OPERATING ROOM SCHEDULING WITH UNCERTAIN SURGERY DURATIONS

by

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ABSTRACT

OPERATING ROOM SCHEDULING WITH UNCERTAIN SURGERY DURATIONS

In this thesis, we consider the day-to-day scheduling problem of a single hospital operating room (OR). We assume that the number and the characteristics of the surgeries to be scheduled for the next day are known in advance, but they have uncertain durations with different means and variances. Our aim is to determine the sequence and scheduled starting times of the surgeries in such a way that a cost function, which is defined as the weighted sum of expected patient waiting times, idle times of the OR, and the end-of-day overtime, is minimized. For the sequencing part of the problem, based on analytical observations of smaller scale problems, we propose ordering surgeries with respect to stochastically increasing durations (roughly corresponding to a smallest variance first sequencing rule). For the determination of the scheduled starting times, we consider three heuristics, each motivated by analytical solutions of approximate models: an expected value based heuristic, a heuristic based on decomposition of surgeries (Myopic heuristic), and a heuristic based on the assumption that the OR is never kept idle (Veteran's heuristic). We test these heuristics by comparing them with the optimal solution found by exhaustive enumeration. Our results reveal that the sequencing rule proposed coupled with the Veteran's heuristic yield the most satisfactory outcome.

ÖZET

RASGELE AMELİYAT SÜRELERİYLE AMELİYAT ODASI ÇİZELGELEMESİ

Bu tez çalışmasında ameliyat odasının günlük çizelgelenmesi problemini ele alıyoruz. Ameliyat sayısının ve karakteristiklerinin önceden bilindiğini, ancak ameliyat sürelerinin farklı beklenen değer ve varyansa sahip rasgele değişkenler olduğunu varsayıyoruz. Amacımız ameliyatların hangi sıralamayla yapılması gerektiğini ve bu sıralamada her bir ameliyatın ne zaman başlaması gerektiğini; hastaların bekleme sürelerinin, ameliyat odasının boşta olduğu sürelerin ve fazla mesai süresinin ağırlıklı toplamı şeklinde ifade edilen maliyet fonksiyonunun minimize edilmesini sağlayacak şekilde belirlenmesi. Ameliyat sıralamasının belirlenmesi için, daha küçük ölçekteki problemlerin çözümündeki analitik gözlemlerimize dayanarak ameliyatların artan stokastik sürelere göre sıralanması kuralını öneriyoruz. Bu kural temel olarak ”en küçük varyans önce” kuralına denk gelmektedir. Ameliyatların başlama zamanlarının belirlenmesi için ise, her biri benzer problemlerin analitik çözümlerine dayanan üç farklı sezgisel yöntem öneriyoruz: beklenen değere dayalı yöntem, ameliyatların ayrıştırılmasına dayalı Miyopik yöntem ve ameliyat odasının hiçbir zaman boş kalmadığını varsayan Veteran yöntemi. Bu yöntemlerin performanslarını nümerik analizlerle test ettiğimizde, Veteran yönteminin en iyi çalışan yöntem olduğunu gözlemliyoruz.

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LIST OF SYMBOLS

AE_i	Actual ending time of i^{th} surgery
AS_i	Actual starting time of i^{th} surgery
C_i	Cost function of scheduling i surgeries in a sequence
D_i	Decision variables - Duration assigned to i^{th} surgery
$\mathcal{D}$	Vector of assigned durations: $(D_1, D_2, ..., D_n)$
f_i	pdf of the i^{th} surgery
F_i	cdf of the i^{th} surgery
$\mathcal{LST}$	List of surgeries to be scheduled
O_n	Objective function for a the given sequence
SE_i	Scheduled ending time of i^{th} surgery
SS_i	Scheduled starting time of i^{th} surgery
$\mathcal{SEQ}$	List of surgeries in the order of appearance in a given sequence
T_i	Random variable describing the actual duration of the i^{th} surgery
$\mathcal{T}$	Vector of actual durations: $(T_1, T_2, ..., T_n)$
λ_i	Rate parameter of the i^{th} surgery, when surgery durations are assumed to be distributed exponentially
$\mathcal{L}$	Vector of rate parameters of the surgeries: $(\lambda_1, \lambda_2, ..., \lambda_n)$
α_1	Unit cost of idle time
α_2	Unit cost of patient waiting time
α_3	Unit cost of overtime

LIST OF ACRONYMS/ABBREVIATIONS

DO	Decreasing Order of rate parameters
IO	Increasing Order of rate parameters
OR	Operating Room
SVF	Shortest Variance First

1. INTRODUCTION

It is reported that health care expenditures in the United States exceeded USD 2.8 trillion in 2012, approximately 17% of the Gross Domestic Product [10]. A third of this total amount is accounted for hospital expenditures [10], and surgeries are known to generate more than 40% of a hospital's total expenses and revenues [23, 35]. A recent joint study by the National Academy of Engineering and the Institute of Medicine [43] highlights the importance of health care and engineering partnership, and points out the need for joint research work in scheduling of health care delivery systems. Some early works in this field [7, 8, 39, 50] describe possibilities of applying mathematical programming methods to solve the problems in health care delivery systems. Nevertheless, a great portion of the studies in the literature have been conducted in and after 2000 [9]. The uprising interest in the field can be explained by the change of health care expenditures in the period between 1960-2000. According to statistics of Centers for Medicare and Medicaid Services (CMS) [10], see Table 1.1, total health care expenditures (and proportion of health expenditures in total GDP) in US increased from USD 27.4 billion (5%) in 1960 to USD 1.37 trillion (13.4%) in 2000.

Table 1.1. Change of National Health Expenditures and GDP (billion $) in US.

	1960	**1970**	**1980**	**1990**	**2000**
National Health Expenditures	27.4	74.9	255.8	724.3	1,377.2
GDP	543	1,076	2,863	5,980	10,290
Proportion	5.0%	6.9%	8.9%	12.1%	13.4%

Providing health services to patients in hospitals has become an important managerial concern. Hospitals often find themselves in need for managing two conflicting aspects of health services they provide: Being a financial institution, they want to reduce costs and improve their financial outcomes. On the other, hand they also want to maximize the level of patient satisfaction. Operating rooms are units of particular interest due to constituting a large portion of hospital's total expenses and revenues, as mentioned earlier. They have a major impact on the performance of the hospital as a whole. However, managing the operating rooms is hard due to the conflicting priorities and the preferences of its stakeholders [29]. Another source of difficulty is the scarcity of costly resources. These factors point out to the need for adequate planning and scheduling procedures. Dexter *et al.* [13, 14, 15, 17, 18, 21] examine how efficient planning and scheduling contributes to improvements in financial performance measures.

An operating room consists of many different resources including a surgical suite, equipment/material resources, and human resources such as surgeons, nurses, and anesthesiologists. Designing surgery schedules is a challenging problem due to its combinatorial nature and inherent uncertainty of surgery durations. Furthermore, often, there are multiple (usually conflicting) performance criteria which need to be optimized simultaneously. An OR session usually lasts 8-9 hours per day. An overtime cost is incurred if an OR is used beyond this period. In addition to overtime cost, there are also less tangible costs such as the costs of surgeon idle time, OR idle time, and patient waiting time. At many hospitals, surgeons use OR only during a certain block of time in a week [3]. Although, this simplifies the OR scheduling problem to some extent, block scheduling is also a very challenging problem.

Surgeries may be either elective or non-elective (urgent or emergent). Elective surgeries are planned in advance, whereas non-elective surgeries typically arise unexpectedly during the day and they need to be added to the existing schedule. It is worth to mention that most of the research that appeared in or after 2000 is directed to the planning and scheduling of elective patients [9]. In our work, we also consider the scheduling of elective surgeries.

A very important aspect of OR scheduling is the high uncertainty of surgery durations [20, 22, 52, 53]. However, often in practice, surgeries are scheduled based on the expected values of surgery durations. This results in high expected overtime and surgeon idle time [11]. In this thesis work, we also confirm that disregarding the uncertainty in surgery durations (i.e., scheduling based on expected durations) results in significantly poorer performance than when uncertainty is incorporated to the problem. However, although the incorporation of uncertainty is more realistic, many researchers prefer the deterministic approach due to computational complexity [9].

OR scheduling is handled in three different levels: Strategic, tactical and operational. These levels can be thought of as planning of long-term, medium-term and day-to-day decisions, respectively. Although a clear distinction between strategic, tactical and operational level is hard to make, we can consider daily scheduling as operational, weekly (repeating) scheduling as tactical, and monthly (or longer) scheduling as strategic scheduling levels. Furthermore, there is a distinction between the information used while planning/scheduling in different levels. For example, Van Houdenhoven *et al.* (2007) [55] state that strategic planning uses patient forecasts and/or historical information, while tactical planning, like operational planning, deals with actual/expected patients. In our study, we focus on operational (i.e., day-to-day) scheduling problem.

OR planning and scheduling decisions affect facilities throughout the entire hospital. Therefore it is sensible to consider impact on other facilities, such as the ICU or PACU, in the decision process and try to improve the overall performance. Otherwise, improving the operating room schedule may lead to reduced efficiency of those related facilities. Several studies (See, for example [5, 19, 18, 44]) are conducted on the OR scheduling problem integrated with other facilities, whereas majority of the contributions in OR scheduling

[2, 6, 12, 26, 38, 47] assume that operating rooms are isolated from other facilities.

In this thesis work, we consider the day-to-day scheduling problem of a single OR. We assume that all surgeries are elective (therefore, the number of surgeries to be scheduled is known prior to scheduling) and have uncertain durations. We further assume that surgeries do not start earlier than planned even if the preceding surgery ends earlier; but they may start later than planned if the preceding surgery ends with a delay. The last assumption comes from the real life implementations in most of the hospitals, as arrival/preparation of patient usually makes it impossible for a surgery to start before its planned time. Our objective is to minimize the expected weighted sum of three performance criteria: idle time cost, waiting time cost, overtime cost. We first find a more compact expression for the objective function, which gets increasingly complex as the number of surgeries to schedule increases. We, then, solve a non-linear optimization problem finding the optimum durations to assign for each surgery, for a given sequence. Finally we develop heuristics to propose a solution to combinatorial problem of finding the optimum sequence. To our knowledge, non-linear optimization approach for this problem has not been considered in literature.

Chapter 2 provides a literature review on the OR scheduling problem, focusing more on the studies that are related to our work. Chapter 3 formally describes the problem and introduces mathematical notations. In Chapter 4, we present our analytical findings for the problem with smaller instances (2-3 surgeries) as well as the most general form of the problem. Chapter 5 describes the heuristics we developed to solve the optimum sequencing problem. In Chapter 6, we present our numerical results. Based on our findings, in Chapter 7, we discuss about possible future research opportunities related to the problem we considered and conclude with the managerial implications of our findings.

2. LITERATURE REVIEW

In this thesis, we consider an OR scheduling problem with uncertain surgery durations. This problem falls under the stochastic scheduling area. Pinedo (2008) [49] provides a thorough overview of problems and solution approaches in the domain of stochastic scheduling. Several approaches such as optimization models ([12, 27, 48]), queuing models ([34, 59]), simulation models ([25, 42]) and heuristics ([12, 32]) have been widely used in literature to study OR scheduling. In our review, we focus on those studies which are directly related to the problem we consider in this thesis work. More extensive reviews can be found in [9, 24, 30]. In order to make it easier to follow the review, we analyze the studies in different sections.

2.1. Performance Measures

Studies can be found in literature aiming to optimize one or more of the following performance measures, simulataneously: Waiting time (surgeon and patient), throughput, utilization, overtime, leveling, makespan, patient deferral, financial [9].

Denton *et al.* [12], for instance, examine how ordering of procedures affects patient waiting time, OR idle time and overtime. They use a two-stage stochastic mixed integer program (MIP) and propose solutions based on heuristics. Dexter et al. ([16, 17]) define a new performance measure, OR efficiency, as a linear combination of the OR under-utilization and over-utilization and evaluate the procedures based on this measure. Marcon and Dexter [41], use discrete-event simulation to examine how standard sequencing rules may improve the pre/post-operative processes. Another way of evaluating the quality of a planning or scheduling procedure is to use the number of deferred, refused or canceled patients. Kim and Horowitz [37], study how to include quotas in the surgery scheduling process in order to streamline the admittance to the ICU.

2.2. Constraints

A substantial part of the literature on operating room planning and scheduling considers linear optimization problems with different types of constraints. First category of constraints are those related to the use of resources. Considering the scarcity and high costs of OR resources, these constraints often have a substantial impact on the feasible and optimal solutions. Second category of constraints is constraints related to time lags. Marcon *et al.* [40], for instance, state that due to contamination risks, it is obliged to schedule infected patients at the end of the surgery day or to insert idle time between surgeries. Pham and Klinkert [48], incorporate many constraints in their OR scheduling problem and model their optimization problem as a multi-mode blocking job shop problem.

