION

Along with the remarkable progress in science and technology, research and development (R&D) activities have started to attract significant attention by the highly competitive technology driven industries. For many companies, the terms "R&D" and "innovation" are the main focus of interest and they treat R&D as an investment rather than a waste of time. Many of them even include these terms in their mission statements. The Walt Disney Company states in its mission that they seek to develop the most creative, innovative and profitable entertainment experiences and related products in the world. Nike, which is one of the most famous athletic and sports brand, also sets its mission as "To bring inspiration and innovation to every athlete in the world". In addition, in today's most of the developed countries, R&D activities play an important role in the national economy and are supported by the government and the educational institutions. Consequently, companies from all around the world are aware of the reality that R&D is one of the representatives of competitive power and is a key to enhance the quality of life.

We observe that firms may invest in R&D and innovative activities for several reasons. One of the most important reasons is the aim of being the market leader and having an edge over the competing rivals. This rivalry may also occur in order to deter the potential entries to the existing markets. Here, especially when the market structure does now allow the follower firms to benefit from an innovation, R&D becomes more of an issue because the leader firm gains all the reward at the end of the successfully managed R&D process. Besides these, firms sometimes may conduct these kinds of activities not to compete with a rival but just to strengthen their brand image in the society, to be the part of a social responsibility program or to improve operational efficiency inside the company.

In the aim of analyzing the investment decisions of competing firms in the R&D rivalry, we model a two-stage dynamic game in this study. In order to illustrate these two stages, we can refer to Choi's story about superconductivity [1]. In 1986, two scientists at the IBM laboratory discovered a substance which is able to induce superconductivity at a temperature of -397 F. With this discovery, the scientists managed to decrease the cooling temperature of superconducting compounds from -460 F to -397 F but it was thought that science could not go any further in inducing superconductivity at a lower temperature. Although, this decrease was a big success, it was useless for practical applications and the actual target was to induce superconductivity at room temperature. We know that this discovery has no value in itself but it was an essential step to reach the target temperature. This period can be thought as our first stage in the R&D game because this intermediate discovery encouraged other scientists to seek after other substances which can superconduct at higher temperatures and made a way to achieve the final discovery. The remaining period to finalize the innovation can be considered as the second stage in our competition game. As a matter of fact, this type of R&D rivalry is modeled previously in the literature. In the existing models of the literature, firms have complete information about their rivals and they can observe the position of their rivals as the game progresses. During our review for the thesis, we concluded that there is a lack of theoretical study about R&D modeling under incomplete information. Our motivation stems from this incomplete information case and our purpose is to observe the competing firms' investment behavior in different stages of the game under information asymmetry.

In the first model in Chapter 3, we consider a two-stage dynamic game in which two firms compete to be the first in the market in terms of successfully launching an innovation under complete information. It is assumed that there is uncertainty on the time that is needed to complete the innovation and this time is a random variable. We note that this model is also studied by several authors in the R&D literature (Please see Chapter 2 for a thorough review). At the initial point of the competition, firms simultaneously invest in order to complete the first stage and the firm that manages to complete the first stage becomes the leader. We assume that the innovation time decreases by increasing the R&D investment. In the aim of understanding the relationship between the cost of investment and the time needed to complete an innovation, Mansfield [2] conducted an empirical study among 60 Japanese and American firms for their innovations launched between 1975-1985. His study shows that Japanese firms devote twice as many resources to R&D activities as American firms in order to shorten the time needed to launch an innovation. Mansfield [2] states that the reason of this difference arises from Japanese firms' belief that the discounted value of the expected profits from an innovation decrease more rapidly when there are delays in the innovation process. We model this dependence by assuming that a higher investment results in a probabilistically shorter time to pass to the next technology level. In our model, each firm can observe its rival's technological position in the race and depending on its position; it can intensify or lower its R&D effort. However, research information is not accessible to everyone at the end of stage one. After the first stage is completed by one of the firms, two scenarios may occur. Firstly, the leader firm may complete the second stage of the competition and may win the R&D game. In this case, the leader takes the entire prize and the follower wins nothing. We assume that the prize granted at the end of the game is a constant value. Secondly, the follower firm may also get through the first stage before the leader wins the competition and they may both compete to launch the innovation and to win the prize. After explaining the model details, a numerical example is provided to understand the dynamics of R&D investment when all firm information is known completely.

In the second model in Chapter 3, we draw a more realistic view of the R&D competition and extend the above mentioned game under information asymmetry. The progress of the game is similar to the first one. However, different than the first model, firms do not need to have the same functional form of technological jump rate (hazard rate) in this model. In addition, one of the firms does not know its rival's hazard rate type at the initial stage of the game. In our model, one of the firms believes that its rival's R&D activities are efficient (its rival has a high type hazard rate) with probability p and inefficient (its rival has a low type hazard rate) with probability $1-p$. After the competition begins and the initial investments are made, uncertainty about the hazard rate disappears and the rest of the competition progresses as the same way with the first model. We also provide a detailed numerical analysis for this case in Chapter 4.

To sum up, our aim is to investigate the equilibrium investment rates and payoff values of two competing firms under technology uncertainty and information asymmetry in this study. In the meantime, we also model and analyze the complete information case based on the models studied earlier in the R&D literature. We use a game theoretical approach to solve the models and we use numerical analysis to illustrate the firms' behavior in the R&D rivalry. We believe that this thesis contributes to the literature by (i) allowing different firms to have different hazard rate functions (hence different capability in R&D projects) and (ii) including incompleteness and information asymmetry to the previously studied dynamic R&D competition models.

The remainder of this thesis is organized as follows. In Chapter 2, we provide a literature review about the dynamic R&D competition. In the first section of Chapter 3, we model a two-stage dynamic R&D race when there is no information asymmetry among firms. We also give a numerical illustration of this model in the same section. The first model is extended in the second section of Chapter 3 to model the case of asymmetric information. In Chapter 4, a detailed numerical analysis about the second model is given for 3 different hazard rate levels of the firm with incomplete information. Finally, in Chapter 5 we give our concluding comments.

2. LITERATURE REVIEW

Dynamic R&D competition or more specifically patent races have been investigated and modeled thoroughly from many aspects in the literature. Most of this research use game theoretic approaches and aim to find out the equilibrium investment levels of competing firms for an innovation or optimal timings of these innovative activities. In this context, structures and assumptions (single-multi stage, complete-incomplete information, perfect-imperfect information, stochastic-deterministic games etc.) of the models vary a lot depending on the scenarios. However, there are also studies which focus on a single firm's R&D activities and do not include the effects of rivalry in their models. An in depth review of firms' innovation strategies can be found in the survey of Kamien and Schwartz [3]. They present their survey from many angles of innovative activity based on both empirical and theoretical data. Beath *et al.* [4] also have a more specific survey about dynamic R&D competition. They investigate two main topics; first, the firms' allocation of their resources to innovation and second, the importance and advantages of being a leader in the R&D competition. Therefore, lots of models with different assumptions are analyzed.

Literature concerning dynamic R&D competition mostly examines the technological investment levels of competing firms under uncertain environments where the time to reach a development is generally a random variable. Firms are often fully informed about their rivals' activities in terms of perfectness and completeness. Also, it is quite common to use a two-stage model with a winner-takes-all type awarding approach and the game is seen as a "race" rather than "competition". Because of this, competitors have the chance to see each other's progress as the race continues so that they can increase or decrease their resources depending on whether they are the leader or the follower [5]. Regarding these points, Grossman and Shapiro [5] model a two stage patent race where the stages are at equal difficulty. Here, the stages can be thought of as research and development. According to their model, the firm that completes both of the research and development stages wins the entire prize and as such in the races, competitors are fully informed about the position of their rivals. In their model, success at each stage is subject to uncertainty and the flow probability per unit time of a next stage jump increases by spending more resources to the R&D project. They figure out that competition is most intense when both of the firms reach the intermediate stage. And when the firms are at different stages, the firm that is ahead invests more than its rival. In addition, they analyze the situations when firms reach an agreement on various types of cooperation during the progress of the competition and they find out that these forms of competition increase joint expected profits. Similarly, Park [6] investigates a form of competition which is very close to Grossman and Shapiro's model. Different from their model, in the second part of his research, he allows the follower to switch to a riskier "all or nothing" strategy rather than a two stage path. As it is expected, his findings are exactly the same with Grossman and Shapiro, showing that the player ahead always allocates more resources than does the one behind. Moreover, like in Grossman and Shapiro's results, he finds out that the competition is more intense when both firms have completed the first stage and if two competing firms are at different stages, the leader continues with the second stage whereas the follower chooses the riskier big single step. He also shows that the follower spends more resources to R&D when it chooses the riskier path.

Unlike most of the models in the literature, Choi [1] models a game with imperfect information and presents a more realistic view of the real world innovation activities. There is a two-stage dynamic R&D game in which the players do not have perfect information about their hazard rates. As it is represented in Grossman and Shapiro [5] and Park [6], Choi [1] also assumes that the mid-stage is just a prerequisite to complete the innovation and have no value for the firms. The winner gets a prize of K at the end and players know all the activities of their rivals perfectly, except the hazard rate. Hazard rate is different for two stages and the second stage hazard rate is perfectly known by both firms. Firms spend an expenditure rate of c per time in all stages. Under these assumptions, firms believe that reaching the intermediate stage is infeasible with a probability of p (first stage hazard rate is 0) and if they believe the discovery is feasible they have a first stage hazard rate with a probability of $(1 - p)$, which is greater than zero. When any of the firms complete the first stage, the uncertainty about the hazard rate becomes certain and firms can have the knowledge about this rate. Results show that, when the hazard rate is certain and the rival reaches the intermediate stage first, then the other firm that is behind loses its belief to win and the gap increases. Interestingly, when there is uncertainty about the hazard rate and the rival reaches the intermediate stage first, the other firm believes that the it may not be as difficult as it guesses and it does not give up at once and stays more in the competition.

We use a similar approach with Choi [1] and we model the technology competition as a two-stage dynamic game, also include incompleteness. However, there are differences between the structures of these two models. In Choi's model, both firms have imperfect information about each other's hazard rates. Different than that, only one of the firms has incomplete information in our model and the other firm has private information about the type of its own hazard rate. Another difference is, while Choi lets firms to have different hazard rates for different stages, we do not allow firms to change their hazard rate types after the competition starts. In addition, in Choi's model, firms spend the same cost of c per time for all the stages of the game and they do not alter their investment rates during the game. However, we let firms to increase or decrease their investment rates as the game progresses and we aim to observe the behavior of the firms when they stay ahead or behind in the competition.

Kamien and Schwartz [7] explore the choice of development period and an introduction time for a single innovation under rivalry. In their problem, the firm tries to maximize its expected profit. Their model assumes that,

- Cost increases when the development period shortens,
- Profit opportunities reduce with the lengthening of the development period and
- There is a probability of rival innovation and imitation that will affect the potential rewards to the firm.

The firm in their model does not know the rivals' plans about the introduction date and it selects the optimal development level by only knowing the cost and the subjectively determined probability of being first at any particular time. It may be important to mention that their research is more applicable for improvement of an existing product/process rather than a major new research. Beyond these, the timing selected in the absence of rivalry and the effects of intense rivalry are also examined in their paper. Their results show that if the firm has no rivals at the innovation stage, then the timing of the innovation will always be postponed compared with the cartel date. Moreover, they find that the intensive rivalry will cause the firm to cancel the project or to delay it forever.

Spector and Zuckerman [8] study a stochastic R&D decision model for a single firm which maximizes the expected discounted net return in a competitive environment. It is assumed in their research that there is a stochastic relationship between expenditure rates and the project's status. They tried to determine a stopping time about the introduction date for the new technology and to develop an investment strategy. Their findings show that under a constant expenditure strategy, the optimal stopping time is a control limit policy: stop whenever the project's state exceeds a fixed critical value or when a similar technology is introduced and protected by one of the rival firms, which of them occurs first. In this situation, winner does not take all the reward. They also represent that for an R&D model where the winner takes all, the firm's optimal expenditure rate increases monotonically as a function of the project's state.

Golany and Rothblum [9] focus on only development rather than analyzing research and development together. Uncertainty is present in their model only as an exogenous factor. If competitors win the game, no revenues are received and all the resources that were invested are lost. They found that under stationary uncertainty (where no information is obtained about future progress of the competitors' effort except they win the race) it is optimal to start investing immediately at full capacity and continue until finishing the allocated budget. And for non stationary environments (some information is gained with time), they show that when the budget is large, it is best to eliminate the risk of the investment and start late, and when the budget is small, it is best to start early and minimize the risk of not putting the project into practice.

Lukach *et al.* [10] investigate the R&D problem from a different perspective and analyze the R&D investment decision of a firm where it faces a threat of new technology entry. They present a model in which the incumbent firm can prevent the entry of the other firm by completing the innovation successfully. Their model is a two-stage investment process and the first stage should be carried out in full in order to finalize the project. There is cost uncertainty and for identifying the leader and the follower, they give privilege to the incumbent firm and let it to have a one-period lead over the entrant. They claim that the resulting monopoly after entry prevention is a different type of monopoly because the potential competition makes the incumbent to complete the R&D project, which the firm otherwise would not have done. According to their results, the threat of new entry may reduce welfare in case of an entry deterrence situation.

Grishagin *et al.* [11] analyse a duopoly type patent race where the firms have no information about their relative positions. This uncertainty is because of the fact that research activities are not made publicly in real life and rivals do not have so much idea about the others until the end of the competition. In this paper, firms can observe the initial positions of eachother but can not get any information about the progress of their rivals. A known value of prize is given to the winner at the end of the game. The results point out that if the firms are at the same level at the beginning, then they will both reach the development at the same time. However, if the starting points are not the same, only one of them (leader) succeeds and the winner spends all or most part of the rent, which results from the discovery, in order to keep away the rival from the competition. This means that the winner firm gains zero profit although it successfully completes the development activity. It is also found out that the greater prize value makes the game tougher and it takes shorter time for completing the development.

Loury [12] studies a research and development competition under technology and market uncertainty with n firms in the market. The time to introduce the developed solution is not certain for a firm and also it is not possible to estimate the launch dates of rivals' successful developments. The equilibrium investment level for a specific firm is obtained by maximizing its expected discounted profits for given investment levels of its rivals. It is observed that an increase in the number of racing firms causes a fierce competition and decreases the equilibrium investment levels. It is mentioned this may be the result of a belief that in case of more firms acting in the market, the introduction of the development will be in a far future date.

Lee and Wilde [13] use Loury's basic model of R&D competition which there are n firms those invest on a new technology to get a prize of V. In case of a firm reaches an innovation, it gets all the prize, meaning that the winner takes all. Introduction time is uncertain in the model and it is a random variable that is distributed exponentially. While Loury deals with the effects of fixed costs and pays only a cost of F at the beginning of the competition, this paper also includes the variable flow costs to win the rivalry. Results show that, the investment level in the equilibrium is higher when the variable costs play an important role and there are more firms in the industry. However, when the fixed costs are significantly higher than the variable costs, increase in the number of firms in the competition causes the equilibrium investment level to decrease.

Reinganum [14] analyzes the situation where n firms compete for developing a new innovation in a dynamic competition environment and there is uncertainty about the finishing time of a successful development. As the players' position changes during the game, they have the opportunity to increase or decrease their R&D expenditure levels and the players have private knowledge about their research findings. Analysis shows that an increase in the number of the firms also increases the R&D investment level of the firms under perfect patent protection. If the imitators are let to gain some benefits from the development, then the speed of the game can change.

Harris and Vickers [15] model a winner takes all type patent race with two players. The players have perfect and complete information about each other's activities and the player who reaches the finishing point gets the entire prize. It is not necessary to be in the same starting point. Players spend effort or invest money on an alternate basis. The investment decision is made by considering the firm's own position and the rival's investment level at the previous turn. It is understood that the winner acts like there are no other competitors in the game under perfect and complete information. To get more healthy incentives about this kind of patent races, they suggest constructing a model with a strong underlying structure that also includes imperfect and incomplete information.

Granot and Zuckerman [16] examine a decision theoretic, multi stage R&D project for a single firm in which research and development activities are selected and performed sequentially. In their model, there is a set of alternative R&D activities in each stage and if an activity is completed successfully, the project moves to a new state with a new set of alternative activities. They allow the decision maker to stop the process and receive a prize depending on the project's status or to continue with selecting one of the R&D activities which are possible in the current stage. Their aim is to find a stopping rule that maximizes the expected discounted net return. They figured out that in specific cases, a greedy-optimal resource allocation procedure can be implemented for increasing the expected discounted reward.

