NON-RELATIVISTIC BOSE GAS IN CURVED BACKGROUND

by

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ABSTRACT

NON-RELATIVISTIC BOSE GAS IN CURVED BACKGROUND

In this thesis we focus on the Bose-Einstein Condensation of a pure Bose gas confined in a box placed near the horizon. Then, we formulate the finite size effects in the limit $\beta m \to 0$. We look for the number of excited particles, as well as free energy and entropy. During these processes, we used the Heat kernel expansion and the Mellin Barnes transform.

ÖZET

EĞİK UZAYDA RELATİVİSTİK OLMAYAN BOSE GAZI

Bu tezde Bose gazının Bose-Einstein yogunlaşması incelenmistir. Bu inceleme yapılırken gaz bir kutuya hapsedilmis, ve bu kutu eğik uzay zamanda konumlandırılmıştır. Daha sonra $\beta m \to 0$ limitinde gazın sınırlı boyut özelliklerine bakılmıştır. Bu işlemler esnasında Heat kernel dağılımı ve Mellin transformu kullanılarak uyarılmıs parcacık sayısı, serbest enerji ve entropi hesaplanmıştır.

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LIST OF SYMBOLS

a_j	Heat kernel coefficients
A	Area of the constant surface
F	Free energy
$\hbar$	Reduced planck constant
H	Hamiltonian operator
k_B	Boltzmann constant
m	Mass of a Bose particle
M	Mass of the astronoical object
n	Number of particles per volume
N	Number of particles
T_C	Critical temperature
Tr	Trace operator
β	Temperature parameter
ϵ	Energy of a state
γ	Non-relativistic analogue of the optical metric
λ_T	De Broglie wavelength
μ	Chemical potential

LIST OF ACRONYMS/ABBREVIATIONS

BEC	Bose-Einstein Condensation
BH	Black Hole
GS	Ground State
KGE	Klein Gordon Equation
$\mathcal{LM}$	Laplace Mellin Transform
SE	Schroedinger Equation

1. INTRODUCTION

Hydrogen, helium, lithium and sodium gases display BEC properties in experiments [1]. In addition to these elements, which nebulas may contain plenty of, some compact astronomical objects are thought to possess some matter in the form of BEC [2–4]. The revelation of these facts triggered a zest to start asking questions about BEC in curved background.

Another important problem in modern gravitational physics is the black hole entropy due to Hawking radiation. [5–8] An important step toward a deeper understanding of black hole entropy would be the investigation of the thermodynamics of quantum fields near the black hole. These examples are more than enough to convince us of the central role of curved space-time statistical thermodynamics in modern gravitational physics.

Our aim in this thesis is to study the basic thermodynamic properties of a non-relativistic Bose field in the Schwarzschild background. More precisely, we will study the behavior of mean occupation number, the free energy and the entropy in the high temperature regime. The Bose field will be confined in a box and will be subject to Dirichlet boundary conditions. Since in a curved background the knowledge of the density of states is not available one cannot generalize the methods used in the familiar flat case to the curved case. One alternative is to express the thermodynamics quantities in terms of the trace of the heat kernel. The resulting expressions are of the harmonic sum form, and their high temperature expansions can be derived by Mellin transform techniques. A crucial ingredient of this program is the use of the heat kernel expansion in the calculation of the Mellin transform. One advantage of the use of the heat kernel expansion is that it enables us, in a relatively straightforward manner, to include the finite size effects on the thermodynamics of the field due to the boundaries of the system.

We will use the above mentioned methods to determine the critical temperature of a Bose gas as a function of the number of particles in a finite box in the gravitational field of a non-rotating spherically symmetric object. In general the thermodynamic quantities are divergent as one approaches the event horizon. In order to avoid this complication, we shall stay away from the horizon in this part of the problem. We will also study the free energy and the entropy of the Bose field in the high temperature expansion, and in this part of the thesis we will approach the event horizon and determine the behavior of the horizon divergences.

In the second chapter, we will introduce the basics of a Bose-Einstein condensate on a flat space. Also, the necessity of the Heat kernel expansion for a curved space-time will be demonstrated. Then, in the third chapter, we will interfere with the metric and obtain an analogue optical metric to calculate the non-relativistic Schroedinger equation. Thus, we will satisfy the pure laplacian that the kernel expansion requires. Afterwards, we are going to use the laplacian in the Heat kernel expansion and Mellin transformation. As a result of these techniques, certain coefficients are going to be revealed based on the area and volume of the container. After handling geometrical calculations in the fifth chapter, finally we will be able to find results for the thermodynamic variables in question.

2. IDEAL BOSE EINSTEIN CONDENSATION IN A FLAT SPACE

The main features that distinguish bosons from other particles are having integer spins and symmetric wave function. The symmetry of their wave function enables more than one boson to occupy the same state. Therefore, unlike fermions, they can occupy the same single particle state. The mean occupation number of bosons is given by the Bose-Einstein distribution

$$\langle n_i \rangle = \frac{1}{e^{\beta(\epsilon_i - \mu)} - 1} \tag{2.1}$$

where

$$\beta = \frac{1}{k_B T} \tag{2.2}$$

$$\mu = \mu(V,T) \quad \text{and} \quad \mu < \epsilon_0 \tag{2.3}$$

So it is clear that μ is non-vanishing. The statistical average is calculated by making use of the density of states $D(k)dk$ in the flat space [9], which is

$$\begin{aligned} D(k)dk &= (2s+1)\frac{Vd^3k}{(2\pi)^3} \\ &= \frac{Vk^2dk}{2\pi^2} \end{aligned} \tag{2.4}$$

The s parameter denotes the spin of the boson particles and it is chosen to be zero. For the ideal particles the energy can be expressed in terms of the wave number via

$$\epsilon = \frac{p^2}{2m} \quad \text{and} \quad p = \hbar k \tag{2.5}$$

therefore, the density of the states in terms of energy is

$$D(\epsilon)d\epsilon \;=\; 2\pi V\left(\frac{2m}{h^2}\right)^{3/2}\epsilon^{1/2}d\epsilon \tag{2.6}$$

so the average particle number is

$$\begin{aligned} N &= \int nD(\epsilon)\; d\epsilon \\ &= 2\pi V\left(\frac{2m}{h^2}\right)^{3/2}\int\frac{\epsilon^{1/2}d\epsilon}{e^{\beta(\epsilon_i-\mu)}-1} \end{aligned} \tag{2.7}$$

Let us take a closer look at the mean occupation number. If the total number of bosons is held fixed at the value N, then the condition

$$N \;=\; \sum_i \frac{1}{e^{\beta(\epsilon_i-\mu)}-1} \tag{2.8}$$

can be solved for μ. It turns out that there exists a critical temperature T_c such that for $T < T_c$. Below T_c we expand the denominator and observe the occupation number become macroscopic [10]. Using binomial expansion,

$$\begin{aligned} n_0 &= \frac{1}{e^{-\beta\mu}-1} \\ &= \frac{1}{1+\beta\mu+\frac{1}{2}(\beta\mu)^2+\ldots-1} \\ &= \frac{1}{\beta\mu+\frac{1}{2}(\beta\mu)^2+\ldots} \\ &= \frac{1}{\beta\mu+X} \end{aligned} \tag{2.9}$$

where X is a finite term. So we can express μ as

$$\mu \;=\; \frac{k_B T}{n_0 - X} \tag{2.10}$$

where n is the particle per volume($n_0 = N_0/V$) for the GS. We see that

$$\mu \neq 0 \tag{2.11}$$

Approaching the critical temperature limit, the chemical potential behaves as follows:

$$\mu = O\left(\frac{1}{V}\right) \tag{2.12}$$

Due to the symmetric nature of their wave-function, Bose particles are able to accumulate in a particular state infinitely. Their unrestricted occupancy in a state is a disadvantage rather than an advantage when counting the mean occupation number at the ground state (GS). We will therefore treat the total number of particles into two sets: the ones in the GS, and the rest.

$$N = N_0 + N_e \tag{2.13}$$

$$N_0 = \frac{1}{e^{-\mu\beta} - 1} \tag{2.14}$$

$$N_e = \sum_{i\neq 1}^{\infty} \frac{1}{e^{\beta(\epsilon_i - \mu)} - 1} \tag{2.15}$$

We witness the warped behavior of the N_0 in the $T < T_c$ condition, where N_0 becomes infinite. Note that none of the remaining terms brings about infinity. Therefore, we can identify and find a solution to the unrealistic result of the occupation at the GS. This strange pile-up in the GS originates from the weight of the probability. So where does the condensation take place? Even though BEC is nothing more than a condensation process in the GS, one can associate a temperature limit for the μ corresponding to the N_0 [11]. From [2.14]

$$\mu(T, V) = \frac{1}{\beta} \ln\left(\frac{N_0}{N_0 + 1}\right) \tag{2.16}$$

More precisely using $\mu(T,V) = \frac{1}{\beta}\ln\left(\frac{N_0}{N_0+1}\right)$ for $T < T_c$ we see that $\frac{N_0}{V}$ becomes finite in the $V \to \infty$ limit. In particular, this means that we cannot convert the sums into integral as easily as we thought we could. This is the reason why the ground state should be treated separately. We notice that for the same $\beta \to \infty$ limit μ becomes 0. Furthermore for temperatures less than T_c, the chemical potential remains zero , and the BEC occurs. After its prediction in1925, BEC was first observed in the laboratory by Wieman and Cornell [12] in 1995.

As for the phase transition, after the boson particles cool down to the extreme temperatures (in the case of Rb^{87}, Wieman and Cornell cooled it down to $0.17\mu K$) they freeze in the momentum space.

Let us try to create a rough formula of dimensions of energy from $\hbar$, m, and n, where $\hbar$ is energy times time, n is particle per unit volume, and mass is purely mass. Therefore

$$\frac{\hbar^2 n^{2/3}}{m} \tag{2.17}$$

is an estimate of energy. By dividing this with Boltzmann constant k_B, we find a quantity in the order of temperature times a numerical constant. For the transition to the condensation phase we can interprete this as the T_c.

$$T_c = C\frac{\hbar^2 n^{2/3}}{mk_B} \tag{2.18}$$

An equivalent method to achieve is to match the inter-atomic gap (which is of order $n^{-1/3}$) with the thermal de Broglie wavelength λ_T.

$$\lambda_T = \sqrt{\frac{h^2}{2\pi m k_B T}} \tag{2.19}$$

When we consider the behavioral change of the gas in phase transition we see that it leaves the classical region when the temperature is low enough (T_c). After that, the phenomenon is defined as a quantum degeneracy and our gas becomes a quantum soup (where λ_T is bigger than the gap between particles) [13]. If we decrease the temperature and the particle mass even more, the distinct packets unify into a whole. In a nutshell, macroscopic occupation of single particle states occur when the de Broglie wavelength exceeds the distance between the bosons and violates the wave-particle duality.