2.3. Solution Technique

A wide range of solution methodologies that are retrieved from the domains of operations management and operations research can be found in the literature of OR scheduling. A brief introduction to the various solution techniques that are widely used in OR scheduling problems can be found in the studies of Gass and Harris [28] and Winston and Goldberg [58]. Goal programming is a flexible optimization technique when multiple objectives need to be optimized simultaneously. For each objective, a goal is specified and the objective is to minimize the deviations from the targets. Ozkarahan [45], formulates a goal programming approach in which surgeries are assigned to operating rooms and intensive care capabilities or operating room and surgeon preferences are also addressed.

Two simulation approaches, discrete-event simulation and Monte-Carlo simulation, are widely used in the literature of OR scheduling. Discrete-event simulation represents a system as it evolves over discrete or countable points in time (dynamic), whereas Monte-Carlo simulation represents a system at a particular point in time (static) [58]. Lebowitz [38], for instance, applies Monte-Carlo simulation to evaluate the impact of sequencing procedures on waiting time and operating room utilization measures. Sciomachen *et al.* [51] construct a discrete-event simulation model to evaluate the performance based on multiple conflicting criteria.

Heuristic approaches are also very commonly used in the literature of OR scheduling. Guinet and Chaabane [31], present a constructive heuristic that minimizes operating room overtime costs and patient hospitalization costs. Hans *et al.* [33] propose priority-based constructive heuristics to maximize the OR utilization.

2.4. Uncertainty

One of the major problems in OR scheduling is the uncertainty inherent to surgical services. In literature two types of uncertainty are well addressed: arrival uncertainty and duration uncertainty. Khaniyev *et al.* [36] develop a new model based on real data to estimate surgery durations, as well as finding the significant factors influencing the surgery durations. Strum *et al.* [52] compare the accuracies of distribution fit models in estimating surgery durations. Persson and Persson [46] describe a discrete-event simulation model to study how resource allocation policies affect the waiting time and utilization of emergency resources, when both patient arrival uncertainty and surgery duration variability are taken into account.

2.5. Multi/Single-OR Scheduling

Batun [3] considers a multi-OR scheduling problem with multiple surgeons where the surgery durations are uncertain. They formulate a two-stage stochastic mixed-integer program (SMIP) for the problem. The main decisions are the number of ORs to open, the assignment of surgeries to ORs, the sequence of surgeries within each OR, and the times at which surgeons start their first surgery of the day. They also explore a stochastic multi-OR scheduling problem where surgery to OR allocation decisions are allowed to be revised during the day (rescheduling). Testi *et al.* [54] propose a three-phase method to generate weekly schedules for a multi-OR surgical suite.

Weiss [57] considers the problem of minimizing OR idle time and patient waiting time in a single-OR where the surgery durations are uncertain and the decisions are the sequence of surgeries and their start times. His numerical results reveal that the solution highly depends on the cost coefficients for small problem instances. Wang [56] considers a single OR scheduling problem where the surgery durations are assumed to be exponentially distributed. Due to the special properties of exponential distribution, he is able to solve larger instances than Weiss. Denton and Gupta [11] study the single server appointment scheduling problem for fixed sequence of customers where the service durations are stochastic. The objective is to determine optimum appointment times for the customers which minimize the total expected cost of customer waiting time, server idle time and tardiness. Begen and Queyranne [4] consider the single server appointment scheduling problem under the assumption that service durations are independent and discrete. They show that their objective function is L-convex under reasonable conditions on cost coefficients.

Despite the vast amount of literature on OR scheduling, there is still a lack of studies on stochastic OR scheduling problems, especially on those with analytical approaches. Our main contributions to the literature of OR scheduling with this thesis work can be listed as follows:

- We adopt a novel approach where the objective function is defined for a given sequence of procedures. Therefore, one needs to first determine a sequence and then optimize the objective function for the given sequence. Assuming a given sequence allows us to remove sequence variables and define our objective function in terms of assigned durations, our set of decision variables. In this way, we are able to study nonlinear optimization of objective function both analytically and numerically,
- We prove the joint convexity of objective function over the assigned durations for two and three procedure OR scheduling problems, and conjecture that the convexity also holds for any number, n, of procedures,
- We obtain a compact and simplified version of objective function in terms of auxiliary functions which assumes a special pattern,
- We develop several heuristics for both optimum sequence and optimum duration assignments where the analytical solution is difficult to obtain,
- We numerically show the effects of cost coefficients on the performances of our heuristics for a wide range of parameter combinations and different numbers of procedures to schedule.

3. MODEL FORMULATION

In this thesis, we consider the daily (operational) scheduling problem of a finite set of surgeries with uncertain durations in a single operating room (OR). All the surgeries are assumed to be elective and surgery durations have independent distributions. Four auxiliary variables are initially defined to aid mathematical formulation the problem: scheduled starting time (SS_i), scheduled ending time (SE_i), actual starting time (AS_i), and actual ending time (AE_i). The scheduled starting time of each operation can be thought of as the time the patient is scheduled to arrive to OR, thus a surgery may not begin before its scheduled starting time. However, it may begin later than its scheduled starting time in case the actual ending time of the preceding surgery exceeds its scheduled ending time. The objective function consists of three components: expected idle time cost (the time OR is idle while waiting for the next patient to arrive), expected patient waiting time cost (the time a patient must wait between his/her scheduled starting time and actual starting time) and expected overtime cost (an overtime cost is incurred when surgery completions extend past a deadline described as the scheduled ending time of the last surgery). Our purpose is to solve two problems simultaneously: finding the optimum sequence - the order of implementation of surgeries and finding the optimum durations to assign to each surgery for a given sequence. Figure 3.1 is a sample plot of how idle time, waiting time, and overtime are determined.

Before proceeding to mathematical formulation of the problem, let us introduce some notations we will use throughout the rest of this thesis work:

- $\mathcal{LST}$: List of surgeries to be scheduled,
- Roman numbers, $i = I, II, III, ..., N$: i^{th} surgery in $\mathcal{LST}$,

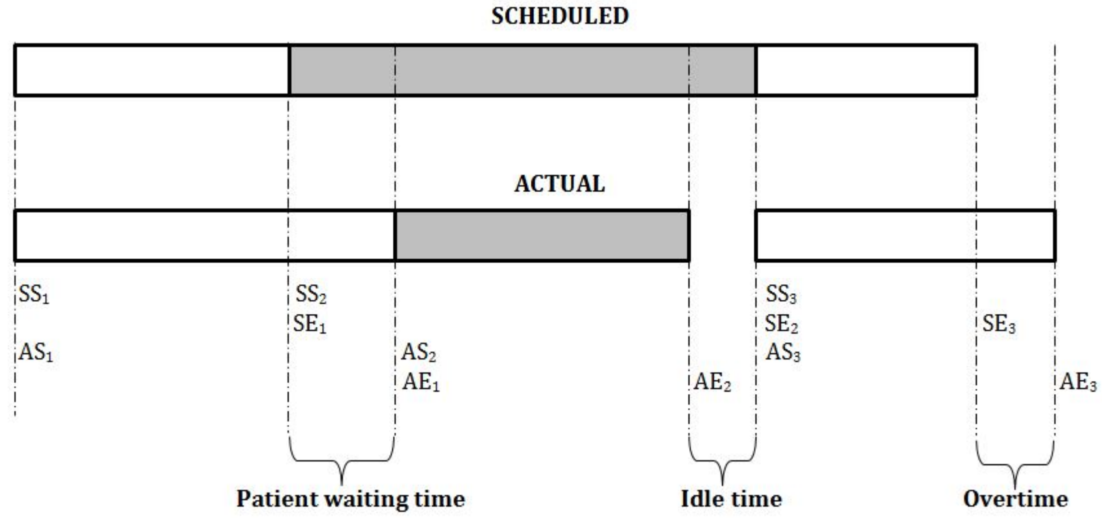

Figure 3.1. An example of schedule in OR.

- $\mathcal{SEQ}$: List of surgeries in the order of appearance in a given sequence,
- Arabic numbers, $i = 1, 2, 3, ..., n$: i^{th} surgery in $\mathcal{SEQ}$,
- $D_i, i = I, II, III, ..., N$: Decision variables - Duration assigned to i^{th} surgery in $\mathcal{LST}$,
- $D_i, i = 1, 2, 3, ..., n$: Decision variables - Duration assigned to i^{th} surgery in $\mathcal{SEQ}$,
- $\mathcal{DL}$: Vector of assigned durations in $\mathcal{LST}$: $(D_I, D_{II}, ..., D_N)$,
- $\mathcal{D}$: Vector of assigned durations in $\mathcal{SEQ}$: $(D_1, D_2, ..., D_n)$,
- $T_i, i = I, II, III, ..., N$: Random variable describing the actual duration of the i^{th} surgery in $\mathcal{LST}$,
- $T_i, i = 1, 2, 3, ..., n$: Random variable describing the actual duration of the i^{th} surgery in $\mathcal{SEQ}$,
- $\mathcal{TL}$: Vector of actual durations in $\mathcal{LST}$: $(T_I, T_{II}, ..., T_N)$,
- $\mathcal{T}$: Vector of actual durations in $\mathcal{SEQ}$: $(T_1, T_2, ..., T_n)$,
- $f_i, F_i, i = I, II, III, ..., N$: pdf, cdf of the i^{th} surgery in $\mathcal{LST}$,
- $f_i, F_i, i = 1, 2, 3, ..., n$: pdf, cdf of the i^{th} surgery in $\mathcal{SEQ}$,
- $\lambda_i, i = I, II, III, ..., N$: rate parameter of the i^{th} surgery in $\mathcal{LST}$, when surgery durations are assumed to be distributed exponentially,
- $\lambda_i, i = 1, 2, 3, ..., n$: rate parameter of the i^{th} surgery in $\mathcal{SEQ}$, when surgery durations

are assumed to be distributed exponentially,

- $\mathcal{LL}$: Vector of rate parameters of the surgeries in $\mathcal{LST}$: $(\lambda_I, \lambda_{II}, ..., \lambda_N)$, when surgery durations are assumed to be distributed exponentially,
- $\mathcal{L}$: Vector of rate parameters of the surgeries in $\mathcal{SEQ}$: $(\lambda_1, \lambda_2, ..., \lambda_n)$, when surgery durations are assumed to be distributed exponentially,
- α_1: Unit cost of idle time,
- α_2: Unit cost of patient waiting time,
- α_3: Unit cost of overtime,
- O_n: Objective function for a given sequence, $\mathcal{SEQ}$. Number of surgeries to schedule is n.
- $(x)^+ = max\{0, x\}$.

We define our objective function, for a given sequence $\mathcal{SEQ}$, as the weighted linear combination of three expected costs; namely, idle time cost, patient waiting time cost, and overtime cost. As depicted in Figure 3.1, an idle time cost is incurred if a surgery ends before its scheduled ending time (equivalently, before the next surgery's scheduled starting time). Similarly, a patient waiting time cost is incurred if a surgery ends after its scheduled ending time (equivalently, after the next surgery's scheduled starting time). Finally, an overtime cost is incurred if the last surgery in the sequence ends after its scheduled ending time. Hence, we can formulate the objective function as follows:

$$O_n = \alpha_1 E\left[\sum_{i=1}^{n}(SS_{i+1} - AE_i)^+\right] + \alpha_2 E\left[\sum_{i=1}^{n-1}(AE_i - SS_{i+1})^+\right] + \alpha_3 E\left[(AE_n - SE_n)^+\right], \tag{3.1}$$

where SS_i is scheduled starting time, AE_i is actual ending time, and SE_i is the scheduled ending time of i^{th} surgery in the sequence, as defined earlier.

As seen in Equation 3.1, the objective function is defined in terms of our auxiliary variables; namely, SS_i, SE_i, AS_i, and AE_i; and not in terms of our decision variables, D_i's.

Therefore, we need to obtain an equivalent expression for the objective function in terms of our decision variables. Lemma 3.1 provides the basis for this modification.

Lemma 3.1. *Let* $a_i = T_i - D_i$ *and* $b_i = a_i + b_{i-1}^{+}$, $b_1 = a_1$ *be random variables. Then, the actual starting time of* i^{th} *surgery (*AS_i*) can be expressed as follows:*

$$AS_i = \sum_{j=1}^{i-1} D_j + b_{i-1}^{+}. \tag{3.2}$$

Proof. Let's write AS_i, AE_i, SS_i and SE_i in terms of D_i and T_i and/or each other: Scheduled starting time of a surgery is simply the sum of assigned durations of the previous surgeries in the sequence.

$$SS_i = \sum_{j=1}^{i-1} D_j,\ SS_1 = 0 \tag{3.3}$$

Scheduled ending time of a surgery is the sum of its scheduled starting time and assigned duration.

$$SE_i = SS_i + D_i = \sum_{j=1}^{i} D_j \tag{3.4}$$

We explained earlier that a surgery may not begin before its scheduled starting time. However, it may begin later than its scheduled starting time in case the previous surgery ends after its scheduled ending time. We can mathematically express this statement as follows:

$$AS_i = max\{SE_{i-1}, AE_{i-1}\},\ AS_1 = 0, \tag{3.5}$$

Actual ending time of a surgery is the sum of its actual starting time and actual duration.

$$AE_i = AS_i + T_i \tag{3.6}$$

$AS_1 = 0$ is given. We will prove the Lemma by induction. Let's first obtain expression for AS_2:

$$AS_2 = max\{SE_1, AE_1\} = max\{D_1, T_1\} = D_1 + (T_1 - D_1)^+ = D_1 + (b_1)^+$$

Therefore, the Lemma holds for $k = 2$.