There are not so many experimental studies on dynamic R&D competition in the literature. One of the guiding studies about this area is conducted by Sbriglia and Hey [17] which represents some results of laboratory experiments regarding R&D competition. The experiments investigate the firms' search strategies to reach a profitable solution, their expenditures at different stages of the R&D competition and use of previous information that affects them to continue or quit the race. Subjects try to discover an unknown combination of elements that will gain them a considerable profit. It is observed that nearly all the winners of the game find out a successful search rule during the experiment and spend more money than the rest of the competitors. When there is uncertainty about the players' performance and there are few players in the game, winners can succeed with less investment and the game lasts longer. Conversely, without the stochastic environment and with many players, investment rate increases and it takes less time to finalize the game.

Another empirical study is about the dynamic R&D competition in the TFT-LCD (Thin Film Transistor Liquid Crystal Display) industry [18]. The authors believe that the LCD industry is good for analyzing the behavior of the competing firms for several reasons. Firstly, due to the dynamic market conditions and intense rivalry, the leader firm constantly changes and every firm finds itself as a follower at some point of the competition. In addition, innovations can be easily defined and measured objectively in the LCD industry. Also, the oligopolistic structure of the market makes it easier to observe and analyze the firms' innovation outcomes. In order to analyze the market dynamics, detailed information like manufacturer name, plant location, investment rate, and production capacity is included in the dataset and the catching up behavior of the follower firms is observed. The empirical study of Lee *et al.* [18] shows that when firms have a moderate distance from the industry best innovation, they are more likely to catch up with the leader and they invest more than the other followers. However, most of the models in the literature state that the followers decrease their technology investments and the intense competition occurs between the leader and the closest followers. As there is no theoretical model explaining this behavior, the authors call for further research on this area. Their second finding shows that as the leader's production capacity for the state of the art technology increases, followers invest more to catch up, which is in line with the theoretical studies in the literature.

Some examples of the models dealing with only a single firm where there is no rivaly are studied by Grossman and Shapiro [19] and Gerchak and Parlar [20]. Grossman and Shapiro [19] has a study about a single firm conducting an R&D program and it does not include any rivalry or competition. Their analysis results indicate that the firm should change its R&D effort as the expected value of the development changes. This expected value gets higher values monotonically with getting closer to the finish line. Therefore, expending the same level of effort to the project is not optimal in this case and the monotonic property of the function makes it possible to spend less cost to reach the same value of prize. In the deterministic case; if firm has initial idea about the difficulty of the project, then its evaluation for the expected value changes and it may cause the firm to increase its effort or to even quit the R&D program. If there is uncertainty about the effort-timeline relation, then as the firm gets closer to finalize the project, it increases its effort and does not consider leaving the program at all. In addition, it is seen that under uncertainty, firms choose riskier projects than risk-free ones.

Gerchak and Parlar [20] consider the decision problem of a firm that allocates a fixed budget among two activities under uncertain environment in order to maximize the expected profit from these activities. Again, the competition is regarded as a race in their model and when the firm reaches the desired breakthrough for an activity, the winner takes the entire prize for that activity. They assume that the probability that a firm wins the competition in an activity is increasing in its allocation to that activity, but decreasing in competitors' allocation to it. They found the optimal allocation level for a specific capture probability function and analyzed the Nash and Stackelberg equilibrium.

Our study is closest to the two-stage dynamic models of Park [6] and Grossman and Shapiro [5] and aims to extend theirs by including incomplete information on the technological jump rates (hazard rates) of the firms. In this respect, our model shows some similarities with the dynamic game modeled by Choi [1]. In all of these 3 models, the competition is a two-stage game and the winner takes the entire prize at the end. Also, there is uncertainty on successfully completing the stages and firms need to invest in the new technology in order to jump to the following stages. In addition, firms are fully informed about the progress of their rivals so that they can adjust their investment rates based on their positions in the game. Briefly, our model differs from the existing models of the literature in including incompleteness on one of the competing firms' hazard rate information. Although, Choi [1] adds hazard rate imperfectness in its model, in its case both firms have private information about their own hazard rates whereas only one of the firms has private information about its own hazard rate in our model.

3. MODELING

3.1. Two-Stage Dynamic R&D Game Model under Complete Information

We consider a two-stage dynamic R&D game in which two firms compete for a reward of W. As it is stated in Grossman and Shapiro [5], these two stages can be thought of as research and development. The whole reward is given only to the winner and the player that finishes the game in the second place does not gain any benefit from the competition, also loses all of its R&D investment. In other words, the game is a winner-takes-all type competition.

Initially, both firms start the race from the technology level 0. In order to receive the reward, a firm has to invest on a technology and complete both stages of the game. After completing the first stage, reaching the intermediate level 1 has no actual value to a firm and it can be thought as a prerequisite level to win the race. The first firm that reaches level 2 becomes the winner and gets the entire prize.

The state of the technology levels can be described as the following:

- Both firms are at technology level 0,
- One firm has completed the first stage, but the other not (The leader is at technology level 1; the follower is at technology level 0) and
- Both firms have reached the intermediate technology level.

Let $a_{ij} \in R_+$ denote the investment rate for firm i when both firms are at level 0 $(i \neq j)$. When there is $*$ notation over i or j, it shows that the i^{th} or j^{th} player which has the $*$ notation has completed the first stage of the game. As an instance, a_{i^*j} denotes the i^{th} player's investment rate where the i^{th} player has completed the first stage whereas j^{th} player still has not completed the first stage and stays at level 0. Similarly, $a_{i^*j^*}$ denotes the i^{th} player's rate of investment where both players have completed the first stage.

There is uncertainty on the time which is needed to jump to the next level. We assume that this time is dependent only on the investment rate of the firm and is a random variable τ, distributed according to

$$F_{ij}(t|a_{ij}) \equiv Pr[\tau(a_{ij}) \leq t] = 1 - e^{-h(a_{ij})t} \tag{3.1}$$

where $h(a_{ij})$ describes the speed (hazard rate) of the i^{th} firm or stated in other words, the probabilistic rate to achieve a breakthrough in order to reach to the next technology level when the expenditure is a_{ij}. F_{ij} has the same functional form for all $h(a_{ij})$ and takes different values only when $h(a_{ij})$ changes. Similarly, $h(a_{ij})$ has the same functional form for all a_{ij} and differentiates only when the investment rate varies. We define

$$f_{ij}(t|a_{ij}) \equiv \partial F_{ij}(t|a_{ij})/\partial t \tag{3.2}$$

Here, we assume that $h(.)$ is strictly increasing, concave and $h(0) = 0$. Also, each firm knows the technological state of its rival immediately and can observe whether the other is ahead or behind in the race. However, research information is not accessible to everyone at the end of the first stage. For the base model, we assume that $h(.)$ is identical for both of the players and does not change with the progress of the game.

Each firm discounts its future expenditures and gains at rate r. Let V_{ij} denote the present expected discounted payoff of firm i when both firms are at level 0 ($i \neq j$). As in the investment rate notation, if there is $*$ notation over i or j, it shows that the i^{th} or j^{th} player which has the $*$ notation has completed the first stage of the game. As an instance, V_{i^*j} denotes the i^{th} player's present expected discounted payoff value where the i^{th} player has completed the first stage whereas j^{th} player still has not completed the first stage and stays at level 0. Similarly, $V_{i^*j^*}$ denotes the i^{th} player's payoff where both players have completed the first stage.

Firms choose their investment levels to maximize their present expected discounted net profits. We can use the above mentioned four technology levels to analyze the effect of position (being ahead or back) on the rate of the firms' expenditures. To remind again, these possible levels occur (i) when none of the firms have reached the intermediate level, (ii) when one of the firms has completed the first stage and the other has not and (iii) when both firms succeed in reaching the first level.

By using backward induction, we can solve the game and analyze the subgame perfect Nash equilibrium investment rates.

3.1.1. Analysis of the Last Stage

We begin with the situation when both players have achieved the intermediate technology level. Firm 1's payoff is determined by

$$\begin{aligned} V_{1^*2^*}(a_{1^*2^*};a_{2^*1^*}) = & \int_0^\infty We^{-rt}[1-F_{2^*1^*}(t|a_{2^*1^*})]f_{1^*2^*}(t|a_{1^*2^*})dt \\ & - \int_0^\infty \left\{ \int_0^t a_{1^*2^*}e^{-rs}ds \right\} [1-F_{2^*1^*}(t|a_{2^*1^*})]f_{1^*2^*}(t|a_{1^*2^*})dt \\ & - \int_0^\infty \left\{ \int_0^t a_{1^*2^*}e^{-rs}ds \right\} [1-F_{1^*2^*}(t|a_{1^*2^*})]f_{2^*1^*}(t|a_{2^*1^*})dt \end{aligned} \tag{3.3}$$

The first term in Equation 3.3 refers to the present expected discounted reward and the second term refers to the present expected discounted cost in case Firm 1 wins the race. Last term refers to the present expected discounted cost for Firm 1 when its rival (Firm 2) crosses the finish line before it. When we substitute Equation 3.1 and Equation 3.2 into Equation 3.3, we obtain

$$\begin{aligned} V_{1^*2^*}(a_{1^*2^*};a_{2^*1^*}) = & \int_0^\infty We^{-rt}[1-1+e^{-h(a_{2^*1^*})t}][h(a_{1^*2^*})e^{-h(a_{1^*2^*})t}]dt \\ & - \int_0^\infty \left\{ \int_0^t a_{1^*2^*}e^{-rs}ds \right\} [1-1+e^{-h(a_{2^*1^*})t}][h(a_{1^*2^*}) \\ & e^{-h(a_{1^*2^*})t}]dt \\ & - \int_0^\infty \left\{ \int_0^t a_{1^*2^*}e^{-rs}ds \right\} [1-1+e^{-h(a_{1^*2^*})t}][h(a_{2^*1^*}) \\ & e^{-h(a_{2^*1^*})t}]dt \end{aligned} \tag{3.4}$$

After taking integrals and some algebra, the above equation becomes

$$V_{1^*2^*}(a_{1^*2^*}; a_{2^*1^*}) = \frac{Wh(a_{1^*2^*}) - a_{1^*2^*}}{r + h(a_{1^*2^*}) + h(a_{2^*1^*})} \tag{3.5}$$

For a given $a_{2^*1^*}$, Firm 1 chooses $a_{1^*2^*}$ to maximize its profit. In order Firm 1 to stay in the game, it chooses such an $a_{1^*2^*}$ that makes $V_{1^*2^*}$ greater than zero. This happens when $W > a_{1^*2^*}/h(a_{1^*2^*})$.

Analogous to Equation 3.5, we can determine the other states' payoffs. Grossman and Shapiro [5] state that any equilibrium in this state is symmetric. Therefore, Firm 2's payoff is determined by

$$V_{2^*1^*}(a_{2^*1^*}; a_{1^*2^*}) = \frac{Wh(a_{2^*1^*}) - a_{2^*1^*}}{r + h(a_{2^*1^*}) + h(a_{1^*2^*})} \tag{3.6}$$

For a given $a_{1^*2^*}$, Firm 2 chooses $a_{2^*1^*}$ to maximize its profit. In order Firm 2 to stay in the game, it chooses such an $a_{2^*1^*}$ that makes $V_{2^*1^*}$ greater than zero. This happens when $W > a_{2^*1^*}/h(a_{2^*1^*})$.

3.1.2. Analysis of the Intermediate Stages

Consider the case when one of the firms, say Firm 1, has completed the first stage and the other has not. In this case, Firm 1's payoff is determined by

$$V_{1^*2}(a_{1^*2}; a_{21^*}) = \frac{Wh(a_{1^*2}) + V_{1^*2^*}h(a_{21^*}) - a_{1^*2}}{r + h(a_{1^*2}) + h(a_{21^*})} \tag{3.7}$$

For a given a_{21^*}, Firm 1 chooses a_{1^*2} to maximize its profit. In order Firm 1 to stay in the game, it chooses such an a_{1^*2} that makes a_{1^*2} greater than zero. This happens when $W > a_{1^*2}/h(a_{1^*2})$. Firm 2's payoff is determined by

$$V_{21^*}(a_{21^*}; a_{1^*2}) = \frac{V_{2^*1^*}h(a_{21^*}) - a_{21^*}}{r + h(a_{1^*2}) + h(a_{21^*})} \tag{3.8}$$

For a given a_{1^*2}, Firm 2 chooses a_{21^*} to maximize its profit. In order Firm 2 to stay in the game, it chooses such an a_{21^*} that makes V_{21^*} greater than zero. This happens when $V_{2^*1^*} > a_{21^*}/h(a_{21^*})$.

Similarly, think of the case when Firm 2 has completed the first stage and the other has not. In this case, Firm 2's payoff is determined by

$$V_{2^*1}(a_{2^*1}; a_{12^*}) = \frac{Wh(a_{2^*1}) + V_{2^*1^*}h(a_{12^*}) - a_{2^*1}}{r + h(a_{2^*1}) + h(a_{12^*})} \tag{3.9}$$

For a given a_{12^*}, Firm 2 chooses a_{2^*1} to maximize its profit. In order Firm 2 to stay in the game, it chooses such an a_{2^*1} that makes V_{2^*1} greater than zero. This happens when $W > a_{2^*1}/h(a_{2^*1})$.

Firm 1's payoff is determined by

$$V_{12^*}(a_{12^*}; a_{2^*1}) = \frac{V_{1^*2^*}h(a_{12^*}) - a_{12^*}}{r + h(a_{2^*1}) + h(a_{12^*})} \tag{3.10}$$

For a given a_{2^*1}, Firm 2 chooses a_{12^*} to maximize its profit. In order Firm 2 to stay in the game, it chooses such an a_{12^*} that makes V_{12^*} greater than zero. This happens when $V_{1^*2^*} > a_{12^*}/h(a_{12^*})$.

3.1.3. Analysis of the Initial Stage

Lastly, consider the situation when both players are at technology level 0 and neither of them has achieved the intermediate technology level. Firm 1's payoff is determined by

$$V_{12}(a_{12}; a_{21}) = \frac{V_{1^*2}h(a_{12}) + V_{12^*}h(a_{21}) - a_{12}}{r + h(a_{12}) + h(a_{21})} \tag{3.11}$$

For a given a_{21}, Firm 1 chooses a_{12} to maximize its profit. In order Firm 1 to stay in the game, it chooses such an a_{12} that makes V_{12} greater than zero. This happens when

$V_{1^*2} > a_{12}/h(a_{12})$.

Grossman and Shapiro [5] states that any equilibrium in this state is symmetric when $h(.)$ is the same for both firms. Therefore, Firm 2's payoff is determined by

$$V_{21}(a_{21}; a_{12}) = \frac{V_{2^*1}h(a_{21}) + V_{21^*}h(a_{12}) - a_{21}}{r + h(a_{21}) + h(a_{12})} \tag{3.12}$$

For a given a_{12}, Firm 1 chooses a_{21} to maximize its profit. In order Firm 1 to stay in the game, it chooses such an a_{21} that makes V_{21} greater than zero. This happens when $V_{2^*1} > a_{21}/h(a_{21})$.

3.1.4. Numerical Analysis of the Model under Complete Information

Findings of Grossman and Shapiro [5] indicate that when one of the firms attain the intermediate level of the game, it invests more than the firm that is behind. That is because of the fact that, as the leader senses the victory, it intensifies its investment rate, causing the discouraged follower firm to decrease its R&D efforts. In this case, the follower firm also spends less effort than its investment rate at the initial level of the game where both of them are at level 0. Conversely, for most of the parameter values, the leader firm spends greater effort than its investment rate at the initial level of the game. However, Grossman and Shapiro [5] found some counter examples for small values of e (which is the constant elasticity of variable costs). Secondly, when both firms complete the initial stage of the game, the competition becomes more intense and both firms increase their investment levels.

In order to understand the model properties, we use a simple numerical simulation.

We set the parameters for the prize W, the discount rate r and the hazard rate $h(a)$ as 100, 0.05 and $a^{1/3}$ respectively. In accordance with the analytical findings, we observe that firms spend more of their resources for the new technology when they both

Table 3.1. The Numerical Illustration of the Responses When There is Complete Information.