3. NON-RELATIVISTIC HAMILTONIAN

Since the Schroedinger equation is not a covariant wave equation, and since it is not a priori clear what the non-relativistic Hamiltonian should be on a manifold with Lorentzian sign, we will start with the KGE (which is indeed a covariant wave function) and take its non-relativistic limit. Hereinafter we take $\hbar = c = k_B = 1$ for simplification purposes. On the other hand, this will cause unexpected dimensional results. This feature should be remembered while making a dimensional analysis.

3.1. A General Case

In the case of the static space-times [14] this can be done in a straightforward manner using algebra, which is indeed what we are going to do. On the other hand, it is also possible to get this result more systematically using the Foldy-Wouthuysen approximation [15].The static space-time has the form [16]

$$ds^2 = -F dt^2 + g_{ij} dx^i dx^j \tag{3.1}$$

Schwarzschild time is taken as the time coordinate [17], and F, and g_{ij} are independent of time.

Due to our metric preferences the KGE is in the form

$$(\Box - m^2)\phi = 0 \tag{3.2}$$

where

$$\Box = \frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu$$

and $g^{\mu\nu}$ is the inverse metric operator

$$\begin{aligned}
&= \partial_0 g^{00}\partial_0 + \frac{1}{\sqrt{-g}}\partial_i\sqrt{-g}h^{ij}\partial_j \\
&= g^{00}{\partial_0}^2 + \frac{1}{\sqrt{-g}}\partial_i\sqrt{-g}h^{ij}\partial_j \\
&= \frac{-1}{F}{\partial_0}^2 + \frac{1}{\sqrt{-g}}\partial_i\sqrt{-g}h^{ij}\partial_j
\end{aligned} \tag{3.3}$$

We separate KGE in order to obtain the SE, in which the m^2 term is bigger than the kinetic term in accordance with the desired limit.

$$\begin{aligned}
(\Box - m^2)\phi &= 0 \\
(-\frac{1}{F}{\partial_0}^2 + \frac{1}{\sqrt{-g}}\partial_i\sqrt{-g}h^{ij}\partial_j - m^2)\phi &= 0
\end{aligned} \tag{3.4}$$

We are free to insert identity as $F^{1/2}F^{-1/2}$ and to multiply everything with $F^{1/2}$ from the left.

$$-{\partial_0}^2(F^{-1/2}\phi) = -\frac{F^{1/2}}{\sqrt{-g}}\partial_i\sqrt{-g}h^{ij}\partial_j F^{1/2}(F^{-1/2}\phi) + m^2F(F^{-1/2}\phi) \tag{3.5}$$

Define

$$F^{-1/2}\phi \equiv \psi \tag{3.6}$$

$$-{\partial_0}^2\psi = \left(-\frac{F^{1/2}}{\sqrt{-g}}\partial_i\sqrt{-g}h^{ij}\partial_j F^{1/2} + m^2F\right)\psi \tag{3.7}$$

We wish to obtain the square root of [3.7] for getting the energy operator so that we can relate it to the Hamiltonian

$$i\frac{\partial\psi}{\partial t} = H_0\psi \tag{3.8}$$

$$H_0 = \sqrt{-\frac{F^{1/2}}{\sqrt{-g}}\partial_i\sqrt{-g}h^{ij}\partial_j F^{1/2} + m^2F} \tag{3.9}$$

Within the framework of non-relativistic approximation, m^2 term is manifestly bigger than the kinetic term. Taking $F^{1/2}$ out of the square root

$$H_0 = mF^{1/4}\sqrt{1-\frac{1}{m^2\sqrt{-g}}\partial_i\sqrt{-g}h^{ij}\partial_j F^{1/4}} \tag{3.10}$$

Here we ignored $O(m^2)$ terms. We use this assumption in binomial expansion. Then we find

$$\begin{aligned} H_{NR} &= mF^{1/4}\left(1-\frac{1}{2m^2\sqrt{-g}}\partial_i\sqrt{-g}h^{ij}\partial_j\right)F^{1/4} \\ &= mF^{1/2}-\frac{1}{2m}\frac{F^{1/4}}{\sqrt{-g}}\partial_i\sqrt{-g}h^{ij}\partial_j F^{1/4} \end{aligned} \tag{3.11}$$

Here, we did not take the remaining terms into consideration because their order is beyond our focus. As can be seen, our original KGE has now turned into a first order differential equation. One should notice the unusual terms in the resulting equation which are not present in the original SE. This will be the source of the gravitational effects on the smaller box in the non-relativistic system. Next, we will show the appropriate similarity transformations for obtaining a Hamiltonian composed solely of a laplacian and the potential terms with the help of a non-relativistic analogue of optical metric and similarity transformations [18]. Let us start with

$$H_{NR} = -\frac{1}{2m}\frac{F^{1/4}}{\sqrt{-g}}\partial_i\sqrt{-g}h^{ij}\partial_j F^{1/4}+mF^{1/2} \tag{3.12}$$

We isolated the time dependence of the KGE in order to obtain SE.

$$g = -Fh \tag{3.13}$$

$$\begin{aligned} H_{NR} &= -\frac{1}{2m}\frac{F^{1/4}}{F^{1/2}\sqrt{h}}\partial_i F^{1/2}\sqrt{h}h^{ij}\partial_j F^{1/4}+mF^{1/2} \\ &= -\frac{1}{2m}\frac{F^{-1/4}}{\sqrt{h}}\partial_i F^{1/2}\sqrt{h}h^{ij}\partial_j F^{1/4}+mF^{1/2} \end{aligned} \tag{3.14}$$

We cancel the

$$= F^{-1/4}\left(-\frac{1}{2m}\frac{1}{\sqrt{h}}\partial_i F^{1/2}\sqrt{h}h^{ij}\partial_j F^{1/4}+mF^{1/2}\right)F^{1/4} \tag{3.15}$$

This and the upcoming similarity transformations will help us to write the Hamiltonian in terms of the laplacian. The similarity transformation F transforms H_{NR} to H so that it is rotated to be band diagonal and the operators are unitary equivalent. As in this case, they are widely used to simplify a Hamiltonian.

$$\tilde{H} = F^{1/4} H_{NR} F^{-1/4} \tag{3.16}$$

We arrive at

$$\tilde{H} = -\frac{1}{2m}\frac{1}{\sqrt{h}}\partial_i F^{1/2}\sqrt{h}h^{ij}\partial_j + mF^{1/2} \tag{3.17}$$

Now we introduce the non-relativistic analog of the optical metric γ where a remains to be calculated

$$h_{ij} = F^a \gamma_{ij} \tag{3.18}$$

$$h = F^{3a}\gamma \tag{3.19}$$

input into the $\tilde{H}$ and we obtain

$$H_\gamma = -\frac{1}{2m}\frac{F^{-3a/2}}{\sqrt{\gamma}}\partial_i F^{1/2+a/2}\sqrt{\gamma}\gamma^{ij}\partial_j + mF^{1/2} \tag{3.20}$$

The main idea behind the similarity transformation is to take a set of eigenfunctions to another set of eigenfunctions. Making one more similarity transformation with an arbitrary power b

$$H = F^{-b}H_\gamma F^b = -\frac{1}{2m}\frac{F^{-3a/2-b}}{\sqrt{\gamma}}\{\partial_i F^{1/2+a/2}\sqrt{\gamma}\gamma^{ij}\partial_j F^b\} + mF^{1/2} \tag{3.21}$$

At this step, let us make clear the plan behind all these games. Our intention is to modify the Hamiltonian in a way that it contains the laplacian and potential terms. the power of F is obviously dependent on a and b; therefore, we can manipulate it in accordance with our aim.

$$\{\ldots\} \;=\; \partial_i F^{1/2+a/2}\sqrt{\gamma}\gamma^{ij}\partial_j F^b \tag{3.22}$$

First, differentiating F^b, then differentiating a second time by treating the terms as F and the non-F terms.

$$\begin{aligned}\{\ldots\} &= \partial_i F^{1/2+a/2+b}\sqrt{\gamma}\gamma^{ij}\partial_j + b\partial_i F^{a/2+b-1/2}\partial_i\sqrt{\gamma}\gamma^{ij}\partial_j \\ &= \partial_i\left(F^{1/2+a/2+b}\right)\sqrt{\gamma}\gamma^{ij}\partial_j + F^{1/2+a/2+b}\partial_i\left(\sqrt{\gamma}\gamma^{ij}\partial_j\right) \\ &\quad +\partial_i\left(F^{-1/2+a/2+b}(\partial_j F)b\right)\sqrt{\gamma}\gamma^{ij} + F^{-1/2+a/2+b}(\partial_j F)b\partial_i\left(\sqrt{\gamma}\gamma^{ij}\right) \\ &\quad +F^{-1/2+a/2+b}(\partial_j F)b\sqrt{\gamma}\gamma^{ij}\partial_j \end{aligned} \tag{3.23}$$

We added the first and the last terms and set their coefficients to zero. Via this we found a relation between a and b.

$$\left(\frac{1}{2}+\frac{a}{2}+2b\right)F^{a/2+b-1/2}(\partial_i F)\sqrt{\gamma}\gamma^{ij}\partial_j \;=\; 0 \tag{3.24}$$

We do not want to have this term in the Hamiltonian. So we set its coefficient to be zero

$$\begin{aligned}\frac{1}{2}+\frac{a}{2}+2b &= 0 \\ a &= -4b-1\end{aligned} \tag{3.25}$$

The second term inside the parenthesis is our laplacian; therefore, after multiplication with the term outside, the values of the coefficient must be one. This way we determined values of a and b.

$$-3a/2-b+a/2+b+1/2=0 \tag{3.26}$$

$$a=\frac{1}{2},\quad b=-\frac{3}{8} \tag{3.27}$$

This gamma metric is an analogue optical metric due to the value of a. The original optical metric corresponds to

$$ds^2 = -dt^2 + F^{-1}h_{ij}dx^i dy^j \tag{3.28}$$

Optical metric is the space time where waves move following light geodesics. However, in our study this won't be the case. We chose to work with this metric solelydue to the presence of the laplacian term, which renders it applicable to our case. The remaining terms in the Hamiltonian are potential

$$\begin{aligned} U &= \partial_i \left(bF^{a/2+b-1/2}\partial_j\right)\sqrt{\gamma}\gamma^{ij} + bF^{a/2+b-1/2}\partial_j\partial_i\left(\sqrt{\gamma}\gamma^{ij}\right) \\ &= \partial_i \left(bF^{a/2+b-1/2}\partial_j\sqrt{\gamma}\gamma^{ij}\right) \\ &= \partial_i \left(\frac{-3}{8}F^{-5/8}\partial_j\sqrt{\gamma}\gamma^{ij}\right) \end{aligned} \tag{3.29}$$

Finally our total result in the analogue optical metric corresponds to

$$ds^2 = -dt^2 + F^{-1/2}h_{ij}dx^i dy^j \tag{3.30}$$

Remember that H is assumed to be a non-negative operator. And finally, the laplacian is

$$\begin{aligned} H &= -\frac{1}{2m}\frac{1}{\sqrt{\gamma}}\partial_i\sqrt{\gamma}\gamma^{ij}\partial_j + \frac{1}{2m}\frac{F^{-3/8}}{\sqrt{\gamma}}\partial_i\left(\frac{3}{8}F^{-5/8}\partial_j F\sqrt{\gamma}\gamma^{ij}\right) \\ &= -\frac{1}{2m}\Delta_\gamma + \frac{3}{16m}F^{-3/8}\partial_i\left(F^{-5/8}\partial_j\sqrt{\gamma}\gamma^{ij}\right) + mF^{1/2} \end{aligned} \tag{3.31}$$

Remember that H is assumed to be a non negative operator. And finally, the laplacian is

$$\Delta = \frac{1}{\sqrt{\gamma}}\partial_i\sqrt{\gamma}\gamma^{ij}\partial_j \tag{3.32}$$

Equation 3.31 might seen incorrect in terms of dimensions. However, one should recall that we took $c = \hbar = 1$ from the beginning. This is also evidents in the form of the

KGE.