Now, assume that the Lemma holds for $k = i - 1$. We, need to show that it also holds for $k = i$. Let's state the induction hypothesis as follows:

$$AS_{i-1} = \sum_{j=1}^{i-2} D_j + {b_{i-2}}^+$$

We, now, need to show that $AS_i = \sum_{j=1}^{i-1} D_j + b_{i-1}^+$:

$$\begin{aligned}
AS_i &= max\{SE_{i-1}, AE_{i-1}\} \\
&= max\{SE_{i-1}, AS_{i-1} + T_{i-1}\} \\
&= max\{\sum_{j=1}^{i-1} D_j, \sum_{j=1}^{i-2} D_j + b_{i-2}^+ + T_{i-1}\} \\
&= \sum_{j=1}^{i-1} D_j + max\{0, b_{i-2}^+ + (T_{i-1} - D_{i-1})\} \\
&= \sum_{j=1}^{i-1} D_j + max\{0, b_{i-2}^+ + a_{i-1}\} \\
&= \sum_{j=1}^{i-1} D_j + max\{0, b_{i-1}\} \\
&= \sum_{j=1}^{i-1} D_j + b_{i-1}^+.
\end{aligned}$$

As, seen, the Lemma holds for $k = i$. This concludes the proof. □

Theorem 3.1. *The objective function, O_n can be expressed as follows:*

$$O_n(\mathcal{B}) = \alpha_1 \sum_{i=1}^{n}(D_i - E[T_i]) + \alpha_2 \sum_{i=1}^{n-1} E[(b_i)^+] + (\alpha_1 + \alpha_3)E[(b_n)^+]$$

where $\mathcal{B} = (b_1, b_2, ..., b_n)$, $b_i = a_i + b_{i-1}^+$, $a_i = T_i - D_i$, $b_1 = a_1$.

Proof. Rewriting the terms in Equation (3.1):

$$(SS_{i+1} - AE_i)^+ = \left[\sum_{j=1}^{i} D_j - \left(\sum_{j=1}^{i-1} D_j + b_{i-1}^+ + T_i\right)\right]^+ = \left[-(T_i - D_i) - b_{i-1}^+\right]^+ = (-b_i)^+$$

$$(AE_i - SS_{i+1})^+ = \left[\left(\sum_{j=1}^{i-1} D_j + b_{i-1}^+ + T_i\right) - \sum_{j=1}^{i} D_j\right]^+ = \left[(T_i - D_i) + b_{i-1}^+\right]^+ = (b_i)^+$$

$$(AE_n - SE_n)^+ = \left[\left(\sum_{j=1}^{n-1} D_j + b_{n-1}^+ + T_n\right) - \sum_{j=1}^{n} D_n\right]^+ = \left[(T_n - D_n) + b_{n-1}^+\right]^+ = (b_n)^+$$

Replacing the terms in Equation (3.1), gives the following expression:

$$O_n(\mathcal{B}) = \alpha_1 E\left[\sum_{i=1}^{n}(-b_i)^+\right] + \alpha_2 E\left[\sum_{i=1}^{n-1}(b_i)^+\right] + \alpha_3 E\left[(b_n)^+\right]. \tag{3.7}$$

We can write:

$$(-b_i)^+ = -b_i + (b_i)^+ \tag{3.8}$$

Then,

$$\begin{aligned}
\alpha_1 E\left[\sum_{i=1}^{n}(-b_i)^+\right] &= \alpha_1 E\left[\sum_{i=1}^{n}(-b_i) + \sum_{i=1}^{n}(b_i)^+\right] \\
&= -\alpha_1 E\left[b_1\right] - \alpha_1 E\left[\sum_{i=2}^{n}(b_i - b_{i-1}^+)\right] + \alpha_1 E\left[(b_n)^+\right] \\
&= -\alpha_1 E\left[b_1\right] - \alpha_1 E\left[\sum_{i=2}^{n} a_i\right] + \alpha_1 E\left[(b_n)^+\right] \\
&= \alpha_1 E\left[\sum_{i=1}^{n}(D_i - T_i)\right] + \alpha_1 E\left[(b_n)^+\right]
\end{aligned}$$

$$= \alpha_1 \sum_{i=1}^{n}(D_i - E[T_i]) + \alpha_1 E\left[(b_n)^+\right]. \tag{3.9}$$

The equalities follow from $a_i = b_i - b_{i-1}^+ = T_i - D_i$ and $b_1 = a_1$.

Replace the first expression in (3.7) with the expression in (3.9); then we can write the objective function as:

$$O_n(\mathcal{B}) = \alpha_1 \sum_{i=1}^{n}(D_i - E[T_i]) + \alpha_2 \sum_{i=1}^{n-1} E[(b_i)^+] + (\alpha_1 + \alpha_3)E[(b_n)^+]. \tag{3.10}$$

□

As can be seen from the expression of objective function in (3.10), it is essential to find expectations $E[(b_i)^+], \forall i = 1, 2, .., n$ in order to explicitly state the objective function in terms of D_i's. In the next chapter, one of our main focuses will be in this direction.

4. ANALYTICAL RESULTS

In this chapter we present some of our analytical findings on the special cases of the problem. We first consider the scheduling of two procedures, which constitutes one of the building blocks of the general problem. Analyzing the two-procedure scheduling problem gives us an idea about the complexity of the problem, solution methodology and the characteristics of the solutions. Section 4.1 is dedicated to two-procedure scheduling problem. We begin the section by expressing the objective function explicitly in terms of D_i variables and the distributions of the surgery durations. Later, we prove joint convexity of the objective function over D_i's. We, then, consider an extreme special case, where one of the surgery durations is deterministic and the other one is stochastic. Next, we try to find a more compact expression for objective function, to aid the calculations in forthcoming sections. Using the compact version of objective function, we turn our attention to the case where both surgeries have exponentially distributed random durations. We give optimality equations for this problem. As it would be evident from the structure of these equations, analytical solutions for these optimality equations are very hard to find.

In Section 4.2, we consider the scheduling of three procedures with the motivation of finding a pattern in solution methodology. The flow of this section is constructed in a similar way as of the Section 4.1. We observe that the complexity of expressions increases and the computations become much harder, which gives us a hint about how the problem would evolve when larger numbers of surgeries are considered. In the final section, we consider scheduling problem of n procedures.

4.1. Scheduling of Two Procedures

4.1.1. Objective Function

Objective function for n-procedure problem was given in (3.3) in terms of expectations of b_i random variables, as follows:

$$\alpha_1 \sum_{i=1}^{n} (D_i - E[T_i]) + \alpha_2 \sum_{i=1}^{n-1} E[(b_i)^+] + (\alpha_1 + \alpha_3) E[(b_n)^+]$$

Our purpose is to express the objective function explicitly in terms of D_i, decision variables. Let us denote the objective function of two-procedure problem as $O_2(D_1, D_2)$, where D_1, D_2 are the assigned durations of first and second procedures in the given sequence, respectively. Following the notation introduced in Chapter 3, f_1 and f_2 are the probability density functions of random variables T_1, T_2, actual durations of first and second procedures in the given sequence, respectively. Then,

$$O_2(D_1, D_2) = \alpha_1 (D_1 - E[T_1] + D_2 - E[T_2]) + \alpha_2 E[(b_1)^+] + (\alpha_1 + \alpha_3) E[(b_2)^+] \tag{4.1}$$

where

$$(b_1)^+ = (a_1)^+ = (T_1 - D_1)^+ = \begin{cases} 0 & if \;\; T_1 < D_1 \\ T_1 - D_1 & if \;\; T_1 \geq D_1 \end{cases} \tag{4.2}$$

and

$$(b_2)^+ = \big(a_2 + (b_1)^+\big)^+ = \big(T_2 - D_2 + (T_1 - D_1)^+\big)^+$$

$$= \begin{cases} 0 & if\ \ T_1 < D_1, T_2 < D_2 \\ T_2 - D_2 & if\ \ T_1 < D_1, T_2 \geq D_2 \\ 0 & if\ \ T_1 \geq D_1, T_2 < D_1 + D_2 - T_1 \\ T_2 - D_2 + T_1 - D_1 & if\ \ T_1 \geq D_1, T_2 \geq D_1 + D_2 - T_1 \end{cases} \tag{4.3}$$

Hence, we can write the objective function in integral form as follows:

$$\begin{aligned} O_2(D_1, D_2) &= \alpha_1(D_1 - E[T_1] + D_2 - E[T_2]) \\ &+ \alpha_2 \int_{D_1}^{\infty} (t_1 - D_1) f_1(t_1)\, dt_1 \\ &+ (\alpha_1 + \alpha_3) \int_0^{D_1} \int_{D_2}^{\infty} (t_2 - D_2) f_1(t_1) f_2(t_2)\, dt_2\, dt_1 \\ &+ (\alpha_1 + \alpha_3) \int_{D_1}^{\infty} \int_{D_1 + D_2 - t_1}^{\infty} (t_2 - D_2 + t_1 - D_1) f_1(t_1) f_2(t_2)\, dt_2\, dt_1 \end{aligned} \tag{4.4}$$

4.1.2. Convexity

As seen from (4.4), the objective function is a non-linear function of assigned durations (decision variables), D_1 and D_2. In order to ensure the existence of a global optima for decision variables, the joint convexity of the objective function over D_1 and D_2 is required. In this section, we prove the joint convexity of two-procedure objective function over D_1, D_2 for any given sequence.

Theorem 4.1. *$O_2(D_1, D_2)$ is jointly convex over D_1 and D_2 for any given sequence and any probability density functions f_1, f_2.*

Proof. We begin the proof by finding the first and second order partial derivatives of the objective function with respect to decision variables.

$$\begin{aligned}\frac{\partial O_2(D_1,D_2)}{\partial D_1} &= \alpha_1 - \alpha_2 \int_{D_1}^{\infty} f_1(t_1)\,dt_1 \\ &+ (\alpha_1+\alpha_3)\int_{D_2}^{\infty}(t_2-D_2)f_1(D_1)f_2(t_2)\,dt_2 \\ &- (\alpha_1+\alpha_3)\int_{D_2}^{\infty}(t_2-D_2)f_1(D_1)f_2(t_2)\,dt_2 \\ &- (\alpha_1+\alpha_3)\int_{D_1}^{\infty}\int_{D_1+D_2-t_1}^{\infty} f_1(t_1)f_2(t_2)\,dt_2\,dt_1\end{aligned}$$

$$\begin{aligned}&= \alpha_1 - \alpha_2 \int_{D_1}^{\infty} f_1(t_1)\,dt_1 \\ &- (\alpha_1+\alpha_3)\int_{D_1}^{\infty}\int_{D_1+D_2-t_1}^{\infty} f_1(t_1)f_2(t_2)\,dt_2\,dt_1\end{aligned}$$

$$\begin{aligned}\frac{\partial O_2(D_1,D_2)}{\partial D_2} &= \alpha_1 - (\alpha_1+\alpha_3)\int_{0}^{D_1}\int_{D_2}^{\infty} f_1(t_1)f_2(t_2)\,dt_2\,dt_1 \\ &- (\alpha_1+\alpha_3)\int_{D_1}^{\infty}\int_{D_1+D_2-t_1}^{\infty} f_1(t_1)f_2(t_2)\,dt_2\,dt_1\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 O_2(D_1,D_2)}{\partial D_1^2} &= \alpha_2 f_1(D_1) + (\alpha_1+\alpha_3)\int_{D_2}^{\infty} f_1(D_1)f_2(t_2)\,dt_2 \\ &+ (\alpha_1+\alpha_3)\int_{D_1}^{\infty} f_1(t_1)f_2(D_1+D_2-t_1)\,dt_1\end{aligned}$$

$$\frac{\partial^2 O_2(D_1,D_2)}{\partial D_1 \partial D_2} = (\alpha_1+\alpha_3)\int_{D_1}^{\infty} f_1(t_1)f_2(D_1+D_2-t_1)\,dt_1$$

$$\begin{aligned}\frac{\partial^2 O_2(D_1,D_2)}{\partial D_2^2} &= (\alpha_1+\alpha_3)\int_{0}^{D_1} f_1(t_1)f_2(D_2)\,dt_1 \\ &+ (\alpha_1+\alpha_3)\int_{D_1}^{\infty} f_1(t_1)f_2(D_1+D_2-t_1)\,dt_1\end{aligned}$$