BOTH FIRMS COMPLETED THE FIRST STAGE:
$a_{1^*2^*} = 2.8870$
$a_{2^*1^*} = 2.8870$
$V_{1^*2^*} = 3.9175$
$V_{2^*1^*} = 3.9175$
ONLY FIRM 1 COMPLETED THE FIRST STAGE:
$a_{1^*2} = 1.3090$
$a_{21^*} = 0.7830$
$V_{1^*2} = 6.4101$
$V_{21^*} = 1.3690$
ONLY FIRM 2 COMPLETED THE FIRST STAGE:
$a_{2^*1} = 1.3090$
$a_{12^*} = 0.7830$
$V_{2^*1} = 6.4101$
$V_{12^*} = 1.3690$
NONE OF THEM COMPLETED THE FIRST STAGE:
$a_{12} = 1.0613$
$a_{21} = 1.0613$
$V_{12} = 3.2887$
$V_{21} = 3.2887$

complete the first stage of the competition. When the firms are at different technology levels, we see that the leader's investment rate is greater than the follower's and while the leader increases its investment rate, the following firm decreases its rate which is less than the initial investment rate it chooses when both of them are at the starting point of the game. Table 3.1 shows how the investment rates and payoff values change in case of complete information when the parameters for the prize W, the discount rate r and the hazard rate $h(a)$ are chosen as 100, 0.05 and $a^{1/3}$ respectively.

3.2. Two-Stage Dynamic R&D Game Model under Incomplete Information

In the previous model, we assumed that the speeds of the firms' research activities, namely the hazard rates, have the same functional properties for both firms and do not vary in different technology stages of the competition. We also assumed that firms have common information about their hazard rate functions. In reality, competing firms may not always have the chance to observe their rival's true research and development capabilities during the progress of the rivalry.

In order to represent the real life behavior of competing firms, we add incompleteness on the hazard rates of the players and we model a dynamic R&D game in which one participant (say Firm 1) has incomplete information about the true hazard rate of the opposite party; while the other participant (say Firm 2) has private information about its own hazard rate. In this case, at the beginning of the competition, Firm 1 believes that Firm 2 has a high type hazard rate with probability p and a low type hazard rate with probability $(1-p)$. Firm 2 has no restriction in observing the hazard rate of Firm 1. In addition, it is assumed that the research information is not accessible to the firms at the end of the initial stage. We may consider Firm 1 as a public company that offers its securities for sale to the general public and Firm 2 as a privately held company which does not have to disclose information that can be used for gaining competitive advantage.

The game progresses as follows. At the starting point, firms decide an initial investment rate and compete to complete the first stage of the innovation game. The firm that first gets through the initial stage becomes the leader. As we mentioned before, completing the first stage has no actual value to the firms but it needs to be completed in order to advance to the second stage. After this intermediate state, if the leader firm sustains its success and completes the second stage; it wins the competition and take the entire prize of W. Here, the follower firm gains nothing and loses all of its R&D expenses. On the other hand, if the follower firm catches up with the leader at the intermediate state, the game moves to another technology state and both firms battle for being the winner of the game.

We assume that as soon as the competition begins, uncertainty about Firm 2's hazard rate is resolved and the competition turns into a complete information game. Thus, incompleteness has an impact only at the initial state of technology game where both players have not yet reached the intermediate level.

The main logic about describing the state of the technology levels is the same with the complete information case. However, in our case, when a firm completes the first stage and becomes a leader, it is important whether the leader is Firm 1 or Firm 2 due to information asymmetry. In addition, we will analyze the game separately for the cases where the real hazard rate of Firm 2 is high and low.

Assuming the real hazard rate of Firm 2 is high, we will be interested in the following technology levels:

- Both firms are at technology level 0,
- Firm 1 (that has incomplete information about Firm 2's hazard rate) has completed the first stage, but Firm 1 not,
- Firm 2 (that has private information about its own hazard rate) has completed the first stage, but the Firm 1 not,
- Both firms have reached the intermediate technology level.

Assuming the real hazard rate of Firm 2 is low, we will be interested in the following technology levels:

- Both firms are at technology level 0,
- Firm 1 (that has incomplete information about Firm 2's hazard rate) has completed the first stage, but Firm 1 not,
- Firm 2 (that has private information about its own hazard rate) has completed the first stage, but the Firm 1 not,
- Both firms have reached the intermediate technology level.

Let $a_{ij} \in R_+$ denote the investment rate for firm i when both firms are at level 0 $(i \neq j)$. When there is $*$ notation over i or j, it shows that the i^{th} or j^{th} player which has the $*$ notation has completed the first stage of the game.

When there is H notation over a, it shows that Firm 2 actually has a high type hazard rate. As an instance, $a^H_{i^*j}$ denotes the i^{th} player's investment rate where the i^{th} player has completed the first stage when the hazard rate type of Firm 2 is actually high H. Similarly, $a^H_{i^*j^*}$ denotes the i^{th} player's rate of investment where both players have completed the first stage when the hazard rate type of Firm 2 is actually high H.

When there is L notation over a, it shows that Firm 2 actually has a low type hazard rate. As an instance, $a^L_{i^*j}$ denotes the i^{th} player's investment rate where the i^{th} player has completed the first stage when the hazard rate type of Firm 2 is actually low L. Similarly, $a^L_{i^*j^*}$ denotes the i^{th} player's rate of investment where both players have completed the first stage when the hazard rate type of Firm 2 is actually low L. But note that a^H_{ij} or a^L_{ij} is never used since the uncertainty about Firm 2's hazard rate type is not resolved at the initial stage.

There is uncertainty on the time which is needed to jump to the next level. We assume that this time is dependent only on the investment rate of the firm and is a random variable τ, distributed according to

$$F_{ij}(t|a_{ij}) \equiv Pr[\tau(a_{ij}) \leq t] = 1 - e^{-h(a_{ij})t} \tag{3.13}$$

where $h(a_{ij})$ describes the speed (hazard rate) of the i^{th} firm or stated in other words, the probabilistic rate to achieve a breakthrough in order to reach to the next technology level when the expenditure is a_{ij}. When there is H notation over $h(a_{ij})$, it shows that the hazard rate is a *high* type one and when there is L notation above $h(a_{ij})$, it shows that the hazard rate is a *low* type one.

We define

$$f_{ij}(t|a_{ij}) \equiv \partial F_{ij}(t|a_{ij})/\partial t \tag{3.14}$$

Our assumptions regarding $h(.)$ are still valid. We assume that $h(.)$ is strictly increasing, concave and $h(0) = 0$. Also, each firm knows the technological state of its rival immediately and can observe whether the other is ahead or behind in the race. Different than the first model, here we assume that $h(.)$ does not need to have the same functional property for both of the players. We assume Firm 2 knows the type of its hazard rate -*high* or *low*- under the assumption that Firm 1 does not know this private hazard rate information during the initial stage of the game. Firm 1 has the belief that Firm 2 has a high type hazard rate with probability p and a low type hazard rate with probability $(1 - p)$.

Each firm discounts its future expenditures and gains at rate r. Let V_{ij} denote the present expected discounted payoff of firm i when both firms are at level 0 $(i \neq j)$. As it is stated above, if there is $*$ notation over i or j, it shows that the i^{th} or j^{th} player which has the $*$ notation has completed the first stage of the game.

When there is H notation over V, it shows that Firm 2 has a *high* type hazard rate. As an instance, $V^H_{i^*j}$ denotes the i^{th} player's present expected discounted payoff value where the i^{th} player has completed the first stage when the hazard rate type of Firm 2 is *high*. Similarly, $V^H_{i^*j^*}$ denotes the i^{th} player's payoff where both players have completed the first stage when the hazard rate type of Firm 2 is *high*.

When there is L notation over V, it shows that Firm 2 has a *low* type hazard rate. As an instance, $V^L_{i^*j}$ denotes the i^{th} player's present expected discounted payoff value where the i^{th} player has completed the first stage when the hazard rate type of Firm 2 is *low*. Similarly, $V^L_{i^*j^*}$ denotes the i^{th} player's payoff both players have completed the first stage when the hazard rate type of Firm 2 is *low*.

Firms choose their investment levels to maximize their present expected discounted net profits. We can use the above mentioned technology levels to analyze the effect of position (being ahead or back) on the rate of the firms' expenditures. To remind again, these possible levels occur when (i) Both firms are at technology level 0, (ii) Firm 1 (that has incomplete information about Firm 2's hazard rate) has completed the first stage, but Firm 1 not, (iii) Firm 2 (that has private information about its own hazard rate) has completed the first stage, but the Firm 1 not, (iv) Both firms have reached the intermediate technology level. This time, the hazard rate type of Firm 2 plays an important role in our analysis. We will model the competition separately for the cases where Firm 2's hazard rate type is *high* and *low*.

By using backward induction, we can solve the game and analyze the subgame perfect Nash equilibrium investment rates.

3.2.1. Analysis of the Last Stage

We begin with the situation when both players have achieved the intermediate technology level and the actual hazard rate of Firm 2 is a high type one.

Firm 1's payoff is determined by

$$\begin{aligned} V^H_{1^*2^*}(a^H_{1^*2^*}; a^H_{2^*1^*}) = & \int_0^\infty W e^{-rt}[1 - F^H_{2^*1^*}(t|a^H_{2^*1^*})] f^H_{1^*2^*}(t|a^H_{1^*2^*})dt \\ & - \int_0^\infty \left\{ \int_0^t a^H_{1^*2^*} e^{-rs} ds \right\} [1 - F^H_{2^*1^*}(t|a^H_{2^*1^*})] f^H_{1^*2^*}(t|a^H_{1^*2^*})dt \quad (3.15) \\ & - \int_0^\infty \left\{ \int_0^t a^H_{1^*2^*} e^{-rs} ds \right\} [1 - F^H_{1^*2^*}(t|a^H_{1^*2^*})] f^H_{2^*1^*}(t|a^H_{2^*1^*})dt \end{aligned}$$

The first term in Equation 3.15 refers to the present expected discounted reward and the second term refers to the present expected discounted cost in case Firm 1 wins the race. Last term refers to the present expected discounted cost for Firm 1 when its rival (Firm 2) crosses the finish line before it. When we substitute Equation 3.13 and

Equation 3.14 into Equation 3.15, we obtain

$$
\begin{aligned}
V^H_{1^*2^*}(a^H_{1^*2^*}; a^H_{2^*1^*}) = & \int_0^\infty We^{-rt}[1 - 1 + e^{-h^H(a^H_{2^*1^*})t}][h(a^H_{1^*2^*})e^{-h(a^H_{1^*2^*})t}]dt \\
& - \int_0^\infty \left\{ \int_0^t a^H_{1^*2^*} e^{-rs} ds \right\} [1 - 1 + e^{-h^H(a^H_{2^*1^*})t}][h(a^H_{1^*2^*}) \\
& e^{-h(a^H_{1^*2^*})t}]dt \\
& - \int_0^\infty \left\{ \int_0^t a^H_{1^*2^*} e^{-rs} ds \right\} [1 - 1 + e^{-h(a^H_{1^*2^*})t}][h^H(a^H_{2^*1^*}) \\
& e^{-h^H(a^H_{2^*1^*})t}]dt
\end{aligned}
\tag{3.16}
$$

After taking integrals and some algebra, the above equation becomes

$$
V^H_{1^*2^*}(a^H_{1^*2^*}; a^H_{2^*1^*}) = \frac{Wh(a^H_{1^*2^*}) - a^H_{1^*2^*}}{r + h(a^H_{1^*2^*}) + h^H(a^H_{2^*1^*})} \tag{3.17}
$$

For a given $a^H_{2^*1^*}$, Firm 1 chooses $a^H_{1^*2^*}$ to maximize its profit. In order Firm 1 to stay in the game, it chooses such an $a^H_{1^*2^*}$ that makes $V^H_{1^*2^*}$ greater than zero. This happens when $W > a^H_{1^*2^*}/h(a^H_{1^*2^*})$.

Firm 2's payoff is determined by

$$
\begin{aligned}
V^H_{2^*1^*}(a^H_{2^*1^*}; a^H_{1^*2^*}) = & \int_0^\infty We^{-rt}[1 - F^H_{1^*2^*}(t|a^H_{1^*2^*})]f_{2^*1^*H}(t|a^H_{2^*1^*})dt \\
& - \int_0^\infty \left\{ \int_0^t a^H_{2^*1^*} e^{-rs} ds \right\} [1 - F^H_{1^*2^*}(t|a^H_{1^*2^*})]f^H_{2^*1^*}(t|a^H_{2^*1^*})dt \\
& - \int_0^\infty \left\{ \int_0^t a^H_{2^*1^*} e^{-rs} ds \right\} [1 - F^H_{2^*1^*}(t|a^H_{2^*1^*})]f^H_{1^*2^*}(t|a^H_{1^*2^*})dt
\end{aligned}
\tag{3.18}
$$

Similar to Firm 1's payoff, the first term in Equation 3.18 refers to the present expected discounted reward and the second term refers to the present expected discounted cost in case Firm 2 wins the race. Last term refers to the present expected discounted cost for Firm 2 when its rival (Firm 1) crosses the finish line before it. When we substitute

Equation 3.13 and Equation 3.14 into Equation 3.18, we obtain

$$
\begin{aligned}
V^H_{2^*1^*}(a^H_{2^*1^*}; a^H_{1^*2^*}) = \int_0^\infty & We^{-rt}[1 - 1 + e^{-h(a^H_{1^*2^*})t}][h^H(a^H_{2^*1^*})e^{-h^H(a^H_{2^*1^*})t}]dt \\
& - \int_0^\infty \left\{ \int_0^t a^H_{2^*1^*} e^{-rs} ds \right\} [1 - 1 + e^{-h(a^H_{1^*2^*})t}][h^H(a^H_{2^*1^*}) \\
& e^{-h^H(a^H_{2^*1^*})t}]dt \\
& - \int_0^\infty \left\{ \int_0^t a^H_{2^*1^*} e^{-rs} ds \right\} [1 - 1 + e^{-h^H(a^H_{2^*1^*})t}][h(a^H_{1^*2^*}) \\
& e^{-h(a^H_{1^*2^*})t}]dt
\end{aligned}
\tag{3.19}
$$

Analogous to Firm 1's payoff, the above equation becomes

$$
V^H_{2^*1^*}(a^H_{2^*1^*}; a^H_{1^*2^*}) = \frac{Wh^H(a^H_{2^*1^*}) - a^H_{2^*1^*}}{r + h^H(a^H_{2^*1^*}) + h(a^H_{1^*2^*})} \tag{3.20}
$$

For a given $a^H_{1^*2^*}$, Firm 2 chooses $a^H_{2^*1^*}$ to maximize its profit. In order Firm 2 to stay in the game, it chooses such an $a^H_{2^*1^*}$ that makes $V^H_{2^*1^*}$ greater than zero. This happens when $W > a^H_{2^*1^*}/h^H(a^H_{2^*1^*})$.

When both players have achieved the intermediate technology level and the hazard rate of Firm 2 is actually a low type one,

Firm 1's payoff is determined by

$$
\begin{aligned}
V^L_{1^*2^*}(a^L_{1^*2^*}; a^L_{2^*1^*}) = \int_0^\infty & We^{-rt}[1 - F^L_{2^*1^*}(t|a^L_{2^*1^*})]f^L_{1^*2^*}(t|a^L_{1^*2^*})dt \\
& - \int_0^\infty \left\{ \int_0^t a^L_{1^*2^*} e^{-rs} ds \right\} [1 - F^L_{2^*1^*}(t|a^L_{2^*1^*})]f^L_{1^*2^*}(t|a^L_{1^*2^*})dt \\
& - \int_0^\infty \left\{ \int_0^t a^L_{1^*2^*} e^{-rs} ds \right\} [1 - F^L_{1^*2^*}(t|a^L_{1^*2^*})]f^L_{2^*1^*}(t|a^L_{2^*1^*})dt
\end{aligned}
\tag{3.21}
$$

The first term in Equation 3.21 refers to the present expected discounted reward and the second term refers to the present expected discounted cost in case Firm 1 wins the race. Last term refers to the present expected discounted cost for Firm 1 when its rival (Firm 2) crosses the finish line before it. When we substitute Equation 3.13 and

Equation 3.14 into Equation 3.21, we obtain

$$
\begin{aligned}
V^L_{1^*2^*}(a^L_{1^*2^*}; a^L_{2^*1^*}) = & \int_0^\infty We^{-rt}[1 - 1 + e^{-h^L(a^L_{2^*1^*})t}][h(a^L_{1^*2^*})e^{-h(a^L_{1^*2^*})t}]dt \\
& - \int_0^\infty \left\{ \int_0^t a^L_{1^*2^*} e^{-rs} ds \right\} [1 - 1 + e^{-h^L(a^L_{2^*1^*})t}][h(a^L_{1^*2^*}) \\
& e^{-h(a^L_{1^*2^*})t}]dt \\
& - \int_0^\infty \left\{ \int_0^t a^L_{1^*2^*} e^{-rs} ds \right\} [1 - 1 + e^{-h(a^L_{1^*2^*})t}][h^L(a^L_{2^*1^*}) \\
& e^{-h^L(a^L_{2^*1^*})t}]dt
\end{aligned}
\tag{3.22}
$$

After taking integrals and some algebra, the above equation becomes

$$
V^L_{1^*2^*}(a^L_{1^*2^*}; a^L_{2^*1^*}) = \frac{Wh(a^L_{1^*2^*}) - a^L_{1^*2^*}}{r + h(a^L_{1^*2^*}) + h^L(a^L_{2^*1^*})} \tag{3.23}
$$

For a given $a^L_{2^*1^*}$, Firm 1 chooses $a^L_{1^*2^*}$ to maximize its profit. In order Firm 1 to stay in the game, it chooses such an $a^L_{1^*2^*}$ that makes $V^L_{1^*2^*}$ greater than zero. This happens when $W > a^L_{1^*2^*}/h(a^L_{1^*2^*})$.