Also, the Hamiltonian can be expressed in terms of our h metric

$$\begin{aligned}
\gamma_{ij} &= F^{-1/2} h_{ij} && (3.33) \\
\gamma &= F^{-3/4} h && (3.34) \\
\Delta(h) &= \frac{F^{3/4}}{\sqrt{h}} \partial_i \left(F^{-1/4} \sqrt{h} h^{ij} \partial_j \right) && (3.35) \\
U(h) &= \frac{3}{16m} F^{-3/8} \partial_i \left(F^{-7/8} \partial_j F \sqrt{h} h^{ij} \right) + m F^{1/2} && (3.36) \\
F^{-3/8} H_\gamma F^{3/8} &= \frac{F^{3/4}}{\sqrt{h}} \partial_i \left(F^{-1/4} \sqrt{h} h^{ij} \partial_j \right) \\
&\quad + \frac{3}{16m} F^{-3/8} \partial_i \left(F^{-7/8} \partial_j F \sqrt{h} h^{ij} \right) + m F^{1/2} && (3.37)
\end{aligned}$$

Before passing to the next chapter, we need to mention the metric dependence of our theoretical machine. The Heat kernel coefficients use the metric that is inherited from the laplacian. In physics and geometry the Heat kernel expansion is a customary tool, the main reason for this being its feature of putting eigenvalues aside and expressing everything in terms of geometrical invariants. For illustrative purposes, we will calculate the exact Hamiltonian of the Schwarzschild metric. It is the metric for a static BH solution, which at a distance from the horizon can be taken to be a planet.

4. HIGH TEMPERATURE EXPANSION OF THERMODYNAMIC QUANTATIES VIA MELLIN TRANSFORMATION AND HEAT KERNEL EXPANSION

At this point, the difference between the finite size and thermodynamic picture should be clarified. The thermodynamic limit ($N \to \infty$, $V \to \infty$, $n = N/V \to$finite) was the dominant subject of interest because it is justified in large macroscopic systems and easy to identify in experiments. On the other hand, simulations are usually set up for finite systems. Also the introduction of statistical theory helps to obtain information from a scale-free region around macroscopic phase transitions. Under these circumstances, finite-size effects are taken as a stand-alone theory. These small-sized effects tell the microscopic story based on the statistical nature of thermodynamics. Today these effects are still hard to identify in experiments as they vanish in the thermodynamic limit, yet in theoretical physics they are accessible.

4.1. Number of Excited Particles of the Bose Gas

As once said, for a curved space time the definition of the density of the states is not clear. This is the reason why we express thermodynamic variables in terms of the trace operator. It will help us to proceed in the broadest sense without knowing the eigenvalues of the system. The total number of the system is simply

$$N \;=\; \langle N_i \rangle \tag{4.1}$$

From now on I will not use $\langle\ \rangle$ to indicate the mean value. Where

$$\begin{aligned} N_i &= \frac{1}{e^{\beta(\epsilon_i-\mu)}-1} \\ &= \frac{e^{-\beta(\epsilon_i-\mu)}}{1-e^{-\beta(\epsilon_i-\mu)}} \\ &= e^{-\beta(\epsilon_i-\mu)}\left(1-e^{-\beta(\epsilon_i-\mu)}\right)^{-1} \end{aligned} \tag{4.2}$$

for illustrative purposes we choose

$$x = \beta(\epsilon_i - \mu) \tag{4.3}$$

Now it is easier to make binomial expansion which will create the trace operator out of this series.

$$\begin{aligned} N_i &= e^{-x}(1-e^{-x})^{-1} \\ &= e^{-x} + e^{-2x} + e^{-3x} \ldots \\ &= \sum_{k=1}^{\infty} e^{-kx} \end{aligned} \tag{4.4}$$

which corresponds to

$$N_i = \sum_{k=1}^{\infty} e^{-k\beta(\epsilon_i - \mu)} \tag{4.5}$$

$$\begin{aligned} N_e &= \sum_{i=1}^{\infty} n_i \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} e^{k\beta\mu} e^{-k\beta\epsilon_i} \\ &= \sum_{k=1}^{\infty}\left(e^{k\beta\mu} \sum_{i=1}^{\infty} e^{-k\beta\epsilon_i} \right) \end{aligned} \tag{4.6}$$

Here, we will presume that H is positive i.e. $-\Delta + U$ is always non-negative. Therefore, the final result can be written in terms of the operators as

$$N_e = \sum_{k=1}^{\infty} e^{k\beta\mu} Tr' e^{-Hk\beta} \tag{4.7}$$

We will make some small manipulations to simplify further steps [19].

$$
\begin{aligned}
H &= -\frac{1}{2m}\Delta + U + m - \epsilon_0 \\
H &= -\frac{1}{2m}[\Delta + 2mU + 2m^2 - 2m\epsilon_0] \\
H_1 &\equiv \Delta + 2mU + 2m^2 - 2m\epsilon_0
\end{aligned}
\tag{4.8}
$$

Now [4.7] becomes

$$
N_e = \sum_{k=1}^{\infty} e^{k\beta(\mu-\epsilon_0)} Tr' e^{-\frac{\beta}{2m}H_1 k} \tag{4.9}
$$

We write this directly by sending $\beta \to \beta/2m$ and $\mu \to 2m\mu$. The reason why we need this trick is that the Heat kernel expansion uses bare laplacian. We simply prevented any confusion that might occur due to the extra $2m$ coefficient. Also, the prime on top of the trace indicates that the ground state is excluded. Obviously, the reason is that we are trying to express the number of the particles above the ground state. The series above is of the form $\sum_{k=1}^{\infty} f(k\beta)$. Such sums are called harmonic sums and their $\beta \to 0$ $(T \to \infty)$ asymptotics is given by the Mellin Transform [20].

Now, we will introduce the Mellin Transformation [21] where u is a locally integrable variable of the function $f(u)$ on $(0, \infty)$ defined by the convergence of the integral.

$$
M_{[f:s]} = \tilde{F}(s) = \int_0^{\infty} u^{s-1} f(u)\, du \tag{4.10}
$$

where s is a complex variable.It should always be remembered that in mathematic literature the Mellin transformation is accepted as the natural way to handle an asymptotic expansion of harmonic sums. This is achieved in two steps. First, the poles and the residues of the meromorphic extension of the transform are analyzed. Then, by taking the inverse Mellin transform the desired result is found. The poles and residues of this integral correspond to the asymptotics of the original $f(\beta)$. The integral and the sum can be interchanged so that the Riemann zeta function can be taken out of the general harmonic sum representation.Therefore,

$$\begin{aligned}\tilde{F}(s) &= \int_0^\infty \beta^{s-1} \sum_{k=1}^{\infty} f(k\beta)\, d\beta \\ &= \sum_{k=1}^{\infty} \int_0^\infty \beta^{s-1} f(k\beta)\, d\beta \end{aligned} \tag{4.11}$$

With the change of variable $\frac{\beta}{2m} = u$ and $\frac{d\beta}{2m} = du$. Now we should emphasize the difference between $f(kx)$ and $f(x)$. Although it seems trivial, it may create some confusion if it is missed.

$$f(ku) = Tre^{-\frac{k\beta}{2m}} \tag{4.12}$$

$$f(u) = Tre^{-\frac{\beta}{2m}} \tag{4.13}$$

According to [4.10] we will continue with $f(u)$

$$\begin{aligned}\tilde{F}(s) &= \sum_{k=1}^{\infty} \int_0^\infty \frac{m^{s+1}u^{s-1}}{k^s} f(u)\, du \\ &= \sum_{k=1}^{\infty} k^{-s} \int_0^\infty u^{s-1} f(u) m^{s+1}\, du \\ &= \sum_{k=1}^{\infty} k^{-s} \tilde{f}(s) \end{aligned} \tag{4.14}$$

$\tilde{F}(s)$ is separated as a Riemann Zeta function and a Mellin transformation. As well known, Riemann Zeta function can be illustrated by the integral

$$\zeta(s) = \sum_{k=1}^{\infty} k^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty \frac{u^{s-1}}{e^s - 1}\, du \tag{4.15}$$

Apparently it is a meromorphic function with a single pole at $s = 1$ $\zeta(s)$. Therefore our expression will have a double pole at $s = 1$. Also, remember that $\Gamma(s)$ is the gamma function.

$$\tilde{f}(s) = \int_0^\infty u^{s-1} f(u)\, du \tag{4.16}$$

where

$$f(u) = e^{\mu u}\theta^*(u) \tag{4.17}$$

where the $\theta^*(u)$ function is

$$\theta^*(u) = \left(Tr' e^{-Hu}\right) \tag{4.18}$$

θ^* is the part which we apply the Heat kernel expansion on. This equation is a solution for the differential equations as in the form

$$\frac{\partial K_t}{\partial t} = -HK_t \tag{4.19}$$

where α is a positive constant. Its kernel has the form

$$K_t = e^{-tH} \tag{4.20}$$

After bringing the Mellin transform into the play, we will apply the small βm asymptotic expansion to the number of excited particles and the free energy of the system. In the definition of θ^* function the prime indicates that we omitted the ground state from the trace. Although μ is a well-known negative physical variable for illustrative purposes, it is written as $-|\mu|$.

$$\tilde{F}(u) = \zeta(s) \int_0^\infty u^{s-1} e^{-|\mu| u}\theta^*(u)\, du \tag{4.21}$$

Here, we see the Laplace Mellin transformation.

$$(\mathcal{LM}g)(s,\alpha) \quad = \quad \int_0^\infty u^{s-1} e^{\alpha u} g_\alpha(u)\; du \tag{4.22}$$

now the final result is

$$\tilde{F}(s) \quad = \quad \zeta(s)(\mathcal{LM}Tr' e^{\Delta u})(s, |\mu|) \tag{4.23}$$