The Hessian matrix below should be positive semi-definite for joint convexity.

$$\mathbf{H} = \begin{bmatrix} \frac{\partial^2 O_2}{\partial D_1^2} & \frac{\partial^2 O_2}{\partial D_1 \partial D_2} \\ \frac{\partial^2 O_2}{\partial D_1 \partial D_2} & \frac{\partial^2 O_2}{\partial D_2^2} \end{bmatrix}$$

Let $\mathbf{z}$ be a column vector of length 2 whose elements are $z_1, z_2 \in \mathbb{R}$. Then, the Hessian matrix is positive semi-definite iff $\mathbf{z}^T\mathbf{H}\mathbf{z} \geq 0, \forall z_1, z_2 \in \mathbb{R}$.

$$\begin{aligned}
\mathbf{z}^T\mathbf{H}\mathbf{z} &= z_1^2 \frac{\partial^2 O_2}{\partial D_1^2} + 2z_1 z_2 \frac{\partial^2 O_2}{\partial D_1 \partial D_2} + z_2^2 \frac{\partial^2 O_2}{\partial D_2^2} \\
&= z_1^2 \alpha_2 f_1(D_1) + z_1^2(\alpha_1 + \alpha_3)\left(\int_{D_2}^{\infty} f_1(D_1) f_2(t_2)\, dt_2\right) \\
&+ z_1^2(\alpha_1 + \alpha_3)\left(\int_{D_1}^{\infty} f_1(t_1) f_2(D_1 + D_2 - t_1)\, dt_1\right) \\
&+ 2z_1 z_2(\alpha_1 + \alpha_3)\left(\int_{D_1}^{\infty} f_1(t_1) f_2(D_1 + D_2 - t_1)\, dt_1\right) \\
&+ z_2^2(\alpha_1 + \alpha_3)\left(\int_{0}^{D_1} f_1(t_1) f_2(D_2)\, dt_1\right) \\
&+ z_2^2(\alpha_1 + \alpha_3)\left(\int_{D_1}^{\infty} f_1(t_1) f_2(D_1 + D_2 - t_1)\, dt_1\right)
\end{aligned}$$

$$\begin{aligned}
&= z_1^2 \alpha_2 f_1(D_1) + z_1^2(\alpha_1 + \alpha_3)\left(\int_{D_2}^{\infty} f_1(D_1) f_2(t_2)\, dt_2\right) \\
&+ z_2^2(\alpha_1 + \alpha_3)\left(\int_{0}^{D_1} f_1(t_1) f_2(D_2)\, dt_1\right) \\
&+ (z_1 + z_2)^2(\alpha_1 + \alpha_3)\left(\int_{D_1}^{\infty} f_1(t_1) f_2(D_1 + D_2 - t_1)\, dt_1\right).
\end{aligned}$$

Since $z_1^2, z_2^2, (z_1 + z_2)^2, \alpha_1, \alpha_2, \alpha_3, f_1$ and f_2 are non-negative by definition, $\Rightarrow \mathbf{z}^T\mathbf{H}\mathbf{z} \geq 0$. $\Rightarrow O_2(D_1, D_2)$ is jointly convex over D_1 and D_2. □

4.1.3. Optimum Sequencing For A Special Case

Pinedo [49] describes several sequencing policies for stochastic scheduling problems. However, the policies he describe do not apply directly to the most general version of our problem. In this subsection, we consider a special case which helps us explore a sequencing heuristic for the most general version of the problem.

Consider the two-procedure OR scheduling problem where the duration for Procedure I is deterministic, say M_I, and the duration for Procedure II is a nonnegative and continuous random variable. We will show that it is always optimal to schedule Procedure I first. In order to prove that statement, we first prove two lemmas.

Lemma 4.1. *Assume that Procedure I is scheduled first. Then, it is optimal to set* $D_I = M_I$.

Proof. Following the notation given in (4.4), the objective function can be written as follows, assuming the first procedure to be scheduled is Procedure I:

$$\begin{aligned}
O_2(D_I, D_{II}) = {} & \alpha_1 \int_0^{D_I} (-t_I + D_I) f_I(t_I)\, dt_I \\
& + \alpha_2 \int_{D_I}^{\infty} (t_I - D_I) f_I(t_I)\, dt_I \\
& + \alpha_1 \int_0^{D_I} \int_0^{D_{II}} (-t_{II} + D_{II}) f_I(t_I) f_{II}(t_{II})\, dt_{II}\, dt_I \\
& + \alpha_3 \int_0^{D_I} \int_{D_{II}}^{\infty} (t_{II} - D_{II}) f_I(t_I) f_{II}(t_{II})\, dt_{II}\, dt_I \\
& + \alpha_1 \int_{D_I}^{\infty} \int_0^{D_I + D_{II} - t_I} (-t_{II} + D_{II} - t_I + D_I) f_I(t_I) f_{II}(t_{II})\, dt_{II}\, dt_I \\
& + \alpha_3 \int_{D_I}^{\infty} \int_{D_I + D_{II} - t_I}^{\infty} (t_{II} - D_{II} + t_I - D_I) f_I(t_I) f_{II}(t_{II})\, dt_{II}\, dt_I
\end{aligned}$$

With the assumption that Procedure I has fixed duration M_I, one can simplify this function for two cases:

Case i: If $D_I \le M_I$, then objective function is:

$$\begin{aligned} O_2(D_I, D_{II}) &= \alpha_2(M_I - D_I) \\ &\quad + \alpha_1 \int_0^{D_I + D_{II} - M_I} (-t_{II} + D_{II} - M_I + D_I) f_{II}(t_{II})\, dt_{II} \\ &\quad + \alpha_3 \int_{D_I + D_{II} - M_I}^{\infty} (t_{II} - D_{II} + M_I - D_I) f_{II}(t_{II})\, dt_{II} \end{aligned}$$

It is easy to see that this function attains its minimum value at $D_I = M_I$ (simply define a new variable that is equal to $D_I + D_{II}$).

Case ii: If $D_I \ge M_I$, then objective function is:

$$\begin{aligned} O_2(D_I, D_{II}) &= \alpha_1(-M_I + D_I) \\ &\quad + \alpha_1 \int_0^{D_{II}} (-t_{II} + D_{II}) f_{II}(t_{II})\, dt_{II} \\ &\quad + \alpha_3 \int_{D_{II}}^{\infty} (t_{II} - D_{II}) f_{II}(t_{II})\, dt_{II}, \end{aligned}$$

which is clearly increasing in D_I.

Based on this case analysis, one can show that it is optimal to set $D_I = M_I$. □

Lemma 4.2. *Assume that Procedure II is scheduled first. Then, it is optimal to set* $D_I = M_I$.

Proof. The objective function, $O_2(D_{II}, D_I)$, can be written as follows:

$$\begin{aligned} O_2(D_{II}, D_I) &= \alpha_1 \int_0^{D_{II}} (-t_{II} + D_{II}) f_{II}(t_{II})\, dt_{II} \\ &\quad + \alpha_2 \int_{D_{II}}^{\infty} (t_{II} - D_{II}) f_{II}(t_{II})\, dt_{II} \\ &\quad + \alpha_1 \int_0^{D_{II}} \int_0^{D_I} (-t_I + D_I) f_{II}(t_{II}) f_I(t_I)\, dt_I\, dt_{II} \end{aligned}$$

$$
\begin{aligned}
&+ \alpha_3 \int_0^{D_{II}} \int_{D_I}^{\infty} (t_I - D_I) f_{II}(t_{II}) f_I(t_I) \, dt_I \, dt_{II} \\
&+ \alpha_1 \int_{D_{II}}^{\infty} \int_0^{D_{II}+D_I-t_{II}} (-t_I + D_I - t_{II} + D_{II}) f_{II}(t_{II}) f_I(t_I) \, dt_I \, dt_{II} \\
&+ \alpha_3 \int_{D_{II}}^{\infty} \int_{D_{II}+D_I-t_{II}}^{\infty} (t_I - D_I + t_{II} - D_{II}) f_{II}(t_{II}) f_I(t_I) \, dt_I \, dt_{II}
\end{aligned}
$$

We will again find the optimal value of D_I following a case analysis and assuming Procedure I duration is M_I.

Case *i*: If $D_I \leq M_I$, then the objective function is:

$$
\begin{aligned}
O_2(D_{II}, D_I) = {} & \alpha_1 \int_0^{D_{II}} (-t_{II} + D_{II}) f_{II}(t_{II}) \, dt_{II} \\
&+ \alpha_2 \int_{D_{II}}^{\infty} (t_{II} - D_{II}) f_{II}(t_{II}) \, dt_{II} \\
&+ \alpha_3 \int_0^{D_{II}} (M_I - D_I) f_{II}(t_{II}) \, dt_{II} \\
&+ \alpha_3 \int_{D_{II}}^{\infty} (t_{II} - D_{II} + M_I - D_I) f_{II}(t_{II}) \, dt_{II},
\end{aligned}
$$

which is clearly decreasing in D_I.

Case *ii*: If $D_I \geq M_I$, then the objective function is:

$$
\begin{aligned}
O_2(D_{II}, D_{II}) = {} & \alpha_1 \int_0^{D_{II}} (-t_{II} + D_{II}) f_{II}(t_{II}) \, dt_{II} \\
&+ \alpha_2 \int_{D_{II}}^{\infty} (t_{II} - D_{II}) f_{II}(t_{II}) \, dt_{II} \\
&+ \alpha_1 \int_0^{D_{II}} (-M_I + D_I) f_{II}(t_{II}) \, dt_{II} \\
&+ \alpha_1 \int_{D_{II}}^{D_I+D_{II}-M_I} (-t_{II} - M_I + D_I + D_{II}) f_{II}(t_{II}) \, dt_{II} \\
&+ \alpha_3 \int_{D_I+D_{II}-M_I}^{\infty} (t_{II} - D_{II} + M_I - D_{II}) f_{II}(t_{II}) \, dt_{II},
\end{aligned}
$$

In this case, one can show that:

$$\frac{\partial O_2(D_{II}, D_I)}{\partial D_I} = \alpha_1 F_{II}(D_I + D_{II} - M_I) - \alpha_3(1 - F_{II}(D_I + D_{II} - M_I))$$
$$\frac{\partial O_2(D_{II}, D_I)}{\partial D_{II}} = \alpha_1 F_{II}(D_I + D_{II} - M_I) - \alpha_2(1 - F_{II}(D_{II})) - \alpha_3(1 - F_{II}(D_I + D_{II} - M_I)$$

and $O_2(D_{II}, D_I)$ is jointly convex in D_I and D_{II} (from Theorem 4.1). Observe that:

$$\frac{\partial O_2(D_{II}, D_I)}{\partial D_I} \geq \frac{\partial O_2(D_{II}, D_I)}{\partial D_{II}}.$$

Hence two cases are possible at optimality: either one of these derivatives is equal to 0.

If $\frac{\partial O_2(D_{II}, D_I)}{\partial D_I} = 0$, then we have:

$$D_I + D_{II} = M_I + F_{II}^{-1}\left(\frac{\alpha_3}{\alpha_1 + \alpha_3}\right).$$

In this case, the objective function is decreasing in D_{II}, which means D_{II} has to equal to its maximum value, equivalently D_I must be equal to its minimum value M_I.

Otherwise, $\frac{\partial O_2(D_{II}, D_I)}{\partial D_{II}} = 0$, implying that the objective value is increasing in D_I and that $D_I = M_I$. This completes the proof that it is optimal to set $D_I = M_I$. $\square$

Now, we state the theorem that shows that it is optimal to schedule the procedure with certain duration, Procedure I, first for all parameters.

Theorem 4.2. *Assume there are two procedures to be scheduled and Procedure I has a fixed duration. Then, it is optimal to schedule Procedure I first. It is uniquely so, if* $\alpha_2 > 0$.

Proof. After observing Lemma 4.1 and 4.2, the proof is simply by showing that the objective function for scheduling Procedure I first is lower, i.e. $O_2(M_I, D_{II}) \leq O_2(D_{II}, M_I)$. This is easy to see, once we write these functions explicitly:

$$O_2(M_I, D_{II}) = \alpha_1 \int_0^{D_{II}} (-t_{II} + D_{II}) f_{II}(t_{II})\, dt_{II} + \alpha_3 \int_{D_{II}}^{\infty} (t_{II} - D_{II}) f_{II}(t_{II})\, dt_{II}$$
$$O_2(D_{II}, M_I) = \alpha_1 \int_0^{D_{II}} (-t_{II} + D_{II}) f_{II}(t_{II})\, dt_{II} + (\alpha_2 + \alpha_3) \int_{D_{II}}^{\infty} (t_{II} - D_{II}) f_{II}(t_{II})\, dt_{II}$$

□

Theorem 4.2, is of a particular importance because it gives us a hint about optimum sequencing of different procedures. Theorem applies to the case when we have two procedures: one with a deterministic duration, the other with a continuous, nonnegative random duration. The result tells us that the procedure with deterministic duration should be scheduled first. One can relate this result to the "Shortest Variance First" sequencing policy, described in [49]. Theorem also eliminates other possible sequencing policies, such as "Shortest Expectation First", because even if the deterministic procedure has larger duration than the expectation of the stochastic one; the procedure to be scheduled first is still the one with deterministic duration. Obviously, this result is not sufficient to state that "Shortest Variance First" sequencing policy is always the best sequencing policy; however, it will provide a basis in our numerical analyses while searching for an optimum sequence.