Firm 2's payoff is determined by

$$
\begin{aligned}
V^L_{2^*1^*}(a^L_{2^*1^*}; a^L_{1^*2^*}) = & \int_0^\infty We^{-rt}[1 - F^L_{1^*2^*}(t|a^L_{1^*2^*})]f_{2^*1^*L}(t|a^L_{2^*1^*})dt \\
& - \int_0^\infty \left\{ \int_0^t a^L_{2^*1^*} e^{-rs} ds \right\} [1 - F^L_{1^*2^*}(t|a^L_{1^*2^*})]f^L_{2^*1^*}(t|a^L_{2^*1^*})dt \\
& - \int_0^\infty \left\{ \int_0^t a^L_{2^*1^*} e^{-rs} ds \right\} [1 - F^L_{2^*1^*}(t|a^L_{2^*1^*})]f^L_{1^*2^*}(t|a^L_{1^*2^*})dt
\end{aligned}
\tag{3.24}
$$

Similar to Firm 1's payoff, the first term in Equation 3.24 refers to the present expected discounted reward and the second term refers to the present expected discounted cost in case Firm 2 wins the race. Last term refers to the present expected discounted cost for Firm 2 when its rival (Firm 1) crosses the finish line before it. When we substitute

Equation 3.13 and Equation 3.14 into Equation 3.24, we obtain

$$\begin{aligned} V^L_{2^*1^*}(a^L_{2^*1^*};a^L_{1^*2^*}) = &\int_0^\infty We^{-rt}[1-1+e^{-h(a^L_{1^*2^*})t}][h^L(a^L_{2^*1^*})e^{-h^L(a^L_{2^*1^*})t}]dt \\ &-\int_0^\infty \left\{\int_0^t a^L_{2^*1^*}e^{-rs}ds\right\}[1-1+e^{-h(a^L_{1^*2^*})t}][h^L(a^L_{2^*1^*}) \\ &e^{-h^L(a^L_{2^*1^*})t}]dt \\ &-\int_0^\infty \left\{\int_0^t a^L_{2^*1^*}e^{-rs}ds\right\}[1-1+e^{-h^L(a^L_{2^*1^*})t}][h(a^L_{1^*2^*}) \\ &e^{-h(a^L_{1^*2^*})t}]dt \end{aligned} \tag{3.25}$$

After taking integrals and some algebra, the above equation becomes

$$V^L_{2^*1^*}(a^L_{2^*1^*};a^L_{1^*2^*}) = \frac{Wh^L(a^L_{2^*1^*}) - a^L_{2^*1^*}}{r + h^L(a^L_{2^*1^*}) + h(a^L_{1^*2^*})} \tag{3.26}$$

For a given $a^L_{1^*2^*}$, Firm 2 chooses $a^L_{2^*1^*}$ to maximize its profit. In order Firm 2 to stay in the game, it chooses such an $a^L_{2^*1^*}$ that makes $V^L_{2^*1^*}$ greater than zero. This happens when $W > a^L_{2^*1^*}/h^L(a^L_{2^*1^*})$.

3.2.2. Analysis of the Intermediate Stages

Consider the case when Firm 1 has completed the first stage and the Firm 2 has not. In case of when the type of Firm 2's hazard rate is *high*,

Firm 1's payoff is determined by

$$\begin{aligned} V^H_{1^*2}(a^H_{1^*2};a^H_{21^*}) = &\int_0^\infty We^{-rt}[1-F^H_{21^*}(t|a^H_{21^*})]f^H_{1^*2}(t|a^H_{1^*2})dt \\ &-\int_0^\infty \left\{\int_0^t a^H_{1^*2}e^{-rs}ds\right\}[1-F^H_{21^*}(t|a^H_{21^*})]f^H_{1^*2}(t|a^H_{1^*2})dt \\ &+\int_0^\infty V^H_{1^*2^*}e^{-rt}[1-F^H_{1^*2}(t|a^H_{1^*2})]f^H_{21^*}(t|a^H_{21^*})dt \\ &-\int_0^\infty \left\{\int_0^t a^H_{1^*2}e^{-rs}ds\right\}[1-F^H_{1^*2}(t|a^H_{1^*2})]f^H_{21^*}(t|a^H_{21^*})dt \end{aligned} \tag{3.27}$$

The first term in Equation 3.27 refers to the present expected discounted reward and the second term refers to the present expected discounted cost in case Firm 1 wins the race. Third term refers to the present expected discounted payoff and fourth term refers to the present expected discounted cost when the rival (Firm 2) catches up with the leader and carries the competition to the state we analyzed above where both firms have completed the initial stage. When we substitute Equation 3.13 and Equation 3.14 into Equation 3.27, we obtain

$$\begin{aligned} V^H_{1*2}(a^H_{1*2}; a^H_{21*}) = & \int_0^\infty We^{-rt}[1 - 1 + e^{-h^H(a^H_{21*})t}][h(a^H_{1*2})e^{-h(a^H_{1*2})t}]dt \\ & - \int_0^\infty \left\{ \int_0^t a^H_{1*2} e^{-rs} ds \right\} [1 - 1 + e^{-h^H(a^H_{21*})t}][h(a^H_{1*2}) \\ & e^{-h(a^H_{1*2})t}]dt \\ & + \int_0^\infty V^H_{1*2*} e^{-rt}[1 - 1 + e^{-h(a^H_{1*2})t}][h^H(a^H_{21*})e^{-h^H(a^H_{21*})t}]dt \\ & - \int_0^\infty \left\{ \int_0^t a^H_{1*2} e^{-rs} ds \right\} [1 - 1 + e^{-h(a^H_{1*2})t}][h^H(a^H_{21*}) \\ & e^{-h^H(a^H_{21*})t}]dt \end{aligned} \tag{3.28}$$

After taking integrals and some algebra, the above equation becomes

$$V^H_{1*2}(a^H_{1*2}; a^H_{21*}) = \frac{Wh(a^H_{1*2}) + V^H_{1*2*} h^H(a^H_{21*}) - a^H_{1*2}}{r + h(a^H_{1*2}) + h^H(a^H_{21*})} \tag{3.29}$$

For a given a^H_{21*}, Firm 1 chooses a^H_{1*2} to maximize its profit. In order Firm 1 to stay in the game, it chooses such an a^H_{1*2} that makes a^H_{1*2} greater than zero. This happens when $W > a^H_{1*2}/h(a^H_{1*2})$. Firm 2's payoff is determined by

$$\begin{aligned} V^H_{21*}(a^H_{21*}; a^H_{1*2}) = & \int_0^\infty V^H_{2*1*} e^{-rt}[1 - F^H_{1*2}(t|a^H_{1*2})] f^H_{21*}(t|a^H_{21*}) dt \\ & - \int_0^\infty \left\{ \int_0^t a^H_{21*} e^{-rs} ds \right\} [1 - F^H_{1*2}(t|a^H_{1*2})] f^H_{21*}(t|a^H_{21*}) dt \\ & - \int_0^\infty \left\{ \int_0^t a^H_{21*} e^{-rs} ds \right\} [1 - F^H_{21*}(t|a^H_{21*})] f^H_{1*2}(t|a^H_{1*2}) dt \end{aligned} \tag{3.30}$$

The first term in Equation 3.30 refers to the present expected discounted payoff and the second term refers to the present expected discounted cost in case Firm 2 catches up with the leader. Last term refers to the present expected discounted cost when the leader (Firm 1) wins the game and Firm 2 loses all its research expenditure. When we substitute Equation 3.13 and Equation 3.14 into Equation 3.30, we obtain

$$
\begin{aligned}
V^H_{21^*}(a^H_{21^*};a^H_{1^*2}) = & \int_0^\infty V^H_{2^*1^*}e^{-rt}[1-1+e^{-h(a^H_{1^*2})t}][h^H(a^H_{21^*})e^{-h^H(a^H_{21^*})t}]dt \\
& - \int_0^\infty \left\{\int_0^t a^H_{21^*}e^{-rs}ds\right\}[1-1+e^{-h(a^H_{1^*2})t}][h^H(a^H_{21^*}) \\
& e^{-h^H(a^H_{21^*})t}]dt \\
& - \int_0^\infty \left\{\int_0^t a^H_{21^*}e^{-rs}ds\right\}[1-1+e^{-h^H(a^H_{21^*})t}][h(a^H_{1^*2}) \\
& e^{-h(a^H_{1^*2})t}]dt
\end{aligned}
\tag{3.31}
$$

After taking integrals and some algebra, the above equation becomes

$$
V^H_{21^*}(a^H_{21^*};a^H_{1^*2}) = \frac{V^H_{2^*1^*}h^H(a^H_{21^*}) - a_{21^*H}}{r + h(a^H_{1^*2}) + h^H(a^H_{21^*})} \tag{3.32}
$$

For a given $a^H_{1^*2}$, Firm 2 chooses $a^H_{21^*}$ to maximize its profit. In order Firm 2 to stay in the game, it chooses such an $a^H_{21^*}$ that makes $V^H_{21^*}$ greater than zero. This happens when $V^H_{2^*1^*} > a^H_{21^*}/h^H(a^H_{21^*})$.

Similarly, consider the case when Firm 1 has completed the first stage and the Firm 2 has not when the type of Firm 2's hazard rate is *low*.

Firm 1's payoff is determined by

$$
\begin{aligned}
V^L_{1^*2}(a^L_{1^*2};a^L_{21^*}) = & \int_0^\infty We^{-rt}[1-F^L_{21^*}(t|a^L_{21^*})]f^L_{1^*2}(t|a^L_{1^*2})dt \\
& - \int_0^\infty \left\{\int_0^t a^L_{1^*2}e^{-rs}ds\right\}[1-F^L_{21^*}(t|a^L_{21^*})]f^L_{1^*2}(t|a^L_{1^*2})dt \\
& + \int_0^\infty V^L_{1^*2^*}e^{-rt}[1-F^L_{1^*2}(t|a^L_{1^*2})]f^L_{21^*}(t|a^L_{21^*})dt \\
& - \int_0^\infty \left\{\int_0^t a^L_{1^*2}e^{-rs}ds\right\}[1-F^L_{1^*2}(t|a^L_{1^*2})]f^L_{21^*}(t|a^L_{21^*})dt
\end{aligned}
\tag{3.33}
$$

The first term in Equation 3.33 refers to the present expected discounted reward and the second term refers to the present expected discounted cost in case Firm 1 wins the race. Third term refers to the present expected discounted payoff and fourth term refers to the present expected discounted cost when the rival (Firm 2) catches up with the leader and carries the competition to the state we analyzed above where both firms have completed the initial stage. When we substitute Equation 3.13 and Equation 3.14 into Equation 3.33, we obtain

$$
\begin{aligned}
V^L_{1^*2}(a^L_{1^*2}; a^L_{21^*}) = & \int_0^\infty W e^{-rt}[1 - 1 + e^{-h^L(a^L_{21^*})t}][h(a^L_{1^*2})e^{-h(a^L_{1^*2})t}]dt \\
& - \int_0^\infty \left\{ \int_0^t a^L_{1^*2} e^{-rs} ds \right\} [1 - 1 + e^{-h^L(a^L_{21^*})t}][h(a^L_{1^*2}) \\
& e^{-h(a^L_{1^*2})t}]dt \\
& + \int_0^\infty V^L_{1^*2^*} e^{-rt}[1 - 1 + e^{-h(a^L_{1^*2})t}][h^L(a^L_{21^*})e^{-h^L(a^L_{21^*})t}]dt \\
& - \int_0^\infty \left\{ \int_0^t a^L_{1^*2} e^{-rs} ds \right\} [1 - 1 + e^{-h(a^L_{1^*2})t}][h^L(a^L_{21^*}) \\
& e^{-h^L(a^L_{21^*})t}]dt
\end{aligned}
\tag{3.34}
$$

After taking integrals and some algebra, the above equation becomes

$$
V^L_{1^*2}(a^L_{1^*2}; a^L_{21^*}) = \frac{W h(a^L_{1^*2}) + V^L_{1^*2^*} h^L(a^L_{21^*}) - a^L_{1^*2}}{r + h(a^L_{1^*2}) + h^L(a^L_{21^*})} \tag{3.35}
$$

For a given $a^L_{21^*}$, Firm 1 chooses $a^L_{1^*2}$ to maximize its profit. In order Firm 1 to stay in the game, it chooses such an $a^L_{1^*2}$ that makes $a^L_{1^*2}$ greater than zero. This happens when $W > a^L_{1^*2}/h(a^L_{1^*2})$. Firm 2's payoff is determined by

$$
\begin{aligned}
V^L_{21^*}(a^L_{21^*}; a^L_{1^*2}) = & \int_0^\infty V^L_{2^*1^*} e^{-rt}[1 - F^L_{1^*2}(t|a^L_{1^*2})] f^L_{21^*}(t|a^L_{21^*})dt \\
& - \int_0^\infty \left\{ \int_0^t a^L_{21^*} e^{-rs} ds \right\} [1 - F^L_{1^*2}(t|a^L_{1^*2})] f^L_{21^*}(t|a^L_{21^*})dt \\
& - \int_0^\infty \left\{ \int_0^t a^L_{21^*} e^{-rs} ds \right\} [1 - F^L_{21^*}(t|a^L_{21^*})] f^L_{1^*2}(t|a^L_{1^*2})dt
\end{aligned}
\tag{3.36}
$$

The first term in Equation 3.36 refers to the present expected discounted payoff and the second term refers to the present expected discounted cost in case Firm 2 catches up with the leader. Last term refers to the present expected discounted cost when the leader (Firm 1) wins the game and Firm 2 loses all its research expenditure. When we substitute Equation 3.13 and Equation 3.14 into Equation 3.36, we obtain

$$\begin{aligned} V^L_{21^*}(a^L_{21^*}; a^L_{1^*2}) = &\int_0^\infty V^L_{2^*1^*} e^{-rt}[1-1+e^{-h(a^L_{1^*2})t}][h^L(a^L_{21^*})e^{-h^L(a^L_{21^*})t}]dt \\ &- \int_0^\infty \left\{ \int_0^t a^L_{21^*} e^{-rs} ds \right\} [1-1+e^{-h(a^L_{1^*2})t}][h^L(a^L_{21^*}) \\ &e^{-h^L(a^L_{21^*})t}]dt \\ &- \int_0^\infty \left\{ \int_0^t a^L_{21^*} e^{-rs} ds \right\} [1-1+e^{-h^L(a^L_{21^*})t}][h(a^L_{1^*2}) \\ &e^{-h(a^L_{1^*2})t}]dt \end{aligned} \tag{3.37}$$

After taking integrals and some algebra, the above equation becomes

$$V^L_{21^*}(a^L_{21^*}; a^L_{1^*2}) = \frac{V^L_{2^*1^*} h^L(a^L_{21^*}) - a_{21^*L}}{r + h(a^L_{1^*2}) + h^L(a^L_{21^*})} \tag{3.38}$$

For a given $a^L_{1^*2}$, Firm 2 chooses $a^L_{21^*}$ to maximize its profit. In order Firm 2 to stay in the game, it chooses such an $a^L_{21^*}$ that makes $V^L_{21^*}$ greater than zero. This happens when $V^L_{2^*1^*} > a^L_{21^*}/h^L(a^L_{21^*})$.