We want to find the poles of the equality. For $Tr'e^{\Delta u}$, there is a limit which we will call K for the moment. On the other hand, to find the values where $\tilde{F}(s)$ explodes, we should examine $\mathcal{LM}$ more closely. The integral blows up at the boundaries of the integral. For the infinity the integral is secure explicitly via the domination of the $e^{-\infty}$ term. The lower bound is harder to see; nevertheless, we will use the Heat Kernel expansion which will make it easier to obtain the result. This expansion is a widely used method both in mathematics and theoretical physics (quantum anomalies, string theory etc.). It provides analytical means to the spectrum of operators. The Heat kernel operator [4.18] creates a solution to the problem of not having a simple momentum space-time representation in general curved space-time.

$$\int \frac{d^d p}{(2\pi)^d} e^{-uH} \to \theta(u) \tag{4.24}$$

It turns these expressions to flat space as burying or confining the geometry to the coefficients of the expansion terms rather than the expanding the theory in that environment [22]. In the $u \to 0$ limit, it is defined as

$$Tr' e^{\Delta u} \quad = \quad \frac{1}{(4\pi u)^{3/2}} \left[a_0 + a_{1/2} u^{1/2} + a_1 u + \ldots \right] \tag{4.25}$$

Here a_i's are the previously mentioned coefficients. These terms include the volume, the area, and various curvature scalers. In this work, we will be interested only in the first and the second coefficients a_0 and $a_{1/2}$, which are volume- and area-dependent, respectively. This is how they introduce the metric to the theory. One of the methods to find the Heat kernel coefficients is the De Witt iteration, yet we will follow Gilkey's approach due to practical issues.

$$Tre^{-Ht} = \frac{1}{(4\pi t)^{3/2}}\left[a_0 + a_{1/2}t^{1/2} + a_1 t + \ldots\right] \tag{4.26}$$

Putting the heat kernel expansion aside, we should emphasize the difference between the normal and the primed trace.

$$\begin{aligned} Tr'e^{-Ht} &= Tre^{-Ht} - e^{-\epsilon_0 t} \\ &= \frac{a_0}{t^{3/2}} + \frac{a_{1/2}}{t^{1/2}} + \frac{a_1}{t} + \ldots + \left(1 - \epsilon_0 t - \frac{1}{2}\epsilon_0 t^2 + \ldots\right) \end{aligned} \tag{4.27}$$

Apparently its impact is unimportant so we basically ignore this effect on the expansion. (The coefficients a_0 and $a_{1/2}$ behave like V and A. Consequently, they are the terms that contribute the most.)

$$\mathcal{LM} = \int_0^\infty u^{s-1}e^{-2m|\mu|u}Tr'e^{\Delta u}\, du \tag{4.28}$$

with the Heat kernel expansion, we are able to find the required information with some geometric properties [23]. The derivation of simpler cases points out the key features of the general situation. When the theory becomes problematic (having boundaries, singularities etc.) this specialty makes the Heat kernel handy for researchers. However, it may become inefficient if the material in the theoretical structure is a combination of bosons and fermions (which is invalid in our case) [22]. For the first term

$$\mathcal{LM}f = \int_0^\infty u^{s-1}e^{-2m|\mu|u}\frac{a_0}{(4\pi u)^{3/2}}\, du \tag{4.29}$$

variable u is changed

$$\begin{aligned} -2m|\mu|u &= y \qquad (4.30)\\ \mathcal{L}\mathcal{M}f &= \int_0^\infty \frac{a_0}{(4\pi)^{3/2}} \frac{y}{-2m|\mu|}^{(s-3/2)-1} e^y \frac{-1}{2m|\mu|}\ dy \\ &= \frac{a_0}{(4\pi)^{3/2}(-2m|\mu|)^{s-3/2}} \int_0^\infty y^{(s-3/2)-1} e^y\ dy \\ &= \frac{a_0}{(4\pi)^{3/2}(-2m|\mu|)^{s-3/2}} \Gamma(s-\frac{3}{2}) \qquad (4.31) \end{aligned}$$

$$s-\frac{3}{2}>0 \qquad (4.32)$$

The presence of $|\mu|$ ($\lim_{T\to 0}|\mu| = O(\frac{1}{V})$) in the denominator shouldn't cause any trouble when we take the thermodynamical limit ($V\to\infty$) because these will at least go up to the denominator or simply vanish for each of the terms of the residues. Simple poles of the meromorphic function originate from these gamma functions. If this is the case $s>\frac{3}{2}$ remains in the analytic continuation zone [24].

$$\begin{aligned} \mathcal{L}\mathcal{M}f &= \int_0^\infty u^{s-1} e^{-2m|\mu|u} \left[\left(Tr' e^{\Delta u} - \frac{a_0}{(4\pi u)^{3/2}} \right) + \frac{a_0}{(4\pi u)^{3/2}} \right] du \\ &= \frac{1}{(4\pi u)^{3/2}} \int_0^\infty u^{s-1} e^{-2m|\mu|u} \mathcal{O}\left(u^{-1}\right)\ du + \frac{a_0}{(4\pi)^{3/2}(-2m|\mu|)^{s-3/2}} \qquad (4.33) \end{aligned}$$

The first term converges when $s>3/2$ so this temporary restriction should be dragged to the left to the $s>1$ limit.

$$\begin{aligned} \int_0^\infty u^{(s-3/2)-1} e^{-2m|\mu|u}\ du &= \frac{a_0}{(4\pi)^{3/2}(-2m|\mu|)^{s-3/2}} \Gamma(s-\frac{3}{2}) \\ &\quad + finite\ \ terms\ \ for\ \ s>1 \qquad (4.34) \end{aligned}$$

This time the analytic continuation zone starts from $s>1$. Then we used an asymptotic expansion for each of the gamma functions of the transformation.

$$\Gamma(s-n) \asymp \sum_{m=0}^{\infty} \frac{-1^m}{m!} \frac{1}{s-n+m} \tag{4.35}$$

$$\Gamma(s-\frac{3}{2}) \asymp \frac{1}{s-3/2} - \frac{1}{s-1/2} + \frac{1}{2(s+1/2)} + \dots \tag{4.36}$$

From now on we will input this expansion for the each gamma function of the transformation. The first three terms of the expansion into $\mathcal{LM}$

$$\begin{aligned}\mathcal{LM}f &= \frac{1}{(4\pi u)^{3/2}} \int_0^\infty u^{s-1} e^{-2m|\mu|u} \mathcal{O}\left(u^{-1}\right)\, du \\ &+ \frac{a_0}{(4\pi)^{3/2}(-2m|\mu|)^{s-3/2}} \left[\frac{1}{s-3/2} - \frac{1}{s-1/2} + \frac{1}{2(s+1/2)}\right]\end{aligned} \tag{4.37}$$

These poles of the second term are our the targetted poles which produce the desired results from N_e and A

$$\begin{aligned}\mathcal{LM}f &= \int_0^\infty u^{s-1} e^{-2m|\mu|u} \left[\left(Tr' e^{\Delta u} - \frac{a_0 + a_{1/2}u^{1/2}}{(4\pi u)^{3/2}}\right) + \frac{a_0 + a_{1/2}u^{1/2}}{(4\pi u)^{3/2}}\right] du \\ &= \int_0^\infty u^{s-1} e^{-2m|\mu|u} \mathcal{O}\left(u^{-1/2}\right)\, du + \int_0^\infty u^{s-1} e^{-2m|\mu|u} \frac{a_0 + a_{1/2}u^{1/2}}{(4\pi u)^{3/2}}\, du \\ &= \int_0^\infty u^{s-1} e^{-2m|\mu|u} \mathcal{O}\left(u^{-1/2}\right)\, du \\ &+ \frac{a_0}{(4\pi)^{3/2}(-2m|\mu|)^{s-3/2}} \int_0^\infty u^{(s-3/2)-1} e^{-2m|\mu|u}\, du \\ &+ \frac{a_{1/2}}{(4\pi)^{3/2}(-2m|\mu|)^{s-1}} \int_0^\infty u^{(s-1)-1} e^{-2m|\mu|u}\, du \\ &= \frac{a_0}{(4\pi)^{3/2}(-2m|\mu|)^{s-3/2}} \Gamma(s-\frac{3}{2}) + \frac{a_{1/2}}{(4\pi)^{3/2}(-2m|\mu|)^{s-1}} \Gamma(s-1) \\ &+ finite\ \ terms\ \ for\ \ s>\frac{1}{2}\end{aligned} \tag{4.38}$$

One should notice that each $\mathcal{LM}$ term contains the poles of the prior term, with a bigger convergence interval. Again with the gamma expansion the result is

$$\begin{aligned}\mathcal{LM}f &= \frac{a_0}{(4\pi)^{3/2}(-2m|\mu|)^{s-3/2}}\left[\frac{1}{s-3/2}-\frac{1}{s-1/2}+\frac{1}{2(s+1/2)}\right]\\ &+\frac{a_{1/2}}{(4\pi)^{3/2}(-2m|\mu|)^{s-1}}\left[\frac{1}{s-1}-\frac{1}{s}+\frac{1}{2(s+1)}\right]\\ &+finite\ \ terms\ \ for\ \ s>\frac{1}{2}\end{aligned} \tag{4.39}$$

where

$$s>\frac{1}{2} \tag{4.40}$$

The integral in this line is not meromorphic

$$\begin{aligned}\mathcal{LM}f &= \int_0^\infty u^{s-1}e^{-2m|\mu|u}\left[\left(Tr'e^{\Delta u}-\frac{a_0+a_{1/2}u^{1/2}+a_1u}{(4\pi u)^{3/2}}\right)\right.\\ &\left.+\frac{a_0+a_{1/2}u^{1/2}+a_1u}{(4\pi u)^{3/2}}\right]\ du\\ &= \int_0^\infty u^{s-1}e^{-2m|\mu|u}\mathcal{O}\left(u^0\right)\ du+\int_0^\infty\frac{a_0+a_{1/2}u^{1/2}+a_1u}{(4\pi u)^{3/2}}\ du\end{aligned} \tag{4.41}$$

Now in the final step we will use some expansions, a theoretical tool to read the Mellin transform, to get the perturbed result for the thermodynamic variables.

$$\begin{aligned}(\mathcal{LM}N_e) &= \frac{a_0}{(4\pi)^{3/2}(-2m|\mu|)^{s-3/2}}\left[\frac{1}{s-3/2}-\frac{1}{s-1/2}+\frac{1}{2(s+1/2)}\right]\\ &+\frac{a_{1/2}}{(4\pi)^{3/2}(-2m|\mu|)^{s-1}}\left[\frac{1}{s-1}-\frac{1}{s}+\frac{1}{2(s+1)}\right]\\ &+\frac{a_1}{(4\pi)^{3/2}(-2m|\mu|)^{s-1/2}}\left[\frac{1}{s-1/2}-\frac{1}{s+1/2}+\frac{1}{2(s+3/2)}\right]\\ &+finite\ \ terms\ \ for\ \ s>0\end{aligned} \tag{4.42}$$

Now we return to the initial asymptotics that we are looking for [25]

$$(\mathcal{M}f)(s) \asymp \sum \frac{N_e(w,k)}{(s-w)^{k+1}} \tag{4.43}$$

$$f(s) = \sum_{wk} N_e(w,k)\frac{(-1)^k}{k!}s^{-w}(\log s)^k \tag{4.44}$$

It follows that N_e is given by an asymptotic series of the following form

$$N_e = \zeta(3/2){N_e}^{(3/2)}T^{3/2} + \zeta(1){N_e}^{(1)}T^1 + \dots \tag{4.45}$$

We cut the series at the order we choose. When we use the expansions

$${N_e}^{(3/2)} = \frac{1}{(4\pi)^{3/2}}a_0 \tag{4.46}$$

$${N_e}^{(1)} = \frac{1}{(4\pi)^{3/2}}a_{1/2} \tag{4.47}$$

write these in the asymptotic expansion

$$N_e = \zeta(3/2)\frac{(2m)^{3/2}}{(4\pi)^{3/2}}a_0T^{3/2} + \zeta(1)\frac{2m}{(4\pi)^{3/2}}a_{1/2}T^1 + \dots \tag{4.48}$$

In the next subsection we will repeat the same methodology for the free energy.