4.1.4. Objective Function in Compact Form

Objective function was expressed in terms of expectations of the auxiliary random variables b_i^+'s in (3.10) and in (4.4), an explicit expression in terms of D_1 and D_2 was given in integral form. In this subsection, we simplify the explicit expression even further to aid simplifying the computations in subsequent sections. We prove two lemmas to state our Theorem for the objective function.

Lemma 4.3.

$$E[(b_1)^+] = \int_{D_1}^{\infty} (1 - F_1(t_1))\, dt_1, \tag{4.5}$$

where $F_1(t_1)$ is the cdf of first surgery's duration.

Proof.

$$E[(b_1)^+] = E[(a_1)^+] = E[(T_1 - D_1)^+] = \int_{D_1}^{\infty} (t_1 - D_1) f_1(t_1)\, dt_1,$$

where $f_1(t_1)$ is the pdf of first surgery's duration. Apply integration by parts:

$$\begin{aligned} &\text{Call } u = -(t_1 - D_1) \text{ and } dv = -f_1(t_1) dt_1 \\ &\Rightarrow du = -dt_1,\ v = 1 - F_1(t_1) \end{aligned}$$

$$\begin{aligned} \int_{D_1}^{\infty} (t_1 - D_1) f_1(t_1)\, dt_1 = {} & [-(t_1 - D_1).(1 - F_1(t_1))]_{t_1 = D1}^{\infty} \\ & + \int_{D_1}^{\infty} (1 - F_1(t_1))\, dt_1. \end{aligned}$$

The first term within brackets vanishes at the limits, which yields:

$$E[(b_1)^+] = \int_{D_1}^{\infty} (1 - F_1(t_1))\, dt_1.$$

□

Lemma 4.4.

$$E[(b_2)^+] = \int_{D_2}^{\infty} (1 - F_2(t_2))\, dt_2 + \int_{D_1}^{\infty} \int_{D_1 + D_2 - t_1}^{\infty} (1 - F_1(t_1)) f_2(t_2)\, dt_2\, dt_1 \tag{4.6}$$

where $F_i(t_i)$ is the cdf and $f_i(t_i)$ is the pdf of i^{th} surgery's duration.

Proof.

$$(b_2)^+ = \begin{cases} (T_2 - D_2) & ; T_1 < D1,\ T_2 \geq D_2 \\ (T_2 - D_2 + T_1 - D_1) & ; T_1 \geq D1,\ T_2 \geq D_1 + D_2 - T_1 \\ 0 & ; elsewhere \end{cases}$$

Thus, we can write:

$$\begin{aligned} E[(b_2)^+] = &\int_0^{D_1} \int_{D_2}^{\infty} (t_2 - D_2) f_1(t_1) f_2(t_2)\, dt_2\, dt_1 \\ &+ \int_{D_1}^{\infty} \int_{D_1+D_2-t_1}^{\infty} (t_2 - D_2 + t_1 - D_1) f_1(t_1) f_2(t_2)\, dt_2\, dt_1 \end{aligned}$$

Apply integration by parts to the second integral:

$$\begin{aligned} &\text{Call } u = -\left(\int_{D_1+D_2-t_1}^{\infty} (t_2 - D_2 + t_1 - D_1) f_2(t_2)\, dt_2\right) \text{ and } dv = -f_1(t_1) dt_1 \\ &\Rightarrow du = -\left(\int_{D_1+D_2-t_1}^{\infty} f_2(t_2)\, dt_2\right) dt_1 \\ &\Rightarrow v = 1 - F_1(t_1) \end{aligned}$$

$$\begin{aligned} uv &= \left[-\left(\int_{D_1+D_2-t_1}^{\infty} (t_2 - D_2 + t_1 - D_1) f_2(t_2)\, dt_2\right)(1 - F_1(t_1))\right]_{t_1=D_1}^{\infty} \\ &= (1 - F_1(D_1)) \int_{D_2}^{\infty} (t_2 - D_2) f_2(t_2)\, dt_2 \\ &= \int_{D1}^{\infty} \int_{D_2}^{\infty} (t_2 - D_2) f_1(t_1) f_2(t_2)\, dt_2\, dt_1 \end{aligned}$$

$$\int vdu = -\int_{D_1}^{\infty} \int_{D_1+D_2-t_1}^{\infty} (1 - F_1(t_1)) f_2(t_2)\, dt_2\, dt_1$$

$$\begin{aligned}E[(b_2)^+] &= \int_0^{D_1}\int_{D_2}^{\infty}(t_2 - D_2) f_1(t_1) f_2(t_2)\, dt_2\, dt_1 + \int_{D1}^{\infty}\int_{D_2}^{\infty}(t_2 - D_2) f_1(t_1) f_2(t_2)\, dt_2\, dt_1 \\ &\quad + \int_{D_1}^{\infty}\int_{D_1+D_2-t_1}^{\infty}(1 - F_1(t_1)) f_2(t_2)\, dt_2\, dt_1\end{aligned}$$

$$\begin{aligned}&= \int_{D_2}^{\infty}(t_2 - D_2)\, dt_2 + \int_{D_1}^{\infty}\int_{D_1+D_2-t_1}^{\infty}(1 - F_1(t_1)) f_2(t_2)\, dt_2\, dt_1 \\ &= \int_{D_2}^{\infty}(1 - F_2(t_2))\, dt_2 + \int_{D_1}^{\infty}\int_{D_1+D_2-t_1}^{\infty}(1 - F_1(t_1)) f_2(t_2)\, dt_2\, dt_1\end{aligned}$$

where the last equality follows from Lemma 4.3. □

Now, we state the proposition that shows the objective function for two-procedure problem as a function of decision variables.

Proposition 4.1.

$$\begin{aligned}O_2(D_1, D_2) &= \alpha_1(D_1 - E[T_1] + D_2 - E[T_2]) \\ &\quad + \alpha_2 \int_{D_1}^{\infty}(1 - F_1(t_1))\, dt_1 \\ &\quad + (\alpha_1 + \alpha_3)\int_{D_2}^{\infty}(1 - F_2(t_2))\, dt_2 \\ &\quad + (\alpha_1 + \alpha_3)\int_{D_1}^{\infty}\int_{D_1+D_2-t_1}^{\infty}(1 - F_1(t_1)) f_2(t_2)\, dt_2\, dt_1\end{aligned} \tag{4.7}$$

Proof. Proof is straightforward by replacing the terms in Theorem 3.1, Equation (3.10), with the Equations (4.5) and (4.6). □

4.1.5. Exponentially Distributed Surgery Durations

In this subsection, we consider the special case where both surgeries have exponentially distributed random durations. We first, find the explicit expression for the objective function, then find the optimality equations.

Proposition 4.2.

$$\begin{aligned}O(\lambda_1, \lambda_2; D_1, D_2) = {} & \alpha_1 \left(D_1 - \frac{1}{\lambda_1} + D_2 - \frac{1}{\lambda_2} \right) \\ & + \alpha_2 \frac{1}{\lambda_1} e^{-\lambda_1 D_1} \\ & + (\alpha_1 + \alpha_3) \frac{1}{\lambda_2} e^{-\lambda_2 D_2} \\ & + (\alpha_1 + \alpha_3) \frac{1}{(\lambda_1 - \lambda_2)} e^{-\lambda_1 D_1} e^{-\lambda_2 D_2} \\ & - (\alpha_1 + \alpha_3) \frac{\lambda_2}{\lambda_1(\lambda_1 - \lambda_2)} e^{-\lambda_1 (D_1 + D_2)}\end{aligned} \tag{4.8}$$

$\forall \lambda_1, \lambda_2 > 0$ s.t. $\lambda_1 \neq \lambda_2$, where λ_1, λ_2 are the rate parameters of the distributions of first and second surgeries in sequence, respectively.

Proof. From (4.7), we have:

$$\begin{aligned}O_2(D_1, D_2) = {} & \alpha_1 (D_1 - E[T_1] + D_2 - E[T_2]) \\ & + \alpha_2 \int_{D_1}^{\infty} (1 - F_1(t_1))\, dt_1 \\ & + (\alpha_1 + \alpha_3) \int_{D_2}^{\infty} (1 - F_2(t_2))\, dt_2 \\ & + (\alpha_1 + \alpha_3) \int_{D_1}^{\infty} \int_{D_1 + D_2 - t_1}^{\infty} (1 - F_1(t_1)) f_2(t_2)\, dt_2\, dt_1\end{aligned}$$

Replace the distribution functions with their explicit expressions for exponential distribution:

$$E[T_1] = \frac{1}{\lambda_1},\ E[T_2] = \frac{1}{\lambda_2}$$

$$\begin{aligned}\int_{D_1}^{\infty} (1 - F_1(t_1))\, dt_1 &= \int_{D_1}^{\infty} e^{-\lambda_1 t_1}\, dt_1 = \frac{1}{\lambda_1} e^{-\lambda_1 D_1} \\ \int_{D_2}^{\infty} (1 - F_2(t_2))\, dt_2 &= \int_{D_2}^{\infty} e^{-\lambda_2 t_2}\, dt_2 = \frac{1}{\lambda_2} e^{-\lambda_2 D_2}\end{aligned}$$

$$\begin{aligned}
&\int_{D_1}^{\infty}\int_{D_1+D_2-t_1}^{\infty}(1-F_1(t_1))f_2(t_2)\,dt_2\,dt_1\\
&=\int_{D_1}^{(D_1+D_2)}\int_{D_1+D_2-t_1}^{\infty} e^{-\lambda_1 t_1}\lambda_2 e^{-\lambda_2 t_2}\,dt_2\,dt_1\\
&+\int_{(D_1+D_2)}^{\infty}\int_{0}^{\infty} e^{-\lambda_1 t_1}\lambda_2 e^{-\lambda_2 t_2}\,dt_2\,dt_1
\end{aligned}$$

$$\begin{aligned}
&= e^{-\lambda_2(D_1+D_2)}\int_{D_1}^{(D_1+D_2)} e^{-(\lambda_1-\lambda_2)t_1}\,dt_1+\frac{1}{\lambda_1}e^{-\lambda_1(D_1+D_2)}\\
&=\frac{1}{(\lambda_1-\lambda_2)}e^{-\lambda_1 D_1}e^{-\lambda_2 D_2}+\left[-\frac{1}{(\lambda_1-\lambda_2)}+\frac{1}{\lambda_1}\right]e^{-\lambda_1(D_1+D_2)}\\
&=\frac{1}{(\lambda_1-\lambda_2)}e^{-\lambda_1 D_1}e^{-\lambda_2 D_2}-\frac{\lambda_2}{\lambda_1(\lambda_1-\lambda_2)}e^{-\lambda_1(D_1+D_2)}
\end{aligned}$$

Placing these terms in the objective function, we get (4.8). □

Proposition 4.3.

$$\begin{aligned}
O(\lambda; D_1, D_2) = \alpha_1\left(D_1+D_2-\frac{2}{\lambda}\right)+\alpha_2\frac{1}{\lambda}e^{-\lambda D_1}+(\alpha_1+\alpha_3)\frac{1}{\lambda}e^{-\lambda D_2}\\
+(\alpha_1+\alpha_3)\frac{1}{\lambda}e^{-\lambda(D_1+D_2)}
\end{aligned}\tag{4.9}$$

$\forall\lambda > 0$, where $\lambda_1=\lambda_2=\lambda$.