Consider the case when Firm 2 has completed the first stage and the other has not. In case of when the type of Firm 2's hazard rate is *high*,

Firm 2's payoff is determined by

$$\begin{aligned} V^H_{2^*1}(a^H_{2^*1}; a^H_{12^*}) = &\int_0^\infty W e^{-rt}[1 - F^H_{12^*}(t|a^H_{12^*})] f^H_{2^*1}(t|a^H_{2^*1})dt \\ &- \int_0^\infty \left\{ \int_0^t a^H_{2^*1} e^{-rs} ds \right\} [1 - F^H_{12^*}(t|a^H_{12^*})] f^H_{2^*1}(t|a^H_{2^*1})dt \\ &+ \int_0^\infty V^H_{2^*1^*} e^{-rt}[1 - F^H_{2^*1}(t|a^H_{2^*1})] f^H_{12^*}(t|a^H_{12^*})dt \\ &- \int_0^\infty \left\{ \int_0^t a^H_{2^*1} e^{-rs} ds \right\} [1 - F^H_{2^*1}(t|a^H_{2^*1})] f^H_{12^*}(t|a^H_{12^*})dt \end{aligned} \tag{3.39}$$

The first term in Equation 3.39 refers to the present expected discounted reward and the second term refers to the present expected discounted cost in case Firm 2 wins the race. Third term refers to the present expected discounted payoff and fourth term refers to the present expected discounted cost when the rival (Firm 1) catches up with the leader and carries the competition to the state we analyzed above where both firms have completed the initial stage. When we substitute Equation 3.13 and Equation 3.14 into Equation 3.39, we obtain

$$\begin{aligned} V^H_{2^*1}(a^H_{2^*1}; a^H_{12^*}) = & \int_0^\infty W e^{-rt}[1 - 1 + e^{-h(a^H_{12^*})t}][h^H(a^H_{2^*1})e^{-h^H(a^H_{2^*1})t}]dt \\ & - \int_0^\infty \left\{ \int_0^t a^H_{2^*1} e^{-rs} ds \right\} [1 - 1 + e^{-h(a^H_{12^*})t}][h^H(a^H_{2^*1}) \\ & e^{-h^H(a^H_{2^*1})t}]dt \\ & + \int_0^\infty V^H_{2^*1^*} e^{-rt}[1 - 1 + e^{-h^H(a^H_{2^*1})t}][h(a^H_{12^*})e^{-h(a^H_{12^*})t}]dt \\ & - \int_0^\infty \left\{ \int_0^t a^H_{2^*1} e^{-rs} ds \right\} [1 - 1 + e^{-h^H(a^H_{2^*1})t}][h(a^H_{12^*}) \\ & e^{-h(a^H_{12^*})t}]dt \end{aligned} \tag{3.40}$$

After taking integrals and some algebra, the above equation becomes

$$V^H_{2^*1}(a^H_{2^*1}; a^H_{12^*}) = \frac{W h^H(a^H_{2^*1}) + V^H_{2^*1^*} h(a^H_{12^*}) - a^H_{2^*1}}{r + h^H(a^H_{2^*1}) + h(a^H_{12^*})} \tag{3.41}$$

For a given $a^H_{12^*}$, Firm 2 chooses $a^H_{2^*1}$ to maximize its profit. In order Firm 2 to stay in the game, it chooses such an $a^H_{2^*1}$ that makes $V^H_{2^*1}$ greater than zero. This happens when $W > a^H_{2^*1}/h^H(a^H_{2^*1})$.

Firm 1's payoff is determined by

$$\begin{aligned} V^H_{12^*}(a^H_{12^*}; a^H_{2^*1}) = & \int_0^\infty V^H_{1^*2^*} e^{-rt}[1 - F^H_{2^*1}(t|a^H_{2^*1})] f^H_{12^*}(t|a^H_{12^*})dt \\ & - \int_0^\infty \left\{ \int_0^t a^H_{12^*} e^{-rs} ds \right\} [1 - F^H_{2^*1}(t|a^H_{2^*1})] f^H_{12^*}(t|a^H_{12^*})dt \\ & - \int_0^\infty \left\{ \int_0^t a^H_{12^*} e^{-rs} ds \right\} [1 - F^H_{12^*}(t|a^H_{12^*})] f^H_{2^*1}(t|a^H_{2^*1})dt \end{aligned} \tag{3.42}$$

The first term in Equation 3.42 refers to the present expected discounted payoff and the second term refers to the present expected discounted cost in case Firm 1 catches up with the leader. Last term refers to the present expected discounted cost when the leader (Firm 2) wins the game and Firm 1 loses all its research expenditure. When we substitute Equation 3.13 and Equation 3.14 into Equation 3.42, we obtain

$$\begin{aligned} V^H_{12^*}(a^H_{12^*}; a^H_{2^*1}) = & \int_0^\infty V^H_{1^*2^*} e^{-rt}[1 - 1 + e^{-h^H(a^H_{2^*1})t}][h(a^H_{12^*}) e^{-h(a^H_{12^*})t}] dt \\ & - \int_0^\infty \left\{ \int_0^t a^H_{12^*} e^{-rs} ds \right\} [1 - 1 + e^{-h^H(a^H_{2^*1})t}][h(a^H_{12^*}) \\ & e^{-h(a^H_{12^*})t}] dt \\ & - \int_0^\infty \left\{ \int_0^t a^H_{12^*} e^{-rs} ds \right\} [1 - 1 + e^{-h(a^H_{12^*})t}][h^H(a^H_{2^*1}) \\ & e^{-h^H(a^H_{2^*1})t}] dt \end{aligned} \tag{3.43}$$

After taking integrals and some algebra, the above equation becomes

$$V^H_{12^*}(a^H_{12^*}; a^H_{2^*1}) = \frac{V^H_{1^*2^*} h(a^H_{12^*}) - a^H_{12^*}}{r + h^H(a^H_{2^*1}) + h(a^H_{12^*})} \tag{3.44}$$

For a given $a^H_{2^*1}$, Firm 2 chooses $a^H_{12^*}$ to maximize its profit. In order Firm 2 to stay in the game, it chooses such an $a^H_{12^*}$ that makes $V^H_{12^*}$ greater than zero. This happens when $V^H_{1^*2^*} > a^H_{12^*}/h(a^H_{12^*})$.

Similarly, consider the case when Firm 2 has completed the first stage and the other has not when the type of Firm 2's hazard rate is *low*.

$$\begin{aligned} V^L_{2^*1}(a^L_{2^*1}; a^L_{12^*}) = & \int_0^\infty W e^{-rt}[1 - F^L_{12^*}(t|a^L_{12^*})] f^L_{2^*1}(t|a^L_{2^*1}) dt \\ & - \int_0^\infty \left\{ \int_0^t a^L_{2^*1} e^{-rs} ds \right\} [1 - F^L_{12^*}(t|a^L_{12^*})] f^L_{2^*1}(t|a^L_{2^*1}) dt \\ & + \int_0^\infty V^L_{2^*1^*} e^{-rt}[1 - F^L_{2^*1}(t|a^L_{2^*1})] f^L_{12^*}(t|a^L_{12^*}) dt \\ & - \int_0^\infty \left\{ \int_0^t a^L_{2^*1} e^{-rs} ds \right\} [1 - F^L_{2^*1}(t|a^L_{2^*1})] f^L_{12^*}(t|a^L_{12^*}) dt \end{aligned} \tag{3.45}$$

The first term in Equation 3.45 refers to the present expected discounted reward and the second term refers to the present expected discounted cost in case Firm 2 wins the race. Third term refers to the present expected discounted payoff and fourth term refers to the present expected discounted cost when the rival (Firm 1) catches up with the leader and carries the competition to the state we analyzed above where both firms have completed the initial stage. When we substitute Equation 3.13 and Equation 3.14 into Equation 3.45, we obtain

$$\begin{aligned} V^L_{2^*1}(a^L_{2^*1}; a^L_{12^*}) = & \int_0^\infty We^{-rt}[1-1+e^{-h(a^L_{12^*})t}][h^L(a^L_{2^*1})e^{-h^L(a^L_{2^*1})t}]dt \\ & - \int_0^\infty \left\{ \int_0^t a^L_{2^*1}e^{-rs}ds \right\} [1-1+e^{-h(a^L_{12^*})t}][h^L(a^L_{2^*1}) \\ & e^{-h^L(a^L_{2^*1})t}]dt \\ & + \int_0^\infty V^L_{2^*1^*}e^{-rt}[1-1+e^{-h^L(a^L_{2^*1})t}][h(a^L_{12^*})e^{-h(a^L_{12^*})t}]dt \\ & - \int_0^\infty \left\{ \int_0^t a^L_{2^*1}e^{-rs}ds \right\} [1-1+e^{-h^L(a^L_{2^*1})t}][h(a^L_{12^*}) \\ & e^{-h(a^L_{12^*})t}]dt \end{aligned} \tag{3.46}$$

After taking integrals and some algebra, the above equation becomes

$$V^L_{2^*1}(a^L_{2^*1}; a^L_{12^*}) = \frac{Wh^L(a^L_{2^*1}) + V^L_{2^*1^*}h(a^L_{12^*}) - a^L_{2^*1}}{r + h^L(a^L_{2^*1}) + h(a^L_{12^*})} \tag{3.47}$$

For a given $a^L_{12^*}$, Firm 2 chooses $a^L_{2^*1}$ to maximize its profit. In order Firm 2 to stay in the game, it chooses such an $a^L_{2^*1}$ that makes $V^L_{2^*1}$ greater than zero. This happens when $W > a^L_{2^*1}/h^L(a^L_{2^*1})$.

Firm 1's payoff is determined by

$$\begin{aligned} V^L_{12^*}(a^L_{12^*}; a^L_{2^*1}) = & \int_0^\infty V^L_{1^*2^*}e^{-rt}[1 - F^L_{2^*1}(t|a^L_{2^*1})]f^L_{12^*}(t|a^L_{12^*})dt \\ & - \int_0^\infty \left\{ \int_0^t a^L_{12^*}e^{-rs}ds \right\} [1 - F^L_{2^*1}(t|a^L_{2^*1})]f^L_{12^*}(t|a^L_{12^*})dt \\ & - \int_0^\infty \left\{ \int_0^t a^L_{12^*}e^{-rs}ds \right\} [1 - F^L_{12^*}(t|a^L_{12^*})]f^L_{2^*1}(t|a^L_{2^*1})dt \end{aligned} \tag{3.48}$$

The first term in Equation 3.48 refers to the present expected discounted payoff and the second term refers to the present expected discounted cost in case Firm 1 catches up with the leader. Last term refers to the present expected discounted cost when the leader (Firm 2) wins the game and Firm 1 loses all its research expenditure. When we substitute Equation 3.13 and Equation 3.14 into Equation 3.48, we obtain

$$\begin{aligned} V^L_{12^*}(a^L_{12^*}; a^L_{2^*1}) = & \int_0^\infty V^L_{1^*2^*} e^{-rt} [1 - 1 + e^{-h^L(a^L_{2^*1})t}][h(a^L_{12^*}) e^{-h(a^L_{12^*})t}] dt \\ & - \int_0^\infty \left\{ \int_0^t a^L_{12^*} e^{-rs} ds \right\} [1 - 1 + e^{-h^L(a^L_{2^*1})t}][h(a^L_{12^*}) \\ & e^{-h(a^L_{12^*})t}] dt \\ & - \int_0^\infty \left\{ \int_0^t a^L_{12^*} e^{-rs} ds \right\} [1 - 1 + e^{-h(a^L_{12^*})t}][h^L(a^L_{2^*1}) \\ & e^{-h^L(a^L_{2^*1})t}] dt \end{aligned} \tag{3.49}$$

After taking integrals and some algebra, the above equation becomes

$$V^L_{12^*}(a^L_{12^*}; a^L_{2^*1}) = \frac{V^L_{1^*2^*} h(a^L_{12^*}) - a^L_{12^*}}{r + h^L(a^L_{2^*1}) + h(a^L_{12^*})} \tag{3.50}$$

For a given $a^L_{2^*1}$, Firm 2 chooses $a^L_{12^*}$ to maximize its profit. In order Firm 2 to stay in the game, it chooses such an $a^L_{12^*}$ that makes $V^L_{12^*}$ greater than zero. This happens when $V^L_{1^*2^*} > a^L_{12^*}/h(a^L_{12^*})$.

3.2.3. Analysis of the Initial Stage

Lastly, consider the situation when both players are at technology level 0 and neither of them has achieved the intermediate technology level. Here, as Firm 1 has incomplete information, it gives its response without knowing the actual hazard rate type of Firm 2. Firm 1 only has a belief that Firm 2 is of type *high* with probability p and of type *low* with probability $(1 - p)$.

Firm 1's payoff is determined by

$$\begin{aligned} V_{12}(a_{12}; a_{21}^H; a_{21}^L) =& p \left\{ \int_0^\infty V_{1^*2}^H e^{-rt} [1 - F_{21}^H(t|a_{21}^H)] f_{12}(t|a_{12}) dt \right. \\ & - \int_0^\infty \left\{ \int_0^t a_{12} e^{-rs} ds \right\} [1 - F_{21}^H(t|a_{21}^H)] f_{12}(t|a_{12}) dt \\ & + \int_0^\infty V_{12^*}^H e^{-rt} [1 - F_{12}(t|a_{12})] f_{21}^H(t|a_{21}^H) dt \\ & \left. - \int_0^\infty \left\{ \int_0^t a_{12} e^{-rs} ds \right\} [1 - F_{12}(t|a_{12})] f_{21}^H(t|a_{21}^H) dt \right\} \\ & + (1-p) \left\{ \int_0^\infty V_{1^*2}^L e^{-rt} [1 - F_{21}^L(t|a_{21}^L)] f_{12}(t|a_{12}) dt \right. \\ & - \int_0^\infty \left\{ \int_0^t a_{12} e^{-rs} ds \right\} [1 - F_{21}^L(t|a_{21}^L)] f_{12}(t|a_{12}) dt \\ & + \int_0^\infty V_{12^*}^L e^{-rt} [1 - F_{12}(t|a_{12})] f_{21}^L(t|a_{21}^L) dt \\ & \left. - \int_0^\infty \left\{ \int_0^t a_{12} e^{-rs} ds \right\} [1 - F_{12}(t|a_{12})] f_{21}^L(t|a_{21}^L) dt \right\} \end{aligned} \tag{3.51}$$

The first term in Equation 3.51 refers to the present expected discounted payoff and the second term refers to the present expected discounted cost in case Firm 1 completes the first stage before its rival and reaches the first level. Third term refers to the present expected discounted payoff and fourth term refers to the present expected discounted cost in case of Firm 2 completes the first stage and succeeds in reaching the first level.

When we substitute Equation 3.13 and Equation 3.14 into Equation 3.51, we obtain

$$\begin{aligned} V_{12}(a_{12};a_{21}^H;a_{21}^L) =& p\left\{\int_0^\infty V_{1^*2}^H e^{-rt}[1-1+e^{-h^H(a_{21}^H)t}][h(a_{12})e^{-h(a_{12})t}]dt\right. \\ & -\int_0^\infty \left\{\int_0^t a_{12}e^{-rs}ds\right\}[1-1+e^{-h^H(a_{21}^H)t}][h(a_{12}) \\ & e^{-h(a_{12})t}]dt \\ & +\int_0^\infty V_{12*}^H e^{-rt}[1-1+e^{-h(a_{12})t}][h^H(a_{21}^H)e^{-h^H(a_{21}^H)t}]dt \\ & -\int_0^\infty \left\{\int_0^t a_{12}e^{-rs}ds\right\}[1-1+e^{-h(a_{12})t}][h^H(a_{21}^H) \\ & \left. e^{-h(a_{21})t}]dt\right\} \\ & +(1-p)\left\{\int_0^\infty V_{1^*2}^L e^{-rt}[1-1+e^{-h^L(a_{21}^L)t}][h(a_{12})\right. \\ & e^{-h(a_{12})t}]dt \\ & -\int_0^\infty \left\{\int_0^t a_{12}e^{-rs}ds\right\}[1-1+e^{-h^L(a_{21}^L)t}][h(a_{12}) \\ & e^{-h(a_{12})t}]dt \\ & +\int_0^\infty V_{12*}^L e^{-rt}[1-1+e^{-h(a_{12})t}][h^L(a_{21}^L)e^{-h^L(a_{21}^L)t}]dt \\ & -\int_0^\infty \left\{\int_0^t a_{12}e^{-rs}ds\right\}[1-1+e^{-h(a_{12})t}][h^L(a_{21}^L) \\ & \left. e^{-h^L(a_{21}^L)t}]dt\right\} \end{aligned} \tag{3.52}$$

After taking integrals and some algebra, the above equation becomes

$$\begin{aligned} V_{12}(a_{12};a_{21}^H;a_{21}^L) =& p\left\{\frac{V_{1^*2}^H h(a_{12}) + V_{12*}^H h^H(a_{21}^H) - a_{12}}{r + h(a_{12}) + h^H(a_{21}^H)}\right\} \\ & +(1-p)\left\{\frac{V_{1^*2}^L h(a_{12}) + V_{12*}^L h^L(a_{21}^L) - a_{12}}{r + h(a_{12}) + h^L(a_{21}^L)}\right\} \end{aligned} \tag{3.53}$$

For given a_{21}^H and a_{21}^L, Firm 1 chooses a_{12} to maximize its profit. In order Firm 1 to stay in the game, it chooses such an a_{12} that makes V_{12} greater than zero.