4.2. Free Energy of the Bose Gas

In statistical physics, free energy is the key quantity that helps us to find other thermodynamic variables such as entropy, pressure etc.

$$\mathcal{F} = \frac{2m}{\beta}\sum_{i}^{\infty} \ln[1 - e^{-\frac{\beta}{2m}(\epsilon_i - 2m\mu)}] \tag{4.49}$$

$$-\ln(1-x) = \sum_{k=1}^{\infty}\frac{x^k}{k} \tag{4.50}$$

$$\begin{aligned}
\mathcal{F} &= -\frac{2m}{\beta}\sum_{i}^{\infty}\sum_{k=1}^{\infty}\frac{e^{-k\frac{\beta}{2m}(\epsilon_i-2m\mu)}}{k} \\
&= -\sum_{k=1}^{\infty}\frac{2me^{k\beta\mu}}{k\beta}\sum_{i}^{\infty}e^{-k\frac{\beta}{2m}\epsilon}
\end{aligned} \tag{4.51}$$

Therefore, we get an expression with the trace operator

$$\mathcal{F} = -\sum_{k=1}^{\infty}\frac{2me^{k\beta\mu}Tre^{k\frac{\beta}{2m}\epsilon}}{k\beta} \tag{4.52}$$

with the help of the thermodynamical formula, we can find entropy, S, of the system

$$S = -\left.\frac{\partial\mathcal{F}}{\partial T}\right|_V \tag{4.53}$$

Notice that we didn't put a prime sign above the trace, because we integrate over each particle.

$$F(s) = -\zeta(k)\int_0^{\infty} u^{(s-1)-1}e^{-2m|\mu|u}Tre^{\Delta u}\,du \tag{4.54}$$

Again with the heat kernel expansion

$$\begin{aligned}
\mathcal{LMF} &= -\int_0^{\infty} u^{(s-1)-1}e^{-2m|\mu|u}\frac{a_0}{(4\pi u)^{3/2}}\,du \\
&= -\frac{a_0}{(4\pi)^{3/2}(-2m|\mu|)^{s-5/2}}\Gamma(s-\frac{5}{2})
\end{aligned} \tag{4.55}$$

$$s-\frac{5}{2}>0 \tag{4.56}$$

So $s>\frac{5}{2}$ remains in the analytic continuation zone.

$$\begin{aligned}
\mathcal{LMF} &= -\int_0^\infty u^{(s-1)-1}e^{-2m|\mu|u}\left[\left(Tr'e^{\Delta u}-\frac{a_0}{(4\pi u)^{3/2}}\right)+\frac{a_0}{(4\pi u)^{3/2}}\right]\,du \\
&= -\frac{1}{(4\pi u)^{3/2}}\int_0^\infty u^{(s-1)-1}e^{-2m|\mu|u}\mathcal{O}\left(u^{-1}\right)\,du \\
&\quad +\frac{a_0}{(4\pi)^{3/2}(-2m|\mu|)^{s-5/2}}\int_0^\infty u^{(s-5/2)-1}e^{-2m|\mu|u}\,du \\
&= -\frac{a_0}{(4\pi)^{3/2}(-2m|\mu|)^{s-5/2}}\Gamma(s-\frac{5}{2})+finite\ \ terms\ \ for\ \ s>2
\end{aligned} \tag{4.57}$$

This time the analytic continuation zone starts from $s>2$. Then we used an asymptotic expansion for the gamma function

$$\Gamma(s-\frac{5}{2}) \asymp \frac{1}{s-5/2}-\frac{1}{s-3/2}+\frac{1}{2(s-1/2)}+\ldots \tag{4.58}$$

Similarly as before we deal only with the first three terms, and put those only

$$\begin{aligned}
\mathcal{LMF} &= -\int_0^\infty u^{(s-2)-1}e^{-2m|\mu|u}\,du \\
&\quad +\frac{a_0}{(4\pi)^{3/2}(-2m|\mu|)^{s-5/2}}\left[\frac{1}{s-5/2}-\frac{1}{s-3/2}+\frac{1}{2(s-1/2)}\right]
\end{aligned} \tag{4.59}$$

$$\mathcal{LMF} = -\int_0^\infty u^{(s-1)-1}e^{-2m|\mu|u}\left[\left(Tr'e^{\Delta u}-\frac{a_0+a_{1/2}u^{1/2}}{(4\pi u)^{3/2}}\right)+\frac{a_0+a_{1/2}u^{1/2}}{(4\pi u)^{3/2}}\right]\,du \tag{4.60}$$

$$
\begin{aligned}
\mathcal{LMF} &= -\int_0^\infty u^{(s-1)-1} e^{-2m|\mu|u} \mathcal{O}\left(u^{-1/2}\right)\ du + \int_0^\infty u^{(s-1)-1} e^{-2m|\mu|u} \frac{a_0 + a_{1/2} u^{1/2}}{(4\pi u)^{3/2}}\ du \\
&= -\int_0^\infty u^{(s-1)-1} e^{-2m|\mu|u} \mathcal{O}\left(u^{-1/2}\right)\ du \\
&\quad + \frac{a_0}{(4\pi)^{3/2}(-2m|\mu|)^{s-5/2}} \int_0^\infty u^{(s-5/2)-1} e^{-2m|\mu|u}\ du \\
&\quad + \frac{a_{1/2}}{(4\pi)^{3/2}(-2m|\mu|)^{s-2}} \int_0^\infty u^{(s-2)-1} e^{-2m|\mu|u}\ du \\
&= -\frac{a_0}{(4\pi)^{3/2}(-2m|\mu|)^{s-5/2}} \left[\frac{1}{s-5/2} - \frac{1}{s-3/2} + \frac{1}{2(s-1/2)}\right] \\
&\quad + \frac{a_{1/2}}{(4\pi)^{3/2}(-2m|\mu|)^{s-2}} \left[\frac{1}{s-2} - \frac{1}{s-2} + \frac{1}{2s}\right] \\
&\quad + finite\ \ terms\ \ for\ \ s > \frac{3}{2} \qquad (4.61)
\end{aligned}
$$

by applying the same methods we obtain

$$
\begin{aligned}
\mathcal{LMF} &= \frac{a_0}{(4\pi)^{5/2}(-2m|\mu|)^{s-5/2}} \left[\frac{1}{s-5/2} - \frac{1}{s-3/2} + \frac{1}{2(s-1/2)}\right] \\
&\quad + \frac{a_{1/2}}{(4\pi)^{3/2}(-2m|\mu|)^{s-2}} \left[\frac{1}{s-2} - \frac{1}{s-2} + \frac{1}{2s}\right] \\
&\quad + \frac{a_1}{(4\pi)^{3/2}(-2m|\mu|)^{s-3/2}} \left[\frac{1}{s-3/2} - \frac{1}{s-1/2} + \frac{1}{2(s+1/2)}\right] \\
&\quad + finite\ \ terms\ \ for\ \ s > 1
\end{aligned}
$$

It follows immediately that the free energy can be expressed as an asymptotic series

$$
\begin{aligned}
\mathcal{F} &= \zeta(5/2)\mathcal{F}^{(5/2)}T^{5/2} + \zeta(2)\mathcal{F}^{(2)}T^2 + \zeta(3/2)\mathcal{F}^{(3/2)}T^{3/2} + \zeta(1)\mathcal{F}^{(1)}T^1 \\
&\quad + \zeta(1/2)\mathcal{F}^{(1/2)}T^{1/2} + \dots \qquad (4.62)
\end{aligned}
$$

$$
\mathcal{F}^{(5/2)} = \frac{1}{(4\pi)^{3/2}} a_0 \qquad (4.63)
$$

$$
\mathcal{F}^{(2)} = \frac{1}{(4\pi)^{3/2}} a_{1/2} \qquad (4.64)
$$

When we use the same expansions [4.43] and [4.44] we get

$$\mathcal{F} = \zeta(5/2)\frac{(2m)^{5/2}}{(4\pi)^{3/2}}a_0 T^{5/2} + \zeta(2)\frac{(2m)^2}{(4\pi)^{3/2}}a_{1/2}T^2 + \ldots \tag{4.65}$$

With the help of the trace operator, we are left to calculate the coefficients and to use the dictionary for finding the N_e and $\mathcal{F}$ [26]. Where the coefficients are

$$a_0 = \frac{V}{(4\pi)^{3/2}} \tag{4.66}$$

$$a_{1/2} = -\frac{A}{16\pi} \tag{4.67}$$

5. APPLICATIONS OF FLAT AND SCHWARZSCHILD METRICS

5.1. Similarity Transformation for the Schwarzschild Metric

Our box full of Bose gas is positioned near a planet; therefore, in this section we will restrict our metric to the Schwarzschild. We should emphasize that our radius in these applications is taken to be bigger than the $3M$ boundary (where M is the mass of the gravitational source and $3M$ is the nearest stable orbit). Hence, all orbits are stable and our box can orbit along a stable path without the danger of falling into the object mass. Also, the event horizon is no danger due to the fact that it stays hidden inside the surface of the planet. The line element of the Schwarzschild metric has the following form,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{5.1}$$

$$= -Fdt^2 + F^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{5.2}$$

$$F = \left(1 - \frac{2M}{r}\right) \tag{5.3}$$

One should remember that the capital mass term belongs to the metric, so it is there to represent the mass of the planet. However, the mass term in the Hamiltonian belongs to the Bose gas inside the box. For the non-relativistic analogue of the Schwarzschild metric, we obtain the metric below

$$\gamma_{ij} = \begin{pmatrix} F^{-3/2} & 0 & 0 \\ 0 & F^{-1/2}r^2 & 0 \\ 0 & 0 & F^{-1/2}r^2\sin^2\theta \end{pmatrix} \tag{5.4}$$

$$\gamma_{ij} = F^{-1/2}h_{ij} = F^{-3/2}dr^2 + F^{-1/2}r^2 d\theta^2 + F^{-1/2}r^2\sin^2\theta d\phi^2 \tag{5.5}$$