Proof. From the proof of Proposition 4.2, we have:

$$\begin{aligned}
\int_{D_1}^{\infty}\int_{D_1+D_2-t_1}^{\infty}(1-F_1(t_1))f_2(t_2)\,dt_2\,dt_1 &= e^{-\lambda_2(D_1+D_2)}\int_{D_1}^{(D_1+D_2)} e^{-(\lambda_1-\lambda_2)t_1}\,dt_1+\frac{1}{\lambda_1}e^{-\lambda_1(D_1+D_2)}\\
&=\frac{1}{\lambda}e^{-\lambda(D_1+D_2)}
\end{aligned}$$

This term is replaces the two terms in the fourth and fifth lines of the objective function in (4.8), we get (4.9). □

Given the explicit objective function and the fact that objective function was shown to be jointly convex over D_1 and D_2, we can say that this objective function has a global optimum, which can be found by solving the optimality equations obtained taking the partial derivatives with respect to D_1 and D_2. Let us, now, state these optimality equations:

$$\begin{aligned}\frac{\partial O_2(D_1,D_2)}{\partial D_1} &= \alpha_1 - \alpha_2 e^{-\lambda_1 D_1} \\ &- (\alpha_1+\alpha_3)\frac{\lambda_1}{(\lambda_1-\lambda_2)}e^{-\lambda_1 D_1}e^{-\lambda_2 D_2} + (\alpha_1+\alpha_3)\frac{\lambda_2}{(\lambda_1-\lambda_2)}e^{-\lambda_1(D_1+D_2)} = 0\end{aligned} \tag{4.10}$$

$$\begin{aligned}\frac{\partial O_2(D_1,D_2)}{\partial D_2} &= \alpha_1 - (\alpha_1+\alpha_3) e^{-\lambda_2 D_2} \\ &- (\alpha_1+\alpha_3)\frac{\lambda_2}{(\lambda_1-\lambda_2)}e^{-\lambda_1 D_1}e^{-\lambda_2 D_2} + (\alpha_1+\alpha_3)\frac{\lambda_2}{(\lambda_1-\lambda_2)}e^{-\lambda_1(D_1+D_2)} = 0\end{aligned} \tag{4.11}$$

Subtracting the (4.10) from (4.11) we get:

$$\begin{aligned}&\alpha_2 e^{-\lambda_1 D_1} - (\alpha_1+\alpha_3)e^{-\lambda_2 D_2} + (\alpha_1+\alpha_3)e^{-\lambda_1 D_1}e^{-\lambda_2 D_2} = 0 \\ &\Rightarrow e^{-\lambda_1 D_1} = \frac{(\alpha_1+\alpha_3)e^{-\lambda_2 D_2}}{\alpha_2 + (\alpha_1+\alpha_3)e^{-\lambda_2 D_2}} \\ &or, equivalently: \\ &\Rightarrow e^{-\lambda_2 D_2} = \frac{\alpha_2 e^{-\lambda_1 D_1}}{(\alpha_1+\alpha_3)(1-e^{-\lambda_1 D_1})}\end{aligned} \tag{4.12}$$

(4.12) gives us the relationship between the optimum values of D_1 and D_2, even if they do not give the explicit solutions. One can see that analytical solution of this set of two non-linear equations is quite hard to find. However, it is easily solvable via numerical methods. Therefore, in this thesis work, we do not focus on analytical solutions, but we focus more on the implications of these optimum solutions. One implication is quite evident from the relationship expressions shown above. As α_2 (the penalty parameter for waiting time) approaches to zero, optimum value of D_1 also approaches to zero, whereas, the optimum value of D_2 approaches to infinity. This result is very intuitive. When there is no cost incurred by waiting time, we assign minimum possible duration (zero) to first surgery in order to decrease the idle time cost, and assign maximum possible duration (infinity) to second surgery in order to decrease the overtime cost.

4.2. Scheduling of Three Procedures

4.2.1. Objective Function

Following the notation given by (3.10), the objective function for three-procedure problem, $O_3(D_1, D_2, D_3)$, can be expressed in terms of decision variables, D_is, as follows:

$$\begin{aligned} O_3(D_1, D_2, D_3) &= \alpha_1(D_1 - E[T_1] + D_2 - E[T_2] + D_3 - E[T_3]) \\ &+ \alpha_2 E[(b_1)^+] \\ &+ \alpha_2 E[(b_2)^+] \\ &+ (\alpha_1 + \alpha_3) E[(b_3)^+] \end{aligned} \tag{4.13}$$

where $(b_1)^+$ and $(b_2)^+$ were given in (4.2) and (4.3), respectively. Let us express $(b_2)^+$, in terms of decision variables, now:

$$\begin{aligned} (b_3)^+ &= \big(a_3 + (b_2)^+\big)^+ \\ &= \Big(T_3 - D_3 + \big(T_2 - D_2 + (T_1 - D_1)^+\big)^+\Big)^+ \end{aligned}$$

$$
= \begin{cases}
0 & if \;\; T_1 < D_1, T_2 < D_2, T_3 < D_3 \\
T_3 - D_3 & if \;\; T_1 < D_1, T_2 < D_2, T_3 \geq D_3 \\
0 & if \;\; T_1 < D_1, T_2 \geq D_2, T_3 < D_2 + D_3 - t_2 \\
T_3 - D_3 + T_2 - D_2 & if \;\; T_1 < D_1, T_2 \geq D_2, T_3 \geq D_2 + D_3 - t_2 \\
0 & if \;\; T_1 \geq D_1, T_2 < D_1 + D_2 - t_1, T_3 < D_3 \\
T_3 - D_3 & if \;\; T_1 \geq D_1, T_2 < D_1 + D_2 - t_1, T_3 \geq D_3 \\
0 & if \;\; T_1 \geq D_1, T_2 \geq D_1 + D_2, \\
 & T_3 < D_1 + D_2 + D_3 - t_1 - t_2 \\
T_3 - D_3 + T_2 - D_2 + T_1 - D_1 & if \;\; T_1 \geq D_1, T_2 \geq D_1 + D_2 - t_1, \\
 & T_3 \geq D_1 + D_2 + D_3 - t_1 - t_2
\end{cases} \tag{4.14}
$$

Hence, we can write the objective function in integral form as follows:

$$
\begin{aligned}
& O_3(D_1, D_2, D_3) \\
& = \alpha_1(D_1 - E[T_1] + D_2 - E[T_2] + D_3 - E[T_3]) \\
& + \alpha_2 \int_{D_1}^{\infty} (t_1 - D_1) f_1(t_1)\, dt_1 \\
& + \alpha_2 \int_{0}^{D_1} \int_{D_2}^{\infty} (t_2 - D_2) f_1(t_1) f_2(t_2)\, dt_2\, dt_1 \\
& + \alpha_2 \int_{D_1}^{\infty} \int_{(D_1+D_2-t_1)}^{\infty} (t_2 - D_2 + t_1 - D_1) f_1(t_1) f_2(t_2)\, dt_2\, dt_1 \\
& + (\alpha_1 + \alpha_3) \int_{0}^{D_1} \int_{0}^{D_2} \int_{D_3}^{\infty} (t_3 - D_3) f_1(t_1) f_2(t_2) f_3(t_3)\, dt_3\, dt_2\, dt_1 \\
& + (\alpha_1 + \alpha_3) \int_{0}^{D_1} \int_{D_2}^{\infty} \int_{(D_2+D_3-t_2)}^{\infty} (t_3 - D_3 + t_2 - D_2) f_1(t_1) f_2(t_2) f_3(t_3)\, dt_3\, dt_2\, dt_1 \\
& + (\alpha_1 + \alpha_3) \int_{D_1}^{\infty} \int_{0}^{(D_1+D_2-t_1)} \int_{D_3}^{\infty} (t_3 - D_3) f_1(t_1) f_2(t_2) f_3(t_3)\, dt_3\, dt_2\, dt_1 \\
& + (\alpha_1 + \alpha_3) \int_{D_1}^{\infty} \int_{t_2^*}^{\infty} \int_{t_3^*}^{\infty} (t_3 - D_3 + t_2 - D_2 + t_1 - D_1) f_1(t_1) f_2(t_2) f_3(t_3)\, dt_3\, dt_2\, dt_1
\end{aligned} \tag{4.15}
$$

where $t_2^* = D_1 + D_2 - t_1$, and $t_3^* = D_1 + D_2 + D_3 - t_1 - t_2$.

4.2.2. Convexity

The following theorem states the joint convexity for the three-procedure problem.

Theorem 4.3. *$O_3(D_1, D_2, D_3)$ is jointly convex over D_1, D_2 and D_3 for any given sequence and any probability density functions f_1, f_2, f_3.*

Proof.

$$\begin{aligned}\frac{\partial O_3(D_1,D_2,D_3)}{\partial D_1} &= \alpha_1 - \alpha_2 \int_{D_1}^{\infty} f_1(t_1)\,dt_1 - \alpha_2 \int_{D_1}^{\infty}\int_{D_1+D_2-t_1}^{\infty} f_1(t_1)f_2(t_2)\,dt_2\,dt_1 \\ &- (\alpha_1+\alpha_3)\int_{D_1}^{\infty}\int_{(D_1+D_2-t_1)}^{\infty}\int_{(D_1+D_2+D_3-t_1-t_2)}^{\infty} f_1(t_1)f_2(t_2)f_3(t_3)\,dt_3\,dt_2\,dt_1\end{aligned}$$

$$\begin{aligned}\frac{\partial O_3(D_1,D_2,D_3)}{\partial D_2} &= \alpha_1 - \alpha_2 \int_{0}^{D_1}\int_{D_2}^{\infty} f_1(t_1)f_2(t_2)\,dt_2\,dt_1 - \alpha_2 \int_{D_1}^{\infty}\int_{D_1+D_2-t_1}^{\infty} f_1(t_1)f_2(t_2)\,dt_2\,dt_1 \\ &- (\alpha_1+\alpha_3)\int_{0}^{D_1}\int_{D_2}^{\infty}\int_{(D_2+D_3-t_2)}^{\infty} f_1(t_1)f_2(t_2)f_3(t_3)\,dt_3\,dt_2\,dt_1 \\ &- (\alpha_1+\alpha_3)\int_{D_1}^{\infty}\int_{(D_1+D_2-t_1)}^{\infty}\int_{(D_1+D_2+D_3-t_1-t_2)}^{\infty} f_1(t_1)f_2(t_2)f_3(t_3)\,dt_3\,dt_2\,dt_1\end{aligned}$$

$$\begin{aligned}\frac{\partial O_3(D_1,D_2,D_3)}{\partial D_3} &= \alpha_1 - (\alpha_1+\alpha_3)\int_{0}^{D_1}\int_{0}^{D_2}\int_{D_3}^{\infty} f_1(t_1)f_2(t_2)f_3(t_3)\,dt_3\,dt_2\,dt_1 \\ &- (\alpha_1+\alpha_3)\int_{0}^{D_1}\int_{D_2}^{\infty}\int_{(D_2+D_3-t_2)}^{\infty} f_1(t_1)f_2(t_2)f_3(t_3)\,dt_3\,dt_2\,dt_1 \\ &- (\alpha_1+\alpha_3)\int_{D_1}^{\infty}\int_{0}^{(D_1+D_2-t_1)}\int_{D_3}^{\infty} f_1(t_1)f_2(t_2)f_3(t_3)\,dt_3\,dt_2\,dt_1 \\ &- (\alpha_1+\alpha_3)\int_{D_1}^{\infty}\int_{(D_1+D_2-t_1)}^{\infty}\int_{(D_1+D_2+D_3-t_1-t_2)}^{\infty} f_1(t_1)f_2(t_2)f_3(t_3)\,dt_3\,dt_2\,dt_1\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 O_3(D_1, D_2, D_3)}{\partial D_1^2} &= \alpha_2 f_1(D_1) + \alpha_2 \int_{D_2}^{\infty} f_1(D_1) f_2(t_2)\, dt_2 + \alpha_2 \int_{D_1}^{\infty} f_1(t_1) f_2(D_1 + D_2 - t_1)\, dt_1 \\
&+ (\alpha_1 + \alpha_3) \int_{D_2}^{\infty} \int_{(D_2 + D_3 - t_2)}^{\infty} f_1(D_1) f_2(t_2) f_3(t_3)\, dt_3\, dt_2 \\
&+ (\alpha_1 + \alpha_3) \int_{D_1}^{\infty} \int_{D_3}^{\infty} f_1(t_1) f_2(D_1 + D_2 - t_1) f_3(t_3)\, dt_3\, dt_1 \\
&+ (\alpha_1 + \alpha_3) \int_{D_1}^{\infty} \int_{(D_1 + D_2 - t_1)}^{\infty} f_1(t_1) f_2(t_2) f_3(D_1 + D_2 + D_3 - t_1 - t_2)\, dt_2\, dt_1
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 O_3(D_1, D_2, D_3)}{\partial D_1 \partial D_2} &= \alpha_2 \int_{D_1}^{\infty} f_1(t_1) f_2(D_1 + D_2 - t_1)\, dt_1 \\
&+ (\alpha_1 + \alpha_3) \int_{D_1}^{\infty} \int_{D_3}^{\infty} f_1(t_1) f_2(D_1 + D_2 - t_1) f_3(t_3)\, dt_3\, dt_1 \\
&+ (\alpha_1 + \alpha_3) \int_{D_1}^{\infty} \int_{(D_1 + D_2 - t_1)}^{\infty} f_1(t_1) f_2(t_2) f_3(D_1 + D_2 + D_3 - t_1 - t_2)\, dt_2\, dt_1
\end{aligned}$$