In case Firm 2's hazard rate is high, Firm 2's payoff is determined by

$$\begin{aligned} V_{21}^H(a_{21}^H;a_{12}) = &\int_0^\infty V_{2^*1}^H e^{-rt}[1-F_{12}(t|a_{12})]f_{21}^H(t|a_{21}^H)dt \\ &-\int_0^\infty \left\{\int_0^t a_{21}^H e^{-rs}ds\right\}[1-F_{12}(t|a_{12})]f_{21}^H(t|a_{21}^H)dt \\ &+\int_0^\infty V_{21^*}^H e^{-rt}[1-F_{21}^H(t|a_{21}^H)]f_{12}(t|a_{12})dt \\ &-\int_0^\infty \left\{\int_0^t a_{21}^H e^{-rs}ds\right\}[1-F_{21}^H(t|a_{21}^H)]f_{12}(t|a_{12})dt \end{aligned} \tag{3.54}$$

The first term in Equation 3.54 refers to the present expected discounted payoff and the second term refers to the present expected discounted cost in case Firm 2 completes the first stage before its rival and reaches the first level. Third term refers to the present expected discounted payoff and fourth term refers to the present expected discounted cost in case Firm 1 completes the first stage and succeeds in reaching the first level. When we substitute Equation 3.13 and Equation 3.14 into Equation 3.54, we obtain

$$\begin{aligned} V_{21}^H(a_{21}^H;a_{12}) = &\int_0^\infty V_{2^*1}^H e^{-rt}[1-1+e^{-h(a_{12})t}][h^H(a_{21}^H)e^{-h^H(a_{21}^H)t}]dt \\ &-\int_0^\infty \left\{\int_0^t a_{21}^H e^{-rs}ds\right\}[1-1+e^{-h(a_{12})t}][h^H(a_{21}^H)e^{-h^H(a_{21}^H)t}]dt \\ &+\int_0^\infty V_{21^*}^H e^{-rt}[1-1+e^{-h^H(a_{21}^H)t}][h(a_{12})e^{-h(a_{12})t}]dt \\ &-\int_0^\infty \left\{\int_0^t a_{21}^H e^{-rs}ds\right\}[1-1+e^{-h^H(a_{21}^H)t}][h(a_{12})e^{-h(a_{12})t}]dt \end{aligned} \tag{3.55}$$

After taking integrals and some algebra, the above equation becomes

$$V_{21}^H(a_{21}^H;a_{12}) = \frac{V_{2^*1}^H h^H(a_{21}^H) + V_{21^*}^H h(a_{12}) - a_{21}^H}{r + h^H(a_{21}^H) + h(a_{12})} \tag{3.56}$$

For a given a_{12}, Firm 1 chooses a_{21}^H to maximize its profit. In order Firm 1 to stay in the game, it chooses such an a_{21}^H that makes V_{21}^H greater than zero. This happens when $V_{2^*1}^H > a_{21}^H/h^H(a_{21}^H)$.

In case Firm 2's hazard rate is low, Firm 2's payoff is determined by

$$\begin{aligned} V_{21}^L(a_{21}^L; a_{12}) = &\int_0^\infty V_{2^*1}^L e^{-rt}[1 - F_{12}(t|a_{12})] f_{21}^L(t|a_{21}^L)dt \\ &- \int_0^\infty \left\{ \int_0^t a_{21}^L e^{-rs} ds \right\} [1 - F_{12}(t|a_{12})] f_{21}^L(t|a_{21}^L)dt \\ &+ \int_0^\infty V_{21^*}^L e^{-rt}[1 - F_{21}^L(t|a_{21}^L)] f_{12}(t|a_{12})dt \\ &- \int_0^\infty \left\{ \int_0^t a_{21}^L e^{-rs} ds \right\} [1 - F_{21}^L(t|a_{21}^L)] f_{12}(t|a_{12})dt \end{aligned} \tag{3.57}$$

The first term in Equation 3.57 refers to the present expected discounted payoff and the second term refers to the present expected discounted cost in case Firm 2 completes the first stage before its rival and reaches the first level. Third term refers to the present expected discounted payoff and fourth term refers to the present expected discounted cost in case Firm 1 completes the first stage and succeeds in reaching the first level. When we substitute Equation 3.13 and Equation 3.14 into Equation 3.57, we obtain

$$\begin{aligned} V_{21}^L(a_{21}^L; a_{12}) = &\int_0^\infty V_{2^*1}^L e^{-rt}[1 - 1 + e^{-h(a_{12})t}][h^L(a_{21}^L) e^{-h^L(a_{21}^L)t}]dt \\ &- \int_0^\infty \left\{ \int_0^t a_{21}^L e^{-rs} ds \right\} [1 - 1 + e^{-h(a_{12})t}][h^L(a_{21}^L) e^{-h^L(a_{21}^L)t}]dt \\ &+ \int_0^\infty V_{21^*}^L e^{-rt}[1 - 1 + e^{-h^L(a_{21}^L)t}][h(a_{12}) e^{-h(a_{12})t}]dt \\ &- \int_0^\infty \left\{ \int_0^t a_{21}^L e^{-rs} ds \right\} [1 - 1 + e^{-h^L(a_{21}^L)t}][h(a_{12}) e^{-h(a_{12})t}]dt \end{aligned} \tag{3.58}$$

After taking integrals and some algebra, the above equation becomes

$$V_{21}^L(a_{21}^L; a_{12}) = \frac{V_{2^*1}^L h^L(a_{21}^L) + V_{21^*}^L h(a_{12}) - a_{21}^L}{r + h^L(a_{21}^L) + h(a_{12})} \tag{3.59}$$

For a given a_{12}, Firm 1 chooses a_{21}^L to maximize its profit. In order Firm 1 to stay in the game, it chooses such an a_{21}^L that makes V_{21}^L greater than zero. This happens when $V_{2^*1}^L > a_{21}^L / h^L(a_{21}^L)$.

In order to see the model properties, we use a numerical simulation and analyze the equilibrium investment rates by using backward induction in the next chapter.

We first find each firms best response function that maximizes its payoff for the given investment rate of its rival and then by substituting one firms best response equation in the other firms best response equation, we get the equilibrium investment rates of the competing firms for different technology levels of the competition.

4. NUMERICAL ANALYSIS

In order to characterize the competing firms' investment decisions under asymmetric hazard rate information, we analyze our two-stage dynamic R&D game numerically in this chapter. We aim to observe the effect of model parameters on the decision variables and the payoff values. We also aim to see how firms change their behavior and investment rates as the technology states of the competition changes over time.

We assume that the value of the prize W that will be won at the end of the competition is a constant value. We set the prize value as 100 in our analysis. Firms discount their earnings and costs at a discount rate r of 0.05. The hazard rate function $h(a)$ for Firm 2 is set as $a^{1/2}$ when its type is high and $a^{1/4}$ when its type is low. It should be noted that Firm 2's hazard rate type cannot change after the competition starts.

We analyze the model for 3 different hazard rate levels of Firm 1. Based on these 3 different hazard rate levels, Firm 1's hazard rate function can take values:

- Greater than the high type hazard rate value of Firm 2,
- Between the high and low type hazard rate values of Firm 2,
- Smaller than the low type hazard rate value of Firm 2.

We use $a^{2/3}$, $a^{1/3}$ or $a^{1/5}$ respectively for the hazard rate levels we mentioned above.

At the beginning of the initial stage, Firm 1 believes that Firm 2 has a high type hazard rate with probability p and a low type hazard rate with probability $(1 - p)$. This probability is varied as $p = 0.2$, $p = 0.5$ and $p = 0.8$ in our analysis. We note again that the type uncertainty about Firm 2's hazard rate reveals just after the competition starts. For this reason, we analyze the model as a complete information game after either of the firms completes the first stage. We begin our numerical analysis with the state where both firms complete the initial stage and we use backward induction to compute the equilibrium investment levels of the different technology states. Our observations are only for a particular parameter set, therefore with another set of parameters, it may be possible to obtain different behaviors from our model.

In parallel with Grossman and Shapiro [5], we observe that when both firms have completed the first stage, it is a common behavior that they spend more resources, than at the beginning of the game when neither of them has finished the initial stage. They also spend more resources, than at the intermediate state when either of them has finished the initial stage. The reason is that the competition becomes more intense at this state and both firms sense that the victory is close; therefore they both intensify their efforts. In this final state, we also observe that both of the firms invest more in the new technology when Firm 2's hazard rate is of type high, compared to the situation when Firm 2's hazard rate is of type low. Because, having a high type hazard rate means, Firm 2 spends more resources on the new technology which causes Firm 1 to respond with a higher investment rate in order not to lose the competition.

After one of the firms has succeeded in jumping to the second stage, the probability that represents Firm 1's belief about the hazard rate type of its rival does not play any role and it is also a common behavior that the firm with the higher hazard rate spends more on the new technology than its competitor.

The conclusions of Grossman and Shapiro [5] show that the leader always dedicates more resources to innovation than does its follower. However, based on our numerical analysis, their finding is not valid for our model. We observe that when Firm 1's hazard rate is greater than Firm 2's high type hazard rate, it is Firm 1 that invests more in R&D, whether Firm 1 is the leader or the follower. Here, as Firm 1 knows that it has a greater hazard rate, it feels confident about catching up with its rival before its rival finishes the game. In this case, Firm 1 even spends more than the situation when it is the leader. Conversely, when Firm 1's hazard rate is smaller than Firm 2's low type hazard rate, it is Firm 2 that invests more on R&D than Firm 1 whether it is the leader or the follower due to the same reason we explained above.

And when Firm 1's hazard rate is between Firm 2's high and low type hazard rate, two scenarios may occur. If Firm 2 is of type high, it is Firm 2 that invests more on R&D whether it is the leader or the follower. Similar to the situation above, as Firm 2 realizes that it has a greater hazard rate after type information of Firm 2 is revealed; it feels confident about jumping to the second stage and believes that catching up with Firm 1 is easy, therefore being the follower does not cause demoralization. And if Firm 2 is of type low, opposite to the first observation; it is Firm 1 that invests more on R&D whether it is the leader or the follower due to the same reasons.

In addition, Grossman and Shapiro [5] find out that a typical behavior for the leader firm is to intensify its R&D investments when it moves to the second stage, while for the other firm to reduce its expenditures. However, in their research they find examples where both the leader and the follower increase (or decrease) their investments after the initial stage. Similar to their findings, we also could not observe a common behavior for this situation in our analysis.

Our model differs from the existing models in including uncertainty during the first stage of the competition. At the beginning of the game, Firm 1 has incomplete information about Firm 2's true hazard rate type and believes that its rival is of type high with probability p and of type low with probability $(1-p)$. In these circumstances, we characterize the competing firms' initial investment rates for the following cases.

4.1. When Firm 1's Hazard Rate is Greater than Firm 2's High Type Hazard Rate

Consider the case when neither of the firms has finished the initial stage and Firm 1 believes that Firm 2 has a high type hazard rate with probability 0.2. In this case, Firm 2 is at a disadvantage no matter what the type of Firm 1 is and we observe that if Firm 2's real hazard rate is of type high, Firm 2 spends more resources on R&D than Firm 1. If Firm 1 becomes a leader after the initial stage, Firm 1 learns the real type information of Firm 2 and understands that it needs to invest more in order to compete with its rival, therefore increases its R&D effort, while Firm 2 decreases.

Table 4.1. The Numerical Illustration of the Initial Stage Responses when $p = 0.2$ and Firm 1's Hazard Rate is Greater than Firm 2's High Type Hazard Rate.

PLAYER 2'S ACTUAL TYPE IS "H":	PLAYER 2'S ACTUAL TYPE IS "L":
$a_{12} = 120.5170$	$a_{12} = 120.5170$
$a_{21}^H = 143.2218$	$a_{21}^L = 3.0828$
$V_{12} = 86.3894$	$V_{12} = 86.3894$
$V_{21}^H = 8.1803$	$V_{21}^L = 0.7773$

Different from our comment for Firm 1, Firm 2 decreases its effort because of the discouragement it feels. However, if Firm 2's real hazard rate is of type low, opposite to the first observation; Firm 1 spends more resources on R&D than Firm 2. If Firm 1 becomes a leader after the initial stage, both Firm 1 and Firm 2 reduce their R&D efforts. As Firm 1 discovers that Firm 2 has a low type hazard rate after the game advances to the intermediate stage, it finds it unnecessary to intensify its innovation effort and even reduces the scope of its expenditures.

If Firm 2 becomes a leader after the first stage, independent of its hazard rate type, it gains motivation to cross the finish line and responds by increasing its R&D expenditures. At this technology state, when only Firm 2 has passed the initial stage, Firm 1 also increases its resources. At this point, although Firm 1 is the one that is behind, it believes that it will catch up with the leader due to its hazard rate power which is higher than Firm 2's high type hazard rate. Furthermore, Firm 1 believes that Firm 2 is of type low so it determines an investment rate relatively small at the beginning. After the actual type of Firm 2 is revealed, Firm 1 makes an adjustment in the intensity of its effort. Table 4.1 shows the initial stage investment rates and payoff values in case of incomplete information when $p = 0.2$ and Firm 1's hazard rate is greater than Firm 2's high type hazard rate.

In order to see the effects of different probability values on the investment rates, we change the p value from 0.2 to 0.5, and then to 0.8. This shows an increasing belief that Firm 2 has a high type hazard rate.

Table 4.2. The Numerical Illustration of the Initial Stage Responses when $p = 0.5$ and Firm 1's Hazard Rate is Greater than Firm 2's High Type Hazard Rate.

PLAYER 2'S ACTUAL TYPE IS "H":	PLAYER 2'S ACTUAL TYPE IS "L":
$a_{12} = 175.0923$	$a_{12} = 175.0923$
$a_{21}^H = 115.3235$	$a_{21}^L = 3.0930$
$V_{12} = 82.3413$	$V_{12} = 82.3413$
$V_{21}^H = 7.1318$	$V_{21}^L = 0.6978$

Table 4.3. The Numerical Illustration of the Initial Stage Responses when $p = 0.8$ and Firm 1's Hazard Rate is Greater than Firm 2's High Type Hazard Rate.

PLAYER 2'S ACTUAL TYPE IS "H":	PLAYER 2'S ACTUAL TYPE IS "L":
$a_{12} = 255.8164$	$a_{12} = 255.8164$
$a_{21}^H = 118.3050$	$a_{21}^L = 3.1371$
$V_{12} = 78.8273$	$V_{12} = 78.8273$
$V_{21}^H = 6.2562$	$V_{21}^L = 0.6337$

When $p = 0.5$ and $p = 0.8$, Firm 1 spends more resources on R&D than Firm 2 independent of the type of Firm 2. We also observe that, as p increases for this hazard rate case, the investment rate of Firm 1 also increases at the initial stage. In addition, if Firm 1 becomes the leader after the initial stage, it reduces its resources; independent of Firm 2 has a high or low type hazard rate. This means that, when p increases, Firm 1 starts the game with a higher investment rate. Then, after it completes the first stage, the type of Firm 2 reveals and Firm 1 learns that it has a greater hazard rate value than Firm 2. Therefore, Firm 1 becomes aware of the situation that there is no need to expend that much effort and reduces its research activities. The initial stage investment rates and payoff values for these two cases when $p = 0.5$ and $p = 0.8$ can be seen in Table 4.2 and Table 4.3 respectively.