So the laplacian ifor the analogue metric is

$$\begin{aligned}\Delta &= \frac{1}{\sqrt{\gamma}}\partial_i\sqrt{\gamma}\gamma^{ij}\partial_j \\ &= \frac{1}{\sqrt{\left(1-\frac{2M}{r}\right)^{-5/2}r^4\sin^2\theta}}\partial_i\sqrt{\left(1-\frac{2M}{r}\right)^{-5/2}r^4\sin^2\theta\gamma^{ij}\partial_j}\end{aligned} \tag{5.6}$$

as the potential term is

$$\begin{aligned}U &= \frac{3}{16m}F^{-3/8}\partial_i\left(F^{-5/8}\partial_j\sqrt{\gamma}\gamma^{ij}\right)+mF^{1/2} \\ &= \frac{3}{16m}\left(1-\frac{2M}{r}\right)^{-3/8}\partial_i\left(\left(1-\frac{2m}{r}\right)^{-5/8}\partial_j\sqrt{\left(1-\frac{2m}{r}\right)^{-5/2}r^4\sin^2\theta\gamma^{ij}}\right) \\ &\quad +m\left(1-\frac{2M}{r}\right)^{1/2}\end{aligned} \tag{5.7}$$

The end result for the Hamiltonian is

$$\begin{aligned}H &= -\frac{1}{2m}\frac{1}{\sqrt{\left(1-\frac{2m}{r}\right)^{-5/2}r^4\sin^2\theta}}\partial_i\sqrt{\left(1-\frac{2M}{r}\right)^{-5/2}r^4\sin^2\theta\gamma^{ij}\partial_j} \\ &\quad +\frac{3}{16m}\left(1-\frac{2M}{r}\right)^{-3/8}\partial_i\left(\left(1-\frac{2M}{r}\right)^{-5/8}\partial_j\sqrt{\left(1-\frac{2M}{r}\right)^{-5/2}r^4\sin^2\theta\gamma^{ij}}\right) \\ &\quad +m\left(1-\frac{2M}{r}\right)^{1/2} \\ &= -\frac{1}{2m}\left[F^{1/2}r^{-2}\left(\frac{M}{2}\partial_r+\cot\theta\partial_\theta+\partial_\theta\partial_\theta\right)+F^{3/2}\left(\frac{2}{r}\partial_r+\partial_r\partial_r\right)\right] \\ &\quad +\frac{3}{16m}\left\{F^{-11/4}\left[-\frac{11}{8}\frac{M^2\sin\theta}{r^2}\right]\right. \\ &\quad +F^{-7/4}\left[-M\sin\theta\left(\frac{1}{4}\partial_r+\frac{3}{2r}\right)+\sin\theta\left(-1+\partial_\theta\partial_\theta+2\cos\theta\partial\theta\right)\right] \\ &\quad \left.+F^{-3/4}\sin\theta\left[2+4r\partial_r+r^2\partial_r\partial_r\right]\right\}\end{aligned} \tag{5.8}$$

5.2. Volume and Area Calculation

N_e and $\mathcal{F}$ should posses hints of the geometry of space-time. These hints are inherited from the heat kernel coefficients that are obtained from the area and the volume of the arbitrary box. We will accomplish this by using $N_{(\xi_1)}$, the normal to the ξ_1 constant surface, and γ, determinant of the optical metric.

$$N_{(i)\mu}dx^{\mu} = (\partial_{\mu}f)dx^{\mu} \quad (5.9)$$

$$N_{(i)\mu} = \partial_{\mu}f \quad (5.10)$$

therefore

$$N_{(\xi_1)i} = \delta_i^{\xi_1} \quad (5.11)$$

$$N_{(\xi_2)i} = \delta_i^{\xi_2} \quad (5.12)$$

$$N_{(\xi_3)i} = \delta_i^{\xi_3} \quad (5.13)$$

where i runs such that (r, θ, ϕ) or (x, y, z) depending on the case. And the normal is

$$\hat{N}_{(i)}^{j}\partial_j = \frac{\gamma^{jk}N_{(i)k}}{\sqrt{N_{(i)}^{k}N_{(i)k}}}\partial_j \quad (5.14)$$

$$= \sqrt{\gamma^{ii}}\partial_i \quad (5.15)$$

One should be careful that we find everything in the gamma metric. As we work through this metric, our space is unaware of any previous kind metric of itself. Also there is no summation over i.

$$dV = \sqrt{\gamma}d\xi^1 \wedge d\xi^2 \wedge d\xi^3 \quad (5.16)$$

$$dA_i = \mathcal{I}_{N_{(i)}}dV \quad (5.17)$$

$$dA_i = \sqrt{\gamma}\hat{N}_i d\xi^j \wedge d\xi^k \quad (5.18)$$

Here A_i represents the surface of the box where the i component is constant of the surface. From this the total area is

$$A = \int dA_r + \int dA_\theta + \int dA_\phi \tag{5.19}$$

For a general metric gamma determinant is

$$\gamma = det\gamma_{ij} \tag{5.20}$$

Whether they are known or unknown, the Heat kernel expansion associates the eigenvalues of the original function to its asymptotics. The small t expansion of the kernel introduces the heat kernel coefficients which contain geometrical features. Although calculating them is hard in high orders [27], at low orders the terms are easy to handle. Luckily, in our estimation due to the perturbation term we will need only the first two coefficients. In the following chapter we will first focus on the spherical flat metric due to pedantic concerns, and then to the Schwarzschild metric.

5.2.1. Area and Volume Calculation for the Flat Metric

We choose to work on the spherical flat metric

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \tag{5.21}$$

From here we find

$$\begin{aligned} V &= \int r^2 \sin\theta \; dr d\theta d\phi \\ &= \frac{1}{3}(r_3^2 - r_1^3)(\cos\theta_1 - \cos\theta_2)\Delta\phi \end{aligned} \tag{5.22}$$

$$\begin{aligned} A_{r_1} &= \int {r_1}^2 \sin\theta \; d\theta d\phi \\ &= {r_1}^2(\cos\theta_1 - \cos\theta_2)\Delta\phi \end{aligned} \tag{5.23}$$

$$
\begin{aligned}
A_{r_1}+A_{r_2} &= ({r_1}^2+{r_2}^2)(\cos\theta_1-\cos\theta_2)\Delta\phi && (5.24)\\
A_{\theta_1} &= \int r\sin\theta \; drd\phi \\
&= \frac{1}{2}({r_2}^2-{r_1}^2)\sin\theta_1\Delta\phi && (5.25)\\
A_{\theta_1}+A_{\theta_2} &= \frac{1}{2}({r_2}^2-{r_1}^2)(\sin\theta_1+\sin\theta_2)\Delta\phi && (5.26)\\
A_{\phi_1} &= \int r \; drd\theta \\
&= \frac{1}{2}({r_2}^2-{r_1}^2)\Delta\theta && (5.27)\\
A_{\phi_1}+A_{\phi_2} &= ({r_2}^2-{r_1}^2)\Delta\theta && (5.28)\\
A &= A_{r_1+r_2}+A_{\theta_1+\theta_2}+A_{\phi_1+\phi_2} && (5.29)\\
&= ({r_1}^2+{r_2}^2)(\cos\theta_1-\cos\theta_2)\Delta\phi \\
&\quad +({r_2}^2-{r_1}^2)\left[\frac{1}{2}(\sin\theta_1+\sin\theta_2)\Delta\phi+\Delta\theta\right] && (5.30)\\
V_{r_2\gg r_1} &= \frac{1}{3}r_2^3(\cos\theta_1-\cos\theta_2)\Delta\phi && (5.31)\\
A_{r_2\gg r_1} &= \frac{1}{2}{r_2}^2[2(\cos\theta_1-\cos\theta_2)\Delta\phi+(\sin\theta_1+\sin\theta_2)\Delta\phi+2\Delta\theta] && (5.32)\\
& && (5.33)
\end{aligned}
$$

When we divide the area to the volume we obtain a small parameter to use as a perturbation parameter.

$$
\left(\frac{A}{V}\right)_{r_2\gg r_1} = \frac{3[2(\cos\theta_1-\cos\theta_2)\Delta\phi+(\sin\theta_1+\sin\theta_2)\Delta\phi+2\Delta\theta]}{2(\cos\theta_1-\cos\theta_2)\Delta\phi}\frac{1}{r_2} \quad (5.34)
$$

5.2.2. Area and Volume Calculation for the Schwarzschild Metric

To determine the boundary effects on the thermodynamics of the Bose gas in the case of BEC, one should find the volume and the area of the spherical cube in the

Schwarzschild metric. Therefore

$$\sqrt{\gamma} = F^{-5/4} r^2 \sin\theta \tag{5.35}$$

$$\hat{N}_r = \sqrt{\gamma^{rr}} = F^{3/4} \tag{5.36}$$

$$\hat{N}_\theta = \sqrt{\gamma^{\theta\theta}} = F^{1/4} r^{-1} \tag{5.37}$$

$$\hat{N}_\phi = \sqrt{\gamma^{\phi\phi}} = F^{1/4} r^{-1} \sin^{-1}\theta \tag{5.38}$$

In the case of BEC we will use a spherical sector defined by

$$r_1 < r < r_2 \tag{5.39}$$

$$\theta_1 < \theta < \theta_2 \tag{5.40}$$

$$\phi_1 < \phi < \phi_2 \tag{5.41}$$

$$\tag{5.42}$$

We will assume the size of the box to be large but finite. We will explain this mathematically by taking $r_2 \gg r_1$. In the case of entropy, we will redefine our container by simply taking a spherical shell,

$$2M + \epsilon < r < r_2 \tag{5.43}$$

since our only concern is the divergences near the event horizon $2M$.

Keeping these in mind, we will compute boundary effects of two different Bose gas containers with different shapes, a spherical box and a spherical shell. Also, we should discuss the integration of these different bodies. The $\sqrt{\gamma}$ term in the integrand leaves us with a very tough integral to solve. Thereupon we expand the problematic term with an approach appropriate to the geometry of the container.