$$\frac{\partial^2 O_3(D_1, D_2, D_3)}{\partial D_1 \partial D_3} = (\alpha_1 + \alpha_3) \int_{D_1}^{\infty} \int_{(D_1 + D_2 - t_1)}^{\infty} f_1(t_1) f_2(t_2) f_3(D_1 + D_2 + D_3 - t_1 - t_2)\, dt_2\, dt_1$$

$$\begin{aligned}
\frac{\partial^2 O_3(D_1, D_2, D_3)}{\partial D_2^2} &= \alpha_2 \int_{0}^{D_1} f_1(t_1) f_2(D_2)\, dt_1 + \alpha_2 \int_{D_1}^{\infty} f_1(t_1) f_2(D_1 + D_2 - t_1)\, dt_1 \\
&+ (\alpha_1 + \alpha_3) \int_{0}^{D_1} \int_{D_3}^{\infty} f_1(t_1) f_2(D_2) f_3(t_3)\, dt_3\, dt_1 \\
&+ (\alpha_1 + \alpha_3) \int_{0}^{D_1} \int_{D_2}^{\infty} f_1(t_1) f_2(t_2) f_3(D_2 + D_3 - t_2)\, dt_2\, dt_1 \\
&+ (\alpha_1 + \alpha_3) \int_{D_1}^{\infty} \int_{D_3}^{\infty} f_1(t_1) f_2(D_1 + D_2 - t_1) f_3(t_3)\, dt_3\, dt_1 \\
&+ (\alpha_1 + \alpha_3) \int_{D_1}^{\infty} \int_{(D_1 + D_2 - t_1)}^{\infty} f_1(t_1) f_2(t_2) f_3(D_1 + D_2 + D_3 - t_1 - t_2)\, dt_2\, dt_1
\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 O_3(D_1,D_2,D_3)}{\partial D_2 \partial D_3} &= (\alpha_1+\alpha_3)\int_0^{D_1}\int_{D_2}^{\infty} f_1(t_1)f_2(t_2)f_3(D_2+D_3-t_2)\,dt_2\,dt_1 \\ &+ (\alpha_1+\alpha_3)\int_{D_1}^{\infty}\int_{(D_1+D_2-t_1)}^{\infty} f_1(t_1)f_2(t_2)f_3(D_1+D_2+D_3-t_1-t_2)\,dt_2\,dt_1\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 O_3(D_1,D_2,D_3)}{\partial D_3^2} &= (\alpha_1+\alpha_3)\int_0^{D_1}\int_0^{D_2} f_1(t_1)f_2(t_2)f_3(D_3)\,dt_2\,dt_1 \\ &+ (\alpha_1+\alpha_3)\int_0^{D_1}\int_{D_2}^{\infty} f_1(t_1)f_2(t_2)f_3(D_2+D_3-t_2)\,dt_2\,dt_1 \\ &+ (\alpha_1+\alpha_3)\int_{D_1}^{\infty}\int_0^{(D_1+D_2-t_1)} f_1(t_1)f_2(t_2)f_3(D_3)\,dt_2\,dt_1 \\ &+ (\alpha_1+\alpha_3)\int_{D_1}^{\infty}\int_{(D_1+D_2-t_1)}^{\infty} f_1(t_1)f_2(t_2)f_3(D_1+D_2+D_3-t_1-t_2)\,dt_2\,dt_1\end{aligned}$$

The Hessian matrix below should be positive semi-definite.

$$\mathbf{H} = \begin{bmatrix} \frac{\partial^2 O_3}{\partial D_1^2} & \frac{\partial^2 O_3}{\partial D_1 \partial D_2} & \frac{\partial^2 O_3}{\partial D_1 \partial D_3} \\ \frac{\partial^2 O_3}{\partial D_1 \partial D_2} & \frac{\partial^2 O_3}{\partial D_2^2} & \frac{\partial^2 O_3}{\partial D_2 \partial D_3} \\ \frac{\partial^2 O_3}{\partial D_1 \partial D_3} & \frac{\partial^2 O_3}{\partial D_2 \partial D_3} & \frac{\partial^2 O_3}{\partial D_3^2} \end{bmatrix}$$

Let $\mathbf{z}$ be a column vector of length 3 whose elements are $z_1, z_2, z_3 \in \mathbb{R}$. Then, the Hessian matrix is positive semi-definite iff $\mathbf{z}^T\mathbf{H}\mathbf{z} \geq 0, \forall z_1, z_2, z_3 \in \mathbb{R}$.

$$\begin{aligned}\mathbf{z}^T\mathbf{H}\mathbf{z} &= z_1^2\frac{\partial^2 O_3}{\partial D_1^2} + z_2^2\frac{\partial^2 O_3}{\partial D_2^2} + z_3^2\frac{\partial^2 O_3}{\partial D_3^2} + 2z_1z_2\frac{\partial^2 O_3}{\partial D_1 \partial D_2} + 2z_1z_3\frac{\partial^2 O_3}{\partial D_1 \partial D_3} + 2z_2z_3\frac{\partial^2 O_3}{\partial D_2 \partial D_3} \\ &= z_1^2\alpha_2 f_1(D_1) + z_1^2\alpha_2\left(\int_{D_2}^{\infty} f_1(D_1)f_2(t_2)\,dt_2\right) + z_2^2\alpha_2\left(\int_0^{D_1} f_1(t_1)f_2(D_2)\,dt_1\right) \\ &+ (z_1+z_2)^2\alpha_2\left(\int_{D_1}^{\infty} f_1(t_1)f_2(D_1+D_2-t_1)\,dt_1\right) \\ &+ z_1^2(\alpha_1+\alpha_3)\left(\int_{D_2}^{\infty}\int_{(D_2+D_3-t_2)}^{\infty} f_1(D_1)f_2(t_2)f_3(t_3)\,dt_3\,dt_2\right)\end{aligned}$$

$$
\begin{aligned}
&+ z_2^2(\alpha_1+\alpha_3)\left(\int_0^{D_1}\int_{D_3}^{\infty} f_1(t_1)f_2(D_2)f_3(t_3)\,dt_3\,dt_1\right)\\
&+ z_3^2(\alpha_1+\alpha_3)\left(\int_0^{D_1}\int_0^{D_2} f_1(t_1)f_2(t_2)f_3(D_3)\,dt_2\,dt_1\right)\\
&+ z_3^2(\alpha_1+\alpha_3)\left(\int_{D_1}^{\infty}\int_0^{(D_1+D_2-t_1)} f_1(t_1)f_2(t_2)f_3(D_3)\,dt_2\,dt_1\right)\\
&+ (z_1+z_2)^2(\alpha_1+\alpha_3)\left(\int_{D_1}^{\infty}\int_{D_3}^{\infty} f_1(t_1)f_2(D_1+D_2-t_1)f_3(t_3)\,dt_3\,dt_1\right)\\
&+ (z_2+z_3)^2(\alpha_1+\alpha_3)\left(\int_0^{D_1}\int_{D_2}^{\infty} f_1(t_1)f_2(t_2)f_3(D_2+D_3-t_2)\,dt_2\,dt_1\right)\\
&+ (z_1+z_2+z_3)^2(\alpha_1+\alpha_3)\left(\int_{D_1}^{\infty}\int_{(D_1+D_2-t_1)}^{\infty} f_1(t_1)f_2(t_2)f_3(D_1+D_2+D_3-t_1-t_2)\,dt_2\,dt_1\right)
\end{aligned}
$$

Since $z_1^2, z_2^2, z_3^2, (z_1+z_2)^2, (z_2+z_3)^2, (z_1+z_2+z_3)^2, \alpha_1, \alpha_2, \alpha_3, f_1, f_2$ and f_3 are non-negative by definition, $\Rightarrow \mathbf{z}^T\mathbf{H}\mathbf{z} \geq 0$.

$\Rightarrow O_3(D_1, D_2, D_3)$ is jointly convex over D_1, D_2 and D_3. □

4.2.3. Objective Function in Compact Form

Lemma 4.5.

$$
\begin{aligned}
E[(b_3)^+] = &\int_{D_3}^{\infty}(1-F_3(t_3))\,dt_3\\
&+\int_{D_2}^{\infty}\int_{D_2+D_3-t_2}^{\infty}(1-F_2(t_2))f_3(t_3)\,dt_3\,dt_2\\
&+\int_{D_1}^{\infty}\int_{D_1+D_2-t_1}^{\infty}\int_{D_1+D_2+D_3-t_1-t_2}^{\infty}(1-F_1(t_1))f_2(t_2)f_3(t_3)\,dt_3\,dt_2\,dt_1
\end{aligned}
\tag{4.16}
$$

where $F_i(t_i)$ is the cdf and $f_i(t_i)$ is the pdf of i^{th} surgery's duration.

Proof. $(b_3)^+$ was given in (4.14). Thus, we can write:

$$\begin{aligned}
E[(b_3)^+] = &\int_0^{D_1}\int_0^{D_2}\int_{D_3}^{\infty}(t_3 - D_3)f_1(t_1)f_2(t_2)f_3(t_3)\,dt_3\,dt_2\,dt_1\\
&+\int_0^{D_1}\int_{D_2}^{\infty}\int_{D_2+D_3-t_2}^{\infty}(t_3 - D_3 + t_2 - D_2)f_1(t_1)f_2(t_2)f_3(t_3)\,dt_3\,dt_2\,dt_1\\
&+\int_{D1}^{\infty}\int_0^{D_1+D_2-t_1}\int_{D_3}^{\infty}(t_3 - D_3)f_1(t_1)f_2(t_2)f_3(t_3)\,dt_3\,dt_2\,dt_1\\
&+\int_{D_1}^{\infty}\int_{t_2^*}^{\infty}\int_{t_3^*}^{\infty}(t_3 - D_3 + t_2 - D_2 + t_1 - D_1)f_1(t_1)f_2(t_2)f_3(t_3)\,dt_3\,dt_2\,dt_1
\end{aligned} \tag{4.17}$$

where $t_2^* = D_1 + D_2 - t_1$, and $t_3^* = D_1 + D_2 + D_3 - t_1 - t_2$.

Apply integration by parts to the fourth integral in (4.17):

$$\begin{aligned}
u =& -\left(\int_{D_1+D_2-t_1}^{\infty}\int_{D_1+D_2+D_3-t_1-t_2}^{\infty}(t_3 - D_3 + t_2 - D_2 + t_1 - D_1)f_2(t_2)f_3(t_3)\,dt_3\,dt_2\right)\\
\Rightarrow du =& -\left(\int_{D_3}^{\infty}(t_3 - D_3)f_2(D_1 + D_2 - t_1)f_3(t_3)\,dt_3\right)dt_1\\
&-\left(\int_{D_1+D_2-t_1}^{\infty}\int_{D_1+D_2+D_3-t_1-t_2}^{\infty}f_2(t_2)f_3(t_3)\,dt_3\,dt_2\right)dt_1
\end{aligned}$$

$$\begin{aligned}
dv =& -f_1(t_1)dt_1\\
\Rightarrow v =& 1 - F_1(t_1)
\end{aligned}$$

$$\begin{aligned}
uv =& \left[(1 - F_1(t_1))\right]_{t_1=D_1}^{\infty}\\
&\cdot\left[-\left(\int_{D_1+D_2-t_1}^{\infty}\int_{D_1+D_2+D_3-t_1-t_2}^{\infty}(t_3 - D_3 + t_2 - D_2 + t_1 - D_1)f_2(t_2)f_3(t_3)\,dt_3\,dt_2\right)\right]_{t_1=D_1}^{\infty}\\
=& (1 - F_1(D_1))\int_{D_2}^{\infty}\int_{D_2+D_3-t_2}^{\infty}(t_3 - D_3 + t_2 - D_2)f_2(t_2)f_3(t_3)\,dt_3\,dt_2\\
=& \int_{D_1}^{\infty}\int_{D_2}^{\infty}\int_{D_2+D_3-t_2}^{\infty}(t_3 - D_3 + t_2 - D_2)f_1(t_1)f_2(t_2)f_3(t_3)\,dt_3\,dt_2\,dt_1
\end{aligned} \tag{4.18}$$

$$\begin{aligned}-\int v du = &\left(\int_{D_1}^{\infty}(1-F_1(t_1))f_2(D_1+D_2-t_1)\,dt_1\right)\int_{D_3}^{\infty}(t_3-D_3)f_3(t_3)\,dt_3\\ &+\int_{D_1}^{\infty}\int_{D_1+D_2-t_1}^{\infty}\int_{D_1+D_2+D_3-t_1-t_2}^{\infty}(1-F_1(t_1))f_2(t_2)f_3(t_3)\,dt_3\,dt_2\,dt_1\end{aligned} \tag{4.19}$$