For $p = 0.8$, when only Firm 2 has passed the initial stage, two scenarios may occur. If Firm 2 is of type high, Firm 1 increases its resources. As we mentioned before, although Firm 1 is the one that is behind, it believes that it will catch up with the leader due to its hazard rate power and increases its investment. However, if Firm 2 is of type low, Firm 1 realizes that it overestimates the rival's studies and spends less resource, than at the beginning of the game when neither of the firms has finished the initial stage.

4.2. When Firm 1's Hazard Rate is Between the Low and High Type Hazard Rate of Firm 2

Consider again the case when neither of the firms has finished the initial stage and Firm 1 believes that Firm 2 has a high type hazard rate with probability 0.2. In this case, two scenarios may occur. If Firm 2's hazard rate is of type high, it spends more resources on R&D than Firm 1. If Firm 1 becomes a leader after the initial stage, Firm 1 learns the real type information of Firm 2 and understands that it needs to invest more in order to compete with its rival, therefore increases its R&D effort. Different than the previous case, Firm 2 also increases its effort even it is the follower firm. On the other hand, if Firm 2 is the leader firm, both firms reduce their R&D efforts.

However, if Firm 2's hazard rate is of type low; Firm 1 spends more resources on R&D than the other. If Firm 1 becomes a leader after the initial stage, both firms reduce their R&D efforts. As Firm 1 discovers that Firm 2 has a low type hazard rate after the game advances to the intermediate stage, it finds it unnecessary to intensify its innovation effort and even reduces the scope of its expenditures. If Firm 2 becomes the leader after the first stage, both firms increase their research activities.

If we compare this case with the previous one, we see a decrease in the investment rate of Firm 1 due to the decrease in its hazard rate value. On the other hand, investment rate of Firm 2 decreases when Firm 2 is of type high and increases when Firm 2 is of type low. Table 4.4 shows the initial stage investment rates and payoff values in case of incomplete information when p is 0.2 and Firm 1's hazard rate is between the low and hype type hazard rate of Firm 2.

Table 4.4. The Numerical Illustration of the Initial Stage Responses when $p = 0.2$ and Firm 1's Hazard Rate is Between the Low and High Type Hazard Rate of Firm 2.

PLAYER 2'S ACTUAL TYPE IS "H":	PLAYER 2'S ACTUAL TYPE IS "L":
$a_{12} = 26.0141$	$a_{12} = 26.0141$
$a_{21}^H = 87.3960$	$a_{21}^L = 15.1892$
$V_{12} = 40.8116$	$V_{12} = 40.8116$
$V_{21}^H = 66.4056$	$V_{21}^L = 24.9430$

Table 4.5. The Numerical Illustration of the Initial Stage Responses when $p = 0.5$ and Firm 1's Hazard Rate is Between the Low and High Type Hazard Rate of Firm 2.

PLAYER 2'S ACTUAL TYPE IS "H":	PLAYER 2'S ACTUAL TYPE IS "L":
$a_{12} = 25.8162$	$a_{12} = 25.8162$
$a_{21}^H = 87.4812$	$a_{21}^L = 15.1723$
$V_{12} = 28.6304$	$V_{12} = 28.6304$
$V_{21}^H = 66.4236$	$V_{21}^L = 24.9657$

In order to see the effects of different probability values on the investment rates, we change the p value from 0.2 to 0.5, and then to 0.8.

For $p = 0.5$ and $p = 0.8$, we see that all mentioned observations for $p = 0.2$ are still valid. Based on our observations, we can conclude that when Firm 1's hazard rate value is between Firm 2's high and low type hazard rate values, probability (i.e., belief

Table 4.6. The Numerical Illustration of the Initial Stage Responses when $p = 0.8$ and Firm 1's Hazard Rate is Between the Low and High Type Hazard Rate of Firm 2.

PLAYER 2'S ACTUAL TYPE IS "H":	PLAYER 2'S ACTUAL TYPE IS "L":
$a_{12} = 25.4767$	$a_{12} = 25.4767$
$a_{21}^H = 87.6372$	$a_{21}^L = 15.1504$
$V_{12} = 16.4456$	$V_{12} = 16.4456$
$V_{21}^H = 66.4548$	$V_{21}^L = 25.0049$

Table 4.7. The Numerical Illustration of the Initial Stage Responses when $p = 0.2$ and Firm 1's Hazard Rate is Smaller than Firm 2's Low Type Hazard Rate.

PLAYER 2'S ACTUAL TYPE IS "H":	PLAYER 2'S ACTUAL TYPE IS "L":
$a_{12} = 8.8394$	$a_{12} = 8.8394$
$a_{21}^{H} = 40.4914$	$a_{21}^{L} = 12.8471$
$V_{12} = 27.5365$	$V_{12} = 27.5365$
$V_{21}^{H} = 77.7042$	$V_{21}^{L} = 45.1055$

in the type of Firm 2) does not have a significant effect on the firms' behaviors under these parameter set. The initial stage investment rates and payoff values for these two cases when $p = 0.5$ and $p = 0.8$ can be seen in Table 4.5 and Table 4.6 respectively.

4.3. When Firm 1's Hazard Rate is Smaller than Firm 2's Low Type Hazard Rate

Again, we start by considering the case when neither of the firms has finished the initial stage and Firm 1 believes that Firm 2 has a high type hazard rate with probability 0.2. In this case, Firm 1 is at a disadvantage no matter what the type of Firm 2 is and we observe that Firm 2 spends more resources on R&D than Firm 1. If Firm 1 manages to be the leader, it increases its efforts; but Firm 2 decreases if it is of type low and increases if it is of type high. If Firm 2 completes the first stage and becomes the leader, Firm 1 decreases its investment rate. For Firm 2, two scenarios may occur. If Firm 2 is of type high, it reduces its R&D efforts while it intensifies them when it is of type low.

Comparing this case with the previous case, we see a decrease in the investment rate of Firm 1 due to the decrease in its hazard rate value. Similarly, investment rate of Firm 2 decreases, either its actual hazard rate type is high or low. Table 4.7 shows the initial stage investment rates and payoff values in case of incomplete information when p is 0.2 and Firm 1's hazard rate is smaller than the low type hazard rate of Firm 2.

Table 4.8. The Numerical Illustration of the Initial Stage Responses when $p = 0.5$ and Firm 1's Hazard Rate is Smaller than Firm 2's Low Type Hazard Rate.

PLAYER 2'S ACTUAL TYPE IS "H":	PLAYER 2'S ACTUAL TYPE IS "L":
$a_{12} = 7.9322$	$a_{12} = 7.9322$
$a_{21}^H = 39.7879$	$a_{21}^L = 12.7052$
$V_{12} = 19.0869$	$V_{12} = 19.0869$
$V_{21}^H = 77.8036$	$V_{21}^L = 45.3306$

Table 4.9. The Numerical Illustration of the Initial Stage Responses when $p = 0.8$ and Firm 1's Hazard Rate is Smaller than Firm 2's Low Type Hazard Rate.

PLAYER 2'S ACTUAL TYPE IS "H":	PLAYER 2'S ACTUAL TYPE IS "L":
$a_{12} = 6.5074$	$a_{12} = 6.5074$
$a_{21}^H = 38.7303$	$a_{21}^L = 12.4473$
$V_{12} = 10.6756$	$V_{12} = 10.6756$
$V_{21}^H = 77.9834$	$V_{21}^L = 45.7414$

In order to see the effects of different probability values on the investment rates, we change the p value from 0.2 to 0.5, and then to 0.8.

For $p = 0.5$, we see that the above mentioned all observations are still valid. However, for $p = 0.8$, when only Firm 2 has passed the initial stage, two scenarios may occur. If Firm 2 is of type high, Firm 1 spends less resource, than at the beginning of the game when neither of the firms has finished the initial stage. However, If Firm 2 is of type low, Firm 1's expenditure increases compared with the beginning of the game when neither of the firms has finished the initial stage. The initial stage investment rates and payoff values for these two cases when $p = 0.5$ and $p = 0.8$ can be seen in Table 4.8 and Table 4.9 respectively.

5. CONCLUSION

In this thesis, we consider a two-stage dynamic R&D game where two firms compete for a prize in a winner-takes-all type competition environment. A literature review consisting of different types of R&D competition is also provided before the modeling part of the thesis. We keep the stochastic structure of the existing patent race models and assume that the success of the stages is subject to uncertainty. In our model, each firm can observe the position of its rival so that it can alter its investment rate depending on its own position. However, research information is not accessible to all parties at the end of stage one. We use backward induction to obtain the solution of the game and as it is not possible to obtain explicit formulas for the subgame perfect Nash equilibrium investment rates, we use numerical analysis to characterize the dynamics of the model.

We focus our attention on how the firms' technology investment rates change during the progress of the competition and extend the two-stage patent race formulation of Park [6] and Grossman and Shapiro [5] in two directions. First, we allow firms to have different hazard rate functions. In the existing models, hazard rate has the same functional property for both of the firms. Second, we include incompleteness and information asymmetry to the previously studied models in the literature.

The general structure of the game is as follows. Firms start the game by simultaneously investing in the new technology. When a firm completes the first stage, it becomes the leader and gets no actual benefit from completing the first stage. After this prerequisite state, if the leader firm gets through the second stage; it wins the competition and receives the entire prize at the end. In this case, the follower firm gains nothing and loses all of its R&D investment. In the meantime, if the follower firm attains the intermediate state before the leader finishes the game, dynamics of the game change and both firms continue the race from the same technology level in the aim of becoming the leader of the whole competition.

We model the above structured game under the case of complete and symmetric information in the first model presented in Chapter 3. Note that the firms' hazard rates have the same functional property in this model. To observe the behavior of the firms, we simulate the model for a particular set of parameters and compare our observations with the results found out in the R&D literature. In our numerical analysis, we observe that when both of the firms complete the initial stage of the game, they invest more in innovation than the amount they invest in the previous stages. When only one of the firms completes the initial stage and attains the intermediate prerequisite state, we observe that the leader firm invests more than the follower. In this case, the leader also invests more than its initial level investment rate whereas the follower decreases its rate of investment and invests less than its initial investment rate. It is seen that these observations are in line with the findings in the literature.

In the second model given in Chapter 3, there is incompleteness and asymmetry on the hazard rate information. While one of the competing firms has private information about its own hazard rate, the other firm has incomplete information about the actual hazard rate type of its rival. In this model, firms may have different hazard rate functions. At the initial point, the firm with incomplete information believes that the other firm has a high type hazard rate with probability p and a low type hazard rate with probability $(1 - p)$. We assume that incompleteness has an effect only at the beginning of the game, therefore as soon as the competition begins, the competition turns into a complete information game. The dynamics of this model are analyzed numerically in Chapter 4.

We observe that when both firms complete the first stage, competition is more intense and firms spend more resources, than at the previous stages of the game when neither/either of them finish the initial stage. In addition, when both/either of the firms complete the first stage, it is a common behavior that the firm with the higher hazard rate invests more in R&D than its competitor, whether the firm is the leader or the follower.

For the initial stage when neither of the firms completes the first stage, we group our observations into three cases; (i) When the hazard rate of the firm with incomplete information is greater than the high type hazard rate of the firm with private information (ii) When the hazard rate of the firm with incomplete information is between the high and low type hazard rate of the firm with private information (iii) When the hazard rate of the firm with incomplete information is smaller than the low type hazard rate of the firm with private information. As we stated before, probability plays an important role in this stage and one of the firms has incomplete information about its rival's actual hazard rate type.

Our main observations for these three cases can be summarized as the following. In the first case, when we increase p from 0.2 to 0.5 and then to 0.8 at the initial stage, we observe that the investment rate of the firm with incomplete information also increases. Furthermore, if the leader is the firm with incomplete information, opposite to the situation for $p = 0.2$, it reduces its R&D resources after the initial stage; independent of its rival has a high or low type hazard rate. We understand that the firm with incomplete information starts the game with an unnecessarily high investment rate. However, after the hazard rate type of the firm with private information reveals, the firm with incomplete information learns that its hazard rate value is much greater than its rival and reduces its research expenditures.

In the second case, when we increase p from 0.2 to 0.5 and then to 0.8 at the initial stage, we observe a decrease in the investment rate of the firm with incomplete information. However, we observe that increasing the probability value does not have an effect on the firms' behaviors and does not change the dynamics of the game under this parameter set. Although our expectation was to observe a significant behavior change in this case, we came across with an opposite behavior.

Lastly in the third case, when we increase p from 0.2 to 0.5 and then to 0.8 at the initial stage, similar to the second case, we observe a decrease in the investment rate of the firm with incomplete information. In this case, when we choose p as 0.5, we observe no change in game dynamics. However, for $p = 0.8$ the dynamics of the game change. When only the firm with private information passes the initial stage, the firm with incomplete information decreases its investment rate if its rival's actual hazard rate type is high and increases its investment rate if its rival's actual hazard rate type is low. As the firm with incomplete information strongly believes that its rival is of type high at the beginning of the competition, it already starts the game with a smaller investment because of the huge gap between the hazard rates. If the actual hazard rate type of the firm with private information reveals as high, the firm with incomplete information learns that its hazard rate value is much smaller than its rival and as it is the follower firm, it does not find it possible to win the game and reduces its R&D investments. On the other hand, if the actual hazard rate type of the firm with private information reveals as low, the firm with incomplete information believes that there is still hope to complete the game although it is the follower and intensifies its R&D investments.

Although Grossman and Shapiro [5] states that in a complete information game, a typical behavior for the leader firm is to intensify its R&D investments, while for the follower firm to reduce its expenditures; we could not observe a common behavior for this situation in our numerical analysis. Our detailed observations and interpretations about the second model can be found in Chapter 4.

For further research, we propose three progress areas. First, the winner-takes-all type approach can be extended to an approach where the loser of the competition also gains some prize. Second, the stages of the game and the number of the firms participating in the competition can be increased. By this way, a more realistic model of the R&D competition can be obtained. Lastly, hazard rate incompleteness can be applied to all stages of the game rather than to apply it only to the initial stage.

APPENDIX A: NUMERICAL ANALYSIS TABLES

Table A.1. The Numerical Illustration of the Responses when $p = 0.2$ and Firm 1's Hazard Rate is Greater than Firm 2's High Type Hazard Rate.

PLAYER 2'S ACTUAL TYPE IS "H":	PLAYER 2'S ACTUAL TYPE IS "L":
Both Players Completed the 1st Stage	Both Players Completed the 1st Stage
$a^H_{1^*2^*} = 532.8110$	$a^L_{1^*2^*} = 265.2797$
$a^H_{2^*1^*} = 327.0915$	$a^L_{2^*1^*} = 61.7223$
$V^H_{1^*2^*} = 72.0198$	$V^L_{1^*2^*} = 87.5264$
$V^H_{2^*1^*} = 17.6664$	$V^L_{2^*1^*} = 4.9519$
P1 Completed the 1st Stage, P2 Not	P1 Completed the 1st Stage, P2 Not
$a^H_{1^*2} = 154.1685$	$a^L_{1^*2} = 26.2793$
$a^H_{21^*} = 50.2999$	$a^L_{21^*} = 1.1878$
$V^H_{1^*2} = 90.0370$	$V^L_{1^*2} = 95.5400$
$V^H_{21^*} = 2.0893$	$V^L_{21^*} = 0.4009$
P2 Completed the 1st Stage, P1 Not	P2 Completed the 1st Stage, P1 Not
$a^H_{12^*} = 437.5860$	$a^L_{12^*} = 226.2143$
$a^H_{2^*1} = 294.7195$	$a^L_{2^*1} = 59.4631$
$V^H_{12^*} = 49.6089$	$V^L_{12^*} = 75.6714$
$V^H_{2^*1} = 32.5999$	$V^L_{2^*1} = 10.0636$
Both Players Are At the Beginning	Both Players Are At the Beginning
$a_{12} = 120.5170$	$a_{12} = 120.5170$
$a^H_{21} = 143.2218$	$a^L_{21} = 3.0828$
$V_{12} = 86.3894$	$V_{12} = 86.3894$
$V^H_{21} = 8.1803$	$V^L_{21} = 0.7773$

Table A.2. The Numerical Illustration of the Responses when $p = 0.5$ and Firm 1's Hazard Rate is Greater than Firm 2's High Type Hazard Rate.