5.2.2.1. Area and Volume of the Unit Cube for the Schwarzschild Metric .

$$V = \int \sqrt{\gamma}\ drd\theta d\phi \tag{5.44}$$
$$= \int F^{-5/4} r^2 \sin\theta\ drd\theta d\phi \tag{5.45}$$
$$= \int \left(1 - \frac{2M}{r}\right)^{-5/4} r^2 \sin\theta\ drd\theta d\phi \tag{5.46}$$

To handle this complicated integral we will use the binomial expansion for the upper limit. By commencing our calculation out of the event horizon, $r > 2M$, we guarantee that $2M/r$ is smaller than 1; consequently, this expansion converges to a finite value.

$$\left(1 - \frac{2M}{r}\right)^{-5/4} = 1 + \frac{5M}{2r} + \frac{35M^2}{8r^2} + \ldots \tag{5.47}$$

inputting the first two terms to the volume integral, we find

$$V = \int [1 + \frac{5M}{2r} + \frac{35M^2}{4r^2} + \ldots] r^2 \sin\theta\ drd\theta d\phi$$
$$= \left[\frac{1}{3}(r_2^3 - r_1^3) + \frac{5M}{4}({r_2}^2 - {r_1}^2) + \frac{35M^2}{4}(r_2 - r_1)\right] (\cos\theta_1 - \cos\theta_2)\Delta\phi \tag{5.48}$$
$$\tag{5.49}$$

Afterwards we will obtain area of the faces at constant distances

$$A_{r_1} = \int \sqrt{\gamma}\hat{N}_r\ d\theta d\phi$$
$$= \int F^{-5/4} {r_1}^2 F^{3/4} \sin\theta\ d\theta d\phi$$
$$= F^{-1/2} {r_1}^2 \int \sin\theta\ d\theta d\phi$$
$$= \left(1 - \frac{2M}{r_1}\right)^{-1/2} {r_1}^2 [\cos\theta_1 - \cos\theta_2]\Delta\phi \tag{5.50}$$

Total A_r is equal to

$$A_{r_1}+A_{r_2} = \left[\left(1-\frac{2M}{r_1}\right)^{-1/2}{r_1}^2+\left(1-\frac{2M}{r_2}\right)^{-1/2}{r_2}^2\right][\cos\theta_1-\cos\theta_2]\Delta\phi \tag{5.51}$$

for the surface at constant θ

$$\begin{aligned} A_{\theta_1} &= \int \sqrt{\gamma}\hat{N}_\theta \; drd\phi & (5.52)\\ &= \int F^{-5/4}r^2\sin\theta_1 F^{1/4}r^{-1} \; drd\phi \\ &= \sin\theta_1\int F^{-1}r \; drd\phi \\ &= \left[4m^2\ln(r-2M)+2Mr+\frac{r^2}{2}\right]\Big|_{r_1}^{r_2}\sin\theta_1\Delta\phi & (5.53)\end{aligned}$$

Finally for the θ angle, we have

$$\begin{aligned} A_{\theta_1}+A_{\theta_2} &= \left[4m^2\ln(r-2M)+2Mr+\frac{r^2}{2}\right]\Big|_{r_1}^{r_2}(\sin\theta_1+\sin\theta_2)\Delta\phi \\ &= \left[4m^2\ln\frac{(r_2-2M)}{(r_1-2M)}+2M(r_2-r_1)+\frac{{r_2}^2-{r_1}^2}{2}\right] \\ &\quad (\sin\theta_1+\sin\theta_2)\Delta\phi & (5.54)\end{aligned}$$

for the surface at constant ϕ

$$\begin{aligned} A_{\phi_1} &= \int \sqrt{\gamma}\hat{N}_\phi \; drd\theta & (5.55)\\ &= \int F^{-5/4}r^2\sin\theta F^{1/4}r^{-1}\sin^{-1}\theta \; drd\theta \\ &= \int F^{-1}r \; drd\theta \\ &= \left[4m^2\ln(r-2M)+2Mr+\frac{r^2}{2}\right]\Big|_{r_1}^{r_2}\Delta\theta & (5.56)\\ A_{\phi_1}+A_{\phi_2} &= 2\left[4m^2\ln\frac{(r_2-2M)}{(r_1-2M)}+2M(r_2-r_1)+\frac{{r_2}^2-{r_1}^2}{2}\right]\Delta\theta & (5.57)\end{aligned}$$

Here $A_{\theta_1}+A_{\theta_2}$ and $A_{\phi_1}+A_{\phi_2}$ have the same r part due to the fact that they are placed in the same radius interval. The total surface of the spherical cube in the Schwarzschild metric is

$$
\begin{aligned}
A &= \left[\left(1-\frac{2M}{r_1}\right)^{-1/2} {r_1}^2 + \left(1-\frac{2M}{r_2}\right)^{-1/2} {r_2}^2\right] [\cos\theta_1 - \cos\theta_2]\Delta\phi \\
&+ \left[4m^2 \ln\frac{(r_2-2M)}{(r_1-2M)} + 2M(r_2-r_1) + \frac{{r_2}^2-{r_1}^2}{2}\right] \\
&[(\sin\theta_1+\sin\theta_2)\Delta\phi 0 + 2\Delta\theta]
\end{aligned}
\tag{5.58}
$$

Finally we have the related material for the heat kernel coefficients. From the equations [4.66] and [4.67] we find

$$
a_0 = \frac{\left[\frac{1}{3}(r_2^3-r_1^3) + \frac{5M}{4}({r_2}^2-{r_1}^2) + \frac{35M^2}{4}(r_2-r_1)\right](\cos\theta_1-\cos\theta_2)\Delta\phi}{(4\pi)^{3/2}} \tag{5.59}
$$

$$
\begin{aligned}
a_{1/2} &= -\frac{1}{16\pi}\left\{\left[\left(1-\frac{2M}{r_1}\right)^{-1/2} {r_1}^2 + \left(1-\frac{2M}{r_2}\right)^{-1/2} {r_2}^2\right] [\cos\theta_1 - \cos\theta_2]\Delta\phi\right. \\
&+ \left[4m^2 \ln\frac{(r_2-2M)}{(r_1-2M)} + 2M(r_2-r_1) + \frac{{r_2}^2-{r_1}^2}{2}\right] \\
&\left.[(\sin\theta_1+\sin\theta_2)\Delta\phi 0 + 2\Delta\theta]\right\}
\end{aligned}
\tag{5.60}
$$

We repeat the same limiting procedure to get a perturbation term for the Schwarzschild metric.

$$
V_{r_2 \gg r_1} = \frac{1}{3} r_2^3 (\cos\theta_1-\cos\theta_2)\Delta\phi {r_2}^3 \tag{5.61}
$$

$$
A_{r_2 \gg r_1} = \frac{1}{2}\left[2[\cos\theta_1-\cos\theta_2]\Delta\phi + (\sin\theta_1+\sin\theta_2) + 2\Delta\theta\right] {r_2}^2 \tag{5.62}
$$

$$
\left(\frac{A}{V}\right)_{r_2 \gg r_1} = \frac{3\left[2[\cos\theta_1-\cos\theta_2]\Delta\phi + (\sin\theta_1+\sin\theta_2) + 2\Delta\theta\right]}{2(\cos\theta_1-\cos\theta_2)\Delta\phi}\frac{1}{r_2} \tag{5.63}
$$

5.2.2.2. Area and Volume of the Shell for the Schwarzschild Metric .

$$\begin{aligned} V &= \int \sqrt{\gamma}\; dr d\theta d\phi \\ &= \int \left(1 - \frac{2M}{r}\right)^{-5/4} r^2 \sin\theta\; dr d\theta d\phi \end{aligned} \tag{5.64}$$

We can express the radial part of the integrand with $z = r - 2M$

$$\left(1 - \frac{2M}{z+2M}\right)^{-5/4} (z+2M)^2 = \frac{(z+2M)^{13/4}}{z^{5/4}} \tag{5.65}$$

where $z < 2M$. We expand this term again with the binomial expansion

$$\frac{(z-2M)^{13/4}}{z^{5/4}} = \frac{1}{2M^{13/4} z^{5/4}} \left[1 + \frac{13z}{8M} + \ldots\right] \tag{5.66}$$

We will input these into the volume integral. We will also take the boundaries according to a spherical shell. The inner radius of this spherical shell is chosen to be far by an amount of ϵ to the event horizon [8] [28].

$$\begin{aligned} V &= \int_{2M+\epsilon}^{r_2} \int_0^{\pi} \int_0^{2\pi} \frac{(2M)^{13/4}}{z^{5/4}} \left[1 + \frac{13z}{8M} + \ldots\right] \sin\theta\; dr d\theta d\phi \\ &= 4\pi (2M)^{13/4} \int_{\epsilon}^{r_2+2M} \left[z^{-5/4} + \frac{13}{8M} z^{-1/4}\right] \\ &= 4\pi (2M)^{13/4} \left[-4r_2 + 2M^{-1/4} + \frac{52}{24M} r_2 + 2M^{3/4} - \left(-4\epsilon^{-1/4} + \frac{52}{24M}\epsilon^{3/4}\right)\right] \end{aligned} \tag{5.67}$$

In this result we should notice that ϵ is very small in comparison to other variables so the third term will dominate the rest.

$$V = 4\pi (2M)^{13/4} 4\epsilon^{-1/4} \tag{5.68}$$

Now we will repeat the same procedure for the area term which is

$$\begin{aligned} A_r &= \int \sqrt{\gamma}\hat{N}_r \; d\theta d\phi \\ &= \int F^{-5/4}{r_1}^2 \sin\theta F^{3/4} \; d\theta d\phi \\ &= \left[\left(1-\frac{2M}{2M+\epsilon}\right)^{-1/2}(2M+\epsilon)^2 + \left(1-\frac{2M}{r_2}\right)^{-1/2}{r_2}^2\right]\int_0^{\pi}\int_0^{2\pi} \; d\theta d\phi \end{aligned} \tag{5.69}$$

After the expansion we will again witness the domination of the first ϵ term in the perturbation and the result will be

$$A_r = 4\pi(2M)^{5/2}\epsilon^{-1/2} \tag{5.70}$$

For the perturbation term this time we find

$$\frac{A}{V} = \frac{1}{4}\epsilon^{-1/2}(2M)^{1/2} \tag{5.71}$$

Besides as we know what the volume and the area is, one can find the heat kernel coefficients for this shape of container from the Heat kernel coefficient equations

$$a_0 = \frac{(2M)^{13/4}4\epsilon^{-1/4}}{(4\pi)^{1/2}} \tag{5.72}$$

$$a_{1/2} = -\frac{(2M)^{5/2}\epsilon^{-1/2}}{4} \tag{5.73}$$

6. ESTIMATION OF ESTIMATION OF THE THERMODYNAIC VARIABLES

6.1. N_e Calculation for the Unit Cube

Since we are interested in the high temperature expansion, we cut out rest of the temperature terms smaller than T. (we cut out the terms that contains higher order terms than $O(\epsilon^2)$). Also μ will be taken as zero being smaller than the perturbation parameter ϵ. Since

$$O\left(\frac{A}{V}\right)_{r_2 \gg r_1} \approx \frac{1}{L} \quad yet \quad \mu = O\left(\frac{1}{V}\right) \approx \frac{1}{L^3} \tag{6.1}$$

Frankly this situation doesn't create any conflict with the claim at [2.11].

$$N_e \;=\; \zeta(3/2)\frac{(2m)^{3/2}}{(4\pi)^{3/2}}a_0 T^{3/2} + \zeta(1)\frac{2m}{(4\pi)^{3/2}}a_{1/2}T^1 \tag{6.2}$$

In this section we will analyze the effects of the boundary on the relation between the temperature and the number of excited particles. Initially we will keep our discussion general without making any restrictions on the geometry of the box other than the assumption

$$O\left(\frac{A}{V}\right) = \frac{1}{L} \tag{6.3}$$

where L is the characteristic length of the box (which is $r_2 - 2M$ for the spherical box).When we divide the expression with the volume to get the perturbation term later

$$n_e \;=\; \zeta(3/2)\frac{1}{(V4\pi)^{3/2}}a_0 T^{3/2} + \zeta(1)\frac{1}{(V4\pi)^{3/2}}a_{1/2}T^1 \tag{6.4}$$

$$n_e = \zeta(3/2)\frac{1}{(4\pi)^{3/2}}\frac{1}{(4\pi)^{3/2}}T^{3/2} - \zeta(1)\frac{1}{(4\pi)^{3/2}}\frac{A}{V16\pi}T^1 \tag{6.5}$$

Now let us start the perturbation mechanism. Take

$$T = T^{(0)} + \epsilon T^{(1)} + \dots \tag{6.6}$$

For the $T^{3/2}$ term we will binomial expansion

$$T = \left[T^{(0)} + \epsilon T^{(1)} + \dots\right]^{3/2} \tag{6.7}$$

$$= (T^{(0)})^{3/2}\left(1 + \frac{3T^{(1)}}{2T^{(0)}}\epsilon + \dots\right) \tag{6.8}$$

we cut $O(\epsilon^2)$ terms. Let's take $r_2 \gg r_1$, and finally input perturbed temperature terms into [6.5]. we get

$$n_e = \zeta(3/2)\frac{1}{(4\pi)^{3/2}}\frac{1}{(4\pi)^{3/2}}T^{3/2} - \zeta(1)\frac{1}{(4\pi)^{3/2}}\frac{A}{V16\pi}T^1 \tag{6.9}$$

Then we get

$$\begin{aligned} n_e = {} & \frac{\zeta(3/2)}{(4\pi)^3}(T^{(0)})^{3/2}\left(1 + \frac{3T^{(1)}}{2T^{(0)}}\epsilon + \dots\right) \\ & -\epsilon\zeta(1)\frac{1}{(4\pi)^{3/2}}\frac{1}{16\pi}(T^{(0)} + \epsilon T^{(1)} + \dots) \end{aligned} \tag{6.10}$$

When its solved

$$n_e = \zeta(3/2)\frac{1}{(4\pi)^3}(T^{(0)})^{3/2} \tag{6.11}$$

$$T^{(0)} = \left[\frac{\zeta(3/2)}{n_e(4\pi)^3}\right]^{2/3} \tag{6.12}$$

For the next order its is

$$
\begin{aligned}
0 &= \frac{3\zeta(3/2)}{2(4\pi)^3}(T^{(0)})^{1/2}T^{(1)} - \zeta(1)\frac{T^{(0)}}{4(4\pi)^{5/2}} && (6.13)\\
& && (6.14)\\
T^{(1)} &= \frac{(4\pi)^{1/2}\zeta(1)}{6\zeta(3/2)}(T^{(0)})^{1/2} && (6.15)\\
T^{(1)} &= \frac{(4\pi)^{1/2}\zeta(1)}{6\zeta(3/2)}\left[\frac{\zeta(3/2)}{n_e(4\pi)^3}\right]^{1/3} && (6.16)
\end{aligned}
$$

Now when we input these terms to [6.6] we finally get the long searched result

$$
\begin{aligned}
T(n_e) &= \left[\frac{\zeta(3/2)}{n_e(4\pi)^3}\right]^{2/3} + \epsilon\frac{(4\pi)^{1/2}\zeta(1)}{6\zeta(3/2)}\left[\frac{\zeta(3/2)}{n_e(4\pi)^3}\right]^{1/3}\\
&= \frac{\zeta(3/2)^{2/3}}{(4\pi)^2}n_e^{-2/3} + \epsilon\frac{\zeta(1)}{6(4\pi)^{1/2}\zeta(3/2)^{2/3}}n_e^{-1/3} && (6.17)
\end{aligned}
$$

6.17 is the temperature of the box in terms of the number of excited particles. By finding this, we proved that for a Bose gas in any box it is impossible to be a pure BEC. Now, let us use the perturbation term for different geometries. For a flat space

$$
\begin{aligned}
T(n_e) &= \frac{\zeta(3/2)^{2/3}}{(4\pi)^2}n_e^{-2/3}\\
&+\frac{\zeta(1)[2(\cos\theta_1-\cos\theta_2)\Delta\phi+(\sin\theta_1+\sin\theta_2)\Delta\phi+2\Delta\theta]}{4(4\pi)^{1/2}\zeta(3/2)^{2/3}(\cos\theta_1-\cos\theta_2)\Delta\phi}\frac{1}{r_2}n_e^{-1/3}\\
& && (6.18)
\end{aligned}
$$

In addition we can express this in the Schwarzschild metric

$$
\begin{aligned}
T(n_e) &= \frac{\zeta(3/2)^{2/3}}{(4\pi)^2}n_e^{-2/3}\\
&+\frac{\zeta(1)\,(2[\cos\theta_1-\cos\theta_2)\Delta\phi+(\sin\theta_1+\sin\theta_2)+2\Delta\theta]}{4(4\pi)^{1/2}\zeta(3/2)^{2/3}(\cos\theta_1-\cos\theta_2)\Delta\phi}\frac{1}{r_2}n_e^{-1/3} && (6.19)
\end{aligned}
$$

where n is the total number density. We should emphasize the relation between BEC and results. From the equation [2.16]

$$\mu = \frac{1}{\beta}\ln\left(\frac{n_0}{n_0+1/V}\right) \tag{6.20}$$

As one can notice the expression inside of the logarithm goes to zero When the particle density at the GS is equal to zero (no BEC)

$$n_0 = 0 \quad and \quad n_e = n \tag{6.21}$$

In this situation μ diverges. On the other hand, if we take the thermodynamic limit

$$V \to \infty \quad as \quad n_0 = n \to finite \tag{6.22}$$

μ becomes zero. This proves that, we will always have a mixing of ground and excited states. So the system will never fall to a phase of a pure Bose Einstein condensate.

6.2. $\mathcal{F}$ and S Calculation for the Spherical Shell

At this point we will assemble our previous results to estimate the free energy and entropy. While we do so we turn our volume box into a shell that is far from the horizon by ϵ

in the thermodynamic limit, it becomes

$$\mathcal{F} = -\zeta(5/2)\frac{(2m)^{5/2}}{(4\pi)^{3/2}}a_0T^{5/2} - \zeta(2)\frac{(2m)^2}{(4\pi)^{3/2}}a_{1/2}T^2 + \ldots \tag{6.23}$$

We cut out the summation after the T^2 term, and focused on the our order of interest. When we input the coefficients

$$\mathcal{F} = -\zeta(5/2)\frac{(2m)^{5/2}}{(4\pi)^{3/2}}a_0T^{5/2} - \zeta(2)\frac{(2m)^2}{(4\pi)^{3/2}}a_{1/2}T^2 \tag{6.24}$$

Let us calculate the entropy S by taking time derivative

$$\begin{aligned} S &= -\left.\frac{\partial \mathcal{F}}{\partial T}\right|_V \\ S &= \frac{5}{2}\zeta(5/2)\frac{(2m)^{5/2}}{(4\pi)^{3/2}}a_0 T^{3/2} + 2\zeta(2)\frac{(2m)^2}{(4\pi)^{3/2}}a_{1/2}T \end{aligned} \tag{6.25}$$

so we get

$$S = 20\zeta(5/2)\frac{2^{1/4}m^{5/2}}{\pi^2}T^{3/2}M^{13/4}\epsilon^{-1/4} - \zeta(2)\frac{2^{1/2}m^2}{\pi^{3/2}}(M)^{5/2}\epsilon^{-1/2}T \tag{6.26}$$

we will apply this result to the horizon divergence problem with proper values for the temperature and the distance from the event horizon. The reader might think that these distinct choices are too specific to be real. On the other hand, this fine-tuning is designed to create a realistic environment for a black hole. When the object at the center is a black hole, and we replace our ordinary T and ϵ by

$$T \to T_H = \frac{1}{8\pi M} \quad and \quad \epsilon \to \frac{\delta^2}{8M} \tag{6.27}$$

So the entropy result turns out to be

$$\begin{aligned} S &= \frac{2^{-7/2}5\zeta(5/2)}{\pi^(7/2)}M^2\delta^{-1/2} - \frac{2^{-15/4}\zeta(2)}{\pi^{5/4}}M^2\delta^{-1} \\ &= \left[\frac{2^{-7/2}5\zeta(5/2)}{\pi^(7/2)}\delta^{-1/2} - \frac{2^{-15/4}\zeta(2)}{\pi^{5/4}}\delta^{-1}\right]M^2 \end{aligned} \tag{6.28}$$

If we use the area of the event horizon

$$A_H = 2\pi(2M)^2 \tag{6.29}$$

$$M^2 = \frac{A_H}{16\pi} \tag{6.30}$$

To express entropy

$$S = \left[\frac{2^{-7/2}5\zeta(5/2)}{\pi^{(7/2)}}\delta^{-1/2} - \frac{2^{-15/4}\zeta(2)}{\pi^{5/4}}\delta^{-1}\right]\frac{A_H}{16\pi} \tag{6.31}$$

Thus we see that the entropy diverges as δ goes to zero. This is one of the example of horizon divergence [29–32]. In the case of the entropy the result is M^2 times a constant as expected. It is clear that the surface A is more important than the bulk.

7. CONCLUSION

Several methods and a wide spectrum of concepts have been used in this master thesis. The Heat Kernel approximation and the Mellin transformation are highly preferred methods for different problems in various fields. With the helping hand of the boundary conditions of a box, which is full of ideal non-relativistic Bose gas, we found perturbative results for the number of excited particles , the free energy, and entropy in an analogue optical metric in the presence of an astronomical object.

In these estimations, we followed different paths. For N_e, we took favor of the finite size effects to show that Bose Einstein in a finite box is a theoretical no-go, in addition to its impractical nature. Then, while we were calculating $\mathcal{F}$ we used the brick wall model. By preferring specific values such as the Hawking temperature, and the proper cut-off, we adapted our scheme for a black hole; thus, we were able to witness the divergence problem for the entropy. As expected from the black hole entropy, the first term goes with the area of the horizon, which indicates that the results of our methodology are consistent with nature.

Bose Einstein condensation in curved space-time is a basis that allows the construction of alternative structures, leading to interesting and perhaps more challenging questions. Therefore, it would be worthwhile to pursue investigations in such a fertile field. This flexible subject can be approached from different angles, such as the subordination technique, or be studied in a different environment, such as a cosmological phase of choice.

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