PLAYER 2'S ACTUAL TYPE IS "H":	PLAYER 2'S ACTUAL TYPE IS "L":
Both Players Completed the 1st Stage	Both Players Completed the 1st Stage
$a^H_{1^*2^*} = 532.8110$	$a^L_{1^*2^*} = 265.2797$
$a^H_{2^*1^*} = 327.0915$	$a^L_{2^*1^*} = 61.7223$
$V^H_{1^*2^*} = 72.0198$	$V^L_{1^*2^*} = 87.5264$
$V^H_{2^*1^*} = 17.6664$	$V^L_{2^*1^*} = 4.9519$
P1 Completed the 1st Stage, P2 Not	P1 Completed the 1st Stage, P2 Not
$a^H_{1^*2} = 154.1685$	$a^L_{1^*2} = 26.2793$
$a^H_{21^*} = 50.2999$	$a^L_{21^*} = 1.1878$
$V^H_{1^*2} = 90.0370$	$V^L_{1^*2} = 95.5400$
$V^H_{21^*} = 2.0893$	$V^L_{21^*} = 0.4009$
P2 Completed the 1st Stage, P1 Not	P2 Completed the 1st Stage, P1 Not
$a^H_{12^*} = 437.5860$	$a^L_{12^*} = 226.2143$
$a^H_{2^*1} = 294.7195$	$a^L_{2^*1} = 59.4631$
$V^H_{12^*} = 49.6089$	$V^L_{12^*} = 75.6714$
$V^H_{2^*1} = 32.5999$	$V^L_{2^*1} = 10.0636$
Both Players Are At the Beginning	Both Players Are At the Beginning
$a_{12} = 175.0923$	$a_{12} = 175.0923$
$a^H_{21} = 115.3235$	$a^L_{21} = 3.0930$
$V_{12} = 82.3413$	$V_{12} = 82.3413$
$V^H_{21} = 7.1318$	$V^L_{21} = 0.6978$

Table A.3. The Numerical Illustration of the Responses when $p = 0.8$ and Firm 1's Hazard Rate is Greater than Firm 2's High Type Hazard Rate.

PLAYER 2'S ACTUAL TYPE IS "H":	PLAYER 2'S ACTUAL TYPE IS "L":
Both Players Completed the 1st Stage	Both Players Completed the 1st Stage
$a^{H}_{1^*2^*} = 532.8110$	$a^{L}_{1^*2^*} = 265.2797$
$a^{H}_{2^*1^*} = 327.0915$	$a^{L}_{2^*1^*} = 61.7223$
$V^{H}_{1^*2^*} = 72.0198$	$V^{L}_{1^*2^*} = 87.5264$
$V^{H}_{2^*1^*} = 17.6664$	$V^{L}_{2^*1^*} = 4.9519$
P1 Completed the 1st Stage, P2 Not	P1 Completed the 1st Stage, P2 Not
$a^{H}_{1^*2} = 154.1685$	$a^{L}_{1^*2} = 26.2793$
$a^{H}_{21^*} = 50.2999$	$a^{L}_{21^*} = 1.1878$
$V^{H}_{1^*2} = 90.0370$	$V^{L}_{1^*2} = 95.5400$
$V^{H}_{21^*} = 2.0893$	$V^{L}_{21^*} = 0.4009$
P2 Completed the 1st Stage, P1 Not	P2 Completed the 1st Stage, P1 Not
$a^{H}_{12^*} = 437.5860$	$a^{L}_{12^*} = 226.2143$
$a^{H}_{2^*1} = 294.7195$	$a^{L}_{2^*1} = 59.4631$
$V^{H}_{12^*} = 49.6089$	$V^{L}_{12^*} = 75.6714$
$V^{H}_{2^*1} = 32.5999$	$V^{L}_{2^*1} = 10.0636$
Both Players Are At the Beginning	Both Players Are At the Beginning
$a_{12} = 255.8164$	$a_{12} = 255.8164$
$a^{H}_{21} = 118.3050$	$a^{L}_{21} = 3.1371$
$V_{12} = 78.8273$	$V_{12} = 78.8273$
$V^{H}_{21} = 6.2562$	$V^{L}_{21} = 0.6337$

Table A.4. The Numerical Illustration of the Responses when $p = 0.2$ and Firm 1's Hazard Rate is Between the Low and High Type Hazard Rate of Firm 2.

PLAYER 2'S ACTUAL TYPE IS "H":	PLAYER 2'S ACTUAL TYPE IS "L":
Both Players Completed the 1st Stage	Both Players Completed the 1st Stage
$a^H_{1^*2^*} = 134.3220$	$a^L_{1^*2^*} = 66.5033$
$a^H_{2^*1^*} = 254.6142$	$a^L_{2^*1^*} = 43.7262$
$V^H_{1^*2^*} = 17.8820$	$V^L_{1^*2^*} = 50.7487$
$V^H_{2^*1^*} = 63.4727$	$V^L_{2^*1^*} = 31.9832$
P1 Completed the 1st Stage, P2 Not	P1 Completed the 1st Stage, P2 Not
$a^H_{1^*2} = 101.4041$	$a^L_{1^*2} = 23.5716$
$a^H_{21^*} = 160.5538$	$a^L_{21^*} = 9.7084$
$V^H_{1^*2} = 34.0251$	$V^L_{1^*2} = 75.3314$
$V^H_{21^*} = 37.0283$	$V^L_{21^*} = 9.9836$
P2 Completed the 1st Stage, P1 Not	P2 Completed the 1st Stage, P1 Not
$a^H_{12^*} = 11.0390$	$a^L_{12^*} = 27.0124$
$a^H_{2^*1} = 53.0799$	$a^L_{2^*1} = 24.6657$
$V^H_{12^*} = 3.0095$	$V^L_{12^*} = 23.7273$
$V^H_{2^*1} = 85.4205$	$V^L_{2^*1} = 55.7214$
Both Players Are At the Beginning	Both Players Are At the Beginning
$a_{12} = 26.0141$	$a_{12} = 26.0141$
$a^H_{21} = 87.3960$	$a^L_{21} = 15.1892$
$V_{12} = 40.8116$	$V_{12} = 40.8116$
$V^H_{21} = 66.4056$	$V^L_{21} = 24.9430$

Table A.5. The Numerical Illustration of the Responses when $p = 0.5$ and Firm 1's Hazard Rate is Between the Low and High Type Hazard Rate of Firm 2.

PLAYER 2'S ACTUAL TYPE IS "H":	PLAYER 2'S ACTUAL TYPE IS "L":
Both Players Completed the 1st Stage	Both Players Completed the 1st Stage
$a^H_{1^*2^*} = 134.3220$	$a^L_{1^*2^*} = 66.5033$
$a^H_{2^*1^*} = 254.6142$	$a^L_{2^*1^*} = 43.7262$
$V^H_{1^*2^*} = 17.8820$	$V^L_{1^*2^*} = 50.7487$
$V^H_{2^*1^*} = 63.4727$	$V^L_{2^*1^*} = 31.9832$
P1 Completed the 1st Stage, P2 Not	P1 Completed the 1st Stage, P2 Not
$a^H_{1^*2} = 101.4041$	$a^L_{1^*2} = 23.5716$
$a^H_{21^*} = 160.5538$	$a^L_{21^*} = 9.7084$
$V^H_{1^*2} = 34.0251$	$V^L_{1^*2} = 75.3314$
$V^H_{21^*} = 37.0283$	$V^L_{21^*} = 9.9836$
P2 Completed the 1st Stage, P1 Not	P2 Completed the 1st Stage, P1 Not
$a^H_{12^*} = 11.0390$	$a^L_{12^*} = 27.0124$
$a^H_{2^*1} = 53.0799$	$a^L_{2^*1} = 24.6657$
$V^H_{12^*} = 3.0095$	$V^L_{12^*} = 23.7273$
$V^H_{2^*1} = 85.4205$	$V^L_{2^*1} = 55.7214$
Both Players Are At the Beginning	Both Players Are At the Beginning
$a_{12} = 25.8162$	$a_{12} = 25.8162$
$a^H_{21} = 87.4812$	$a^L_{21} = 15.1723$
$V_{12} = 28.6304$	$V_{12} = 28.6304$
$V^H_{21} = 66.4236$	$V^L_{21} = 24.9657$

Table A.6. The Numerical Illustration of the Responses when $p = 0.8$ and Firm 1's Hazard Rate is Between the Low and High Type Hazard Rate of Firm 2.

PLAYER 2'S ACTUAL TYPE IS "H":	PLAYER 2'S ACTUAL TYPE IS "L":
Both Players Completed the 1st Stage	Both Players Completed the 1st Stage
$a^H_{1^*2^*} = 134.3220$	$a^L_{1^*2^*} = 66.5033$
$a^H_{2^*1^*} = 254.6142$	$a^L_{2^*1^*} = 43.7262$
$V^H_{1^*2^*} = 17.8820$	$V^L_{1^*2^*} = 50.7487$
$V^H_{2^*1^*} = 63.4727$	$V^L_{2^*1^*} = 31.9832$
P1 Completed the 1st Stage, P2 Not	P1 Completed the 1st Stage, P2 Not
$a^H_{1^*2} = 101.4041$	$a^L_{1^*2} = 23.5716$
$a^H_{21^*} = 160.5538$	$a^L_{21^*} = 9.7084$
$V^H_{1^*2} = 34.0251$	$V^L_{1^*2} = 75.3314$
$V^H_{21^*} = 37.0283$	$V^L_{21^*} = 9.9836$
P2 Completed the 1st Stage, P1 Not	P2 Completed the 1st Stage, P1 Not
$a^H_{12^*} = 11.0390$	$a^L_{12^*} = 27.0124$
$a^H_{2^*1} = 53.0799$	$a^L_{2^*1} = 24.6657$
$V^H_{12^*} = 3.0095$	$V^L_{12^*} = 23.7273$
$V^H_{2^*1} = 85.4205$	$V^L_{2^*1} = 55.7214$
Both Players Are At the Beginning	Both Players Are At the Beginning
$a_{12} = 25.4767$	$a_{12} = 25.4767$
$a^H_{21} = 87.6372$	$a^L_{21} = 15.1504$
$V_{12} = 16.4456$	$V_{12} = 16.4456$
$V^H_{21} = 66.4548$	$V^L_{21} = 25.0049$

Table A.7. The Numerical Illustration of the Responses when $p = 0.2$ and Firm 1's Hazard Rate is Smaller than Firm 2's Low Type Hazard Rate.

PLAYER 2'S ACTUAL TYPE IS "H":	PLAYER 2'S ACTUAL TYPE IS "L":
Both Players Completed the 1st Stage	Both Players Completed the 1st Stage
$a^H_{1^*2^*} = 36.1969$	$a^L_{1^*2^*} = 23.0778$
$a^H_{2^*1^*} = 154.8233$	$a^L_{2^*1^*} = 30.6779$
$V^H_{1^*2^*} = 11.6068$	$V^L_{1^*2^*} = 38.4077$
$V^H_{2^*1^*} = 74.9143$	$V^L_{2^*1^*} = 47.8543$
P1 Completed the 1st Stage, P2 Not	P1 Completed the 1st Stage, P2 Not
$a^H_{1^*2} = 30.1667$	$a^L_{1^*2} = 12.2457$
$a^H_{21^*} = 107.6844$	$a^L_{21^*} = 12.2650$
$V^H_{1^*2} = 23.2135$	$V^L_{1^*2} = 62.9015$
$V^H_{21^*} = 53.9929$	$V^L_{21^*} = 21.6387$
P2 Completed the 1st Stage, P1 Not	P2 Completed the 1st Stage, P1 Not
$a^H_{12^*} = 2.2921$	$a^L_{12^*} = 7.0811$
$a^H_{2^*1} = 22.8488$	$a^L_{2^*1} = 13.2318$
$V^H_{12^*} = 1.8982$	$V^L_{12^*} = 14.4717$
$V^H_{2^*1} = 90.4399$	$V^L_{2^*1} = 72.2488$
Both Players Are At the Beginning	Both Players Are At the Beginning
$a_{12} = 8.8394$	$a_{12} = 8.8394$
$a^H_{21} = 40.4914$	$a^L_{21} = 12.8471$
$V_{12} = 27.5365$	$V_{12} = 27.5365$
$V^H_{21} = 77.7042$	$V^L_{21} = 45.1055$

Table A.8. The Numerical Illustration of the Responses when $p = 0.5$ and Firm 1's Hazard Rate is Smaller than Firm 2's Low Type Hazard Rate.

PLAYER 2'S ACTUAL TYPE IS "H":	PLAYER 2'S ACTUAL TYPE IS "L":
Both Players Completed the 1st Stage	Both Players Completed the 1st Stage
$a^H_{1^*2^*} = 36.1969$	$a^L_{1^*2^*} = 23.0778$
$a^H_{2^*1^*} = 154.8233$	$a^L_{2^*1^*} = 30.6779$
$V^H_{1^*2^*} = 11.6068$	$V^L_{1^*2^*} = 38.4077$
$V^H_{2^*1^*} = 74.9143$	$V^L_{2^*1^*} = 47.8543$
P1 Completed the 1st Stage, P2 Not	P1 Completed the 1st Stage, P2 Not
$a^H_{1^*2} = 30.1667$	$a^L_{1^*2} = 12.2457$
$a^H_{21^*} = 107.6844$	$a^L_{21^*} = 12.2650$
$V^H_{1^*2} = 23.2135$	$V^L_{1^*2} = 62.9015$
$V^H_{21^*} = 53.9929$	$V^L_{21^*} = 21.6387$
P2 Completed the 1st Stage, P1 Not	P2 Completed the 1st Stage, P1 Not
$a^H_{12^*} = 2.2921$	$a^L_{12^*} = 7.0811$
$a^H_{2^*1} = 22.8488$	$a^L_{2^*1} = 13.2318$
$V^H_{12^*} = 1.8982$	$V^L_{12^*} = 14.4717$
$V^H_{2^*1} = 90.4399$	$V^L_{2^*1} = 72.2488$
Both Players Are At the Beginning	Both Players Are At the Beginning
$a_{12} = 7.9322$	$a_{12} = 7.9322$
$a^H_{21} = 39.7879$	$a^L_{21} = 12.7052$
$V_{12} = 19.0869$	$V_{12} = 19.0869$
$V^H_{21} = 77.8036$	$V^L_{21} = 45.3306$

Table A.9. The Numerical Illustration of the Responses when $p = 0.8$ and Firm 1's Hazard Rate is Smaller than Firm 2's Low Type Hazard Rate.

PLAYER 2'S ACTUAL TYPE IS "H":	PLAYER 2'S ACTUAL TYPE IS "L":
Both Players Completed the 1st Stage	Both Players Completed the 1st Stage
$a^H_{1^*2^*} = 36.1969$	$a^L_{1^*2^*} = 23.0778$
$a^H_{2^*1^*} = 154.8233$	$a^L_{2^*1^*} = 30.6779$
$V^H_{1^*2^*} = 11.6068$	$V^L_{1^*2^*} = 38.4077$
$V^H_{2^*1^*} = 74.9143$	$V^L_{2^*1^*} = 47.8543$
P1 Completed the 1st Stage, P2 Not	P1 Completed the 1st Stage, P2 Not
$a^H_{1^*2} = 30.1667$	$a^L_{1^*2} = 12.2457$
$a^H_{21^*} = 107.6844$	$a^L_{21^*} = 12.2650$
$V^H_{1^*2} = 23.2135$	$V^L_{1^*2} = 62.9015$
$V^H_{21^*} = 53.9929$	$V^L_{21^*} = 21.6387$
P2 Completed the 1st Stage, P1 Not	P2 Completed the 1st Stage, P1 Not
$a^H_{12^*} = 2.2921$	$a^L_{12^*} = 7.0811$
$a^H_{2^*1} = 22.8488$	$a^L_{2^*1} = 13.2318$
$V^H_{12^*} = 1.8982$	$V^L_{12^*} = 14.4717$
$V^H_{2^*1} = 90.4399$	$V^L_{2^*1} = 72.2488$
Both Players Are At the Beginning	Both Players Are At the Beginning
$a_{12} = 6.5074$	$a_{12} = 6.5074$
$a^H_{21} = 38.7303$	$a^L_{21} = 12.4473$
$V_{12} = 10.6756$	$V_{12} = 10.6756$
$V^H_{21} = 77.9834$	$V^L_{21} = 45.7414$

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