Apply another integration by parts to the expression in parenthesis of the first integral term in (4.19):

$$\begin{aligned}\text{Call } u' = 1 - F_1(t_1) &\Rightarrow du' = -f_1(t_1)dt_1,\\ dv' = f_2(D_1+D_2-t_1)dt_1 &\Rightarrow v' = -F_2(D_1+D_2-t_1)\end{aligned}$$

$$\begin{aligned}u'v' - \int v'du' &= [-(1-F_1(t_1))F_2(D_1+D_2-t_1)]_{t_1=D_1}^{\infty} - \int_{D_1}^{\infty}F_2(D_1+D_2-t_1)f_1(t_1)\,dt_1\\ &= (1-F_1(D_1))F_2(D_2) - \int_{D_1}^{\infty}\int_{0}^{D_1+D_2-t_1}f_1(t_1)f_2(t_2)\,dt_2\,dt_1\\ &= \int_{D_1}^{\infty}\int_{0}^{D_2}f_1(t_1)f_2(t_2)\,dt_2\,dt_1 - \int_{D_1}^{\infty}\int_{0}^{D_1+D_2-t_1}f_1(t_1)f_2(t_2)\,dt_2\,dt_1\end{aligned}$$

Rewriting the expressions for the fourth integral term in $E[(b_3)^+]$ expression (4.17), we get:

$$\begin{aligned}&\int_{D_1}^{\infty}\int_{0}^{D_2}\int_{D_3}^{\infty}(t_3-D_3)f_1(t_1)f_2(t_2)f_3(t_3)\,dt_3\,dt_2\,dt_1\\ &-\int_{D_1}^{\infty}\int_{0}^{D_1+D_2-t_1}\int_{D_3}^{\infty}(t_3-D_3)f_1(t_1)f_2(t_2)f_3(t_3)\,dt_3\,dt_2\,dt_1\\ &+\int_{D_1}^{\infty}\int_{D_2}^{\infty}\int_{D_2+D_3-t_2}^{\infty}(t_3-D_3+t_2-D_2)f_1(t_1)f_2(t_2)f_3(t_3)\,dt_3\,dt_2\,dt_1\\ &+\int_{D_1}^{\infty}\int_{D_1+D_2-t_1}^{\infty}\int_{D_1+D_2+D_3-t_1-t_2}^{\infty}(1-F_1(t_1))f_2(t_2)f_3(t_3)\,dt_3\,dt_2\,dt_1\end{aligned} \tag{4.20}$$

Summing (4.20) with the first three integral terms in (4.17), we get:

$$\begin{aligned}\Rightarrow E[(b_3)^+] &= \int_0^{D_2}\int_{D_3}^{\infty}(t_3 - D_3)f_2(t_2)f_3(t_3)\,dt_3\,dt_2\\ &+ \int_{D_2}^{\infty}\int_{D_2+D_3-t_2}^{\infty}(t_3 - D_3 + t_2 - D_2)f_2(t_2)f_3(t_3)\,dt_3\,dt_2\\ &+ \int_{D_1}^{\infty}\int_{D_1+D_2-t_1}^{\infty}\int_{D_1+D_2+D_3-t_1-t_2}^{\infty}(1 - F_1(t_1))f_2(t_2)f_3(t_3)\,dt_3\,dt_2\,dt_1\end{aligned}$$

The first two integral expressions above simplifies further, by use of the proof of Lemma 4.4, (Equation 4.6); and we get (4.16): ☐

Proposition 4.4.

$$\begin{aligned}O_3(D_1, D_2, D_3) = \alpha_1(D_1 &- E[T_1] + D_2 - E[T_2] + D_3 - E[T_3])\\ &+ \alpha_2\int_{D_1}^{\infty}(1 - F_1(t_1))\,dt_1\\ &+ \alpha_2\int_{D_2}^{\infty}(1 - F_2(t_2))\,dt_2\\ &+ \alpha_2\int_{D_1}^{\infty}\int_{D_1+D_2-t_1}^{\infty}(1 - F_1(t_1))f_2(t_2)\,dt_2\,dt_1\\ &+ (\alpha_1 + \alpha_3)\int_{D_3}^{\infty}(1 - F_3(t_3))\,dt_3\\ &+ (\alpha_1 + \alpha_3)\int_{D_2}^{\infty}\int_{D_2+D_3-t_2}^{\infty}(1 - F_2(t_2))f_3(t_3)\,dt_3\,dt_2\\ &+ (\alpha_1 + \alpha_3)\int_{D_1}^{\infty}\int_{D_1+D_2-t_1}^{\infty}\int_{D_1+D_2+D_3-t_1-t_2}^{\infty}(1 - F_1(t_1))f_2(t_2)f_3(t_3)\,dt_3\,dt_2\,dt_1\end{aligned} \tag{4.21}$$

Proof. Proof is straightforward by replacing the terms in (3.10) with the expressions in lemmas (4.5), (4.6), and (4.16). ☐

4.2.4. Exponentially Distributed Surgery Durations

In this subsection, we consider the special case where all three surgeries have exponentially distributed random durations. We first, find the explicit expression for the objective function, then find the optimality equations.

Proposition 4.5.

$$
\begin{aligned}
&O(\lambda_1, \lambda_2, \lambda_3; D_1, D_2, D_3) = \alpha_1 \left(D_1 - \frac{1}{\lambda_1} + D_2 - \frac{1}{\lambda_2} + D_3 - \frac{1}{\lambda_3} \right) \\
&+ \alpha_2 \frac{1}{\lambda_1} e^{-\lambda_1 D_1} + \alpha_2 \frac{1}{\lambda_2} e^{-\lambda_2 D_2} + (\alpha_1 + \alpha_3) \frac{1}{\lambda_3} e^{-\lambda_3 D_3} \\
&+ \alpha_2 \frac{1}{(\lambda_1 - \lambda_2)} e^{-\lambda_1 D_1} e^{-\lambda_2 D_2} - \alpha_2 \frac{\lambda_2}{\lambda_1(\lambda_1 - \lambda_2)} e^{-\lambda_1 (D_1 + D_2)} \\
&+ (\alpha_1 + \alpha_3) \frac{1}{(\lambda_2 - \lambda_3)} e^{-\lambda_2 D_2} e^{-\lambda_3 D_3} - (\alpha_1 + \alpha_3) \frac{\lambda_3}{\lambda_2(\lambda_2 - \lambda_3)} e^{-\lambda_2 (D_2 + D_3)} \\
&+ (\alpha_1 + \alpha_3) \frac{\lambda_2}{(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)} e^{-\lambda_1 D_1} e^{-\lambda_2 D_2} e^{-\lambda_3 D_3} \\
&- (\alpha_1 + \alpha_3) \frac{\lambda_3}{(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)} e^{-\lambda_1 D_1} e^{-\lambda_2 D_2} e^{-\lambda_2 D_3} \\
&+ (\alpha_1 + \alpha_3) \frac{\lambda_2 \lambda_3}{\lambda_1(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} e^{-\lambda_1 D_1} e^{-\lambda_1 D_2} e^{-\lambda_1 D_3} \\
&- (\alpha_1 + \alpha_3) \frac{\lambda_2}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} e^{-\lambda_1 D_1} e^{-\lambda_1 D_2} e^{-\lambda_3 D_3}
\end{aligned}
\tag{4.22}
$$

$\forall \lambda_1, \lambda_2, \lambda_3 > 0$ *s.t.* $\lambda_i \neq \lambda_j$, *where* $\lambda_1, \lambda_2, \lambda_3$ *are the rate parameters of the distributions of first, second and third surgeries in sequence, respectively.*

Proof. Proof is straightforward by making the following replacements in (4.21):

$$
\begin{aligned}
&f_1(t_1) = \lambda_1 e^{-\lambda_1 t_1}, f_2(t_2) = \lambda_2 e^{-\lambda_2 t_2}, f_3(t_3) = \lambda_3 e^{-\lambda_3 t_3}, \\
&1 - F_1(t_1) = e^{-\lambda_1 t_1}, 1 - F_2(t_2) = e^{-\lambda_2 t_2}, 1 - F_3(t_3) = e^{-\lambda_3 t_3}.
\end{aligned}
$$

□

Given the explicit objective function and the fact that objective function was shown to be jointly convex over D_1, D_2, and D_3, we can say that this objective function has a global optimum, which can be found by solving the optimality equations obtained taking the partial derivatives with respect to D_1, D_2 and D_3. Since, it is not much relevant to our work that follows, we do not include these optimality equations here. However, They can easily be obtained by replacing the density functions in partial derivative expressions given in Subsection 4.2.2 with their exponential equivalents.

4.3. Scheduling of n Procedures

In this section, we try to present generalizations of our results obtained for two and three procedure OR scheduling problem. We observed certain patterns in the expressions for objective function. Analyzing these patterns, we had an idea of how the expressions for n-procedure OR scheduling problem would look like. Although we have not yet been able to analytically prove the correctness of these expressions, we observed the same patterns for some larger numbers of procedures, and believe that this constitutes sufficient evidence to present our results as Conjectures, without providing the proofs.

Conjecture 4.1. *The expectation,* $E[(b_i)^+]$*, can be given as follows,* $\forall i \geq 1$*:*

$$\begin{aligned} E[(b_i)^+] = & \int_{D_i}^{\infty} (1 - F_i(t_i))\, dt_i + \int_{D_{i-1}}^{\infty} \int_{D_{i-1}+D_i-t_{i-1}}^{\infty} (1 - F_{i-1}(t_{i-1})) f_i(t_i)\, dt_i\, dt_{i-1} + ... \\ & + \int_{D_1}^{\infty} \int_{D_1+D_2-t_1}^{\infty} ... \int_{\sum_{j=1}^{i} D_j - \sum_{j=1}^{i-1} t_j}^{\infty} (1 - F_1(t_1)) f_2(t_2) ... f_i(t_i)\, dt_i ...\, dt_2\, dt_1 \end{aligned} \tag{4.23}$$

where $F_j(t_j)$ *is the cdf and* $f_j(t_j)$ *is the pdf of* j^{th} *surgery's duration.*

Conjecture 4.2. *Define,*

$$\begin{aligned}
&\mathcal{D}_l^u = (D_l, D_{l+1}, ..., D_u),\ s.t.\ l, u \text{ are positive integers, } u \geq l,\\
&C_1(\mathcal{D}_l^l) = \int_{D_l}^{\infty} (1 - F_l(t_l))\, dt_l\\
&C_2(\mathcal{D}_l^{l+1}) = \int_{D_l}^{\infty} \int_{D_l + D_{l+1} - t_l}^{\infty} (1 - F_l(t_l)) f_{l+1}(t_{l+1})\, dt_{l+1}\, dt_l\\
&C_3(\mathcal{D}_l^{l+2}) = \int_{D_l}^{\infty} \int_{D_l + D_{l+1} - t_l}^{\infty} \int_{D_l + D_{l+1} + D_{l+2} - t_l - t_{l+1}}^{\infty} (1 - F_l(t_l)) f_{l+1}(t_{l+1}) f_{l+2}(t_{l+2})\, dt_{l+2}\, dt_{l+1}\, dt_l\\
&C_k(\mathcal{D}_l^{l+k-1}) = \int_{t_0^*}^{\infty} \int_{t_1^*}^{\infty} ... \int_{t_{l+k-1}^*}^{\infty} (1 - F_l(t_l)) f_{l+1}(t_{l+1}) ... f_{l+k-1}(t_{l+k-1})\, dt_{l+k-1} ...\, dt_{l+1}\, dt_l.
\end{aligned}$$

where $t_j^* = \sum_{m=l}^{l+j} D_m - \sum_{m=l}^{l+j-1} t_m,\ t_1^* = D_l$. *Then, We can write:*

$$\begin{aligned}
E[(b_i)^+] &= C_1(\mathcal{D}_i^i) + C_2(\mathcal{D}_{i-1}^i) + C_3(\mathcal{D}_{i-2}^i) + ... + C_{i-1}(\mathcal{D}_2^i) + C_i(\mathcal{D}_1^i)\\
&= \sum_{k=1}^{i} C_k(\mathcal{D}_{i-k+1}^i).
\end{aligned} \tag{4.24}$$

Conjecture 4.3. *The objective function, $O_n(\mathcal{D})$, for n-procedure OR scheduling problem can be given as follows:*

$$O_n(\mathcal{D}) = \alpha_1 \sum_{i=1}^{n} (D_i - E[T_i]) + \alpha_2 \sum_{i=1}^{n-1} \sum_{k=1}^{i} C_k(\mathcal{D}_{i-k+1}^i) + (\alpha_1 + \alpha_3) \sum_{k=1}^{n} C_k(\mathcal{D}_{n-k+1}^n) \tag{4.25}$$

$\forall n \geq 1$ *where* $\mathcal{D} = \mathcal{D}_1^n = (D_1, D_2, ..., D_n)$.

Assuming that the Conjecture 1 and 2 are true, then the proof of Conjecture 3 follows from replacing $E[(b_i)^+]$ terms in (3.10), by the expression found in (4.24).

One way to interpret the functions $C_k(\mathcal{D}_{i-k+1}^i)$ is that they represent the cost of putting $(i-k+1)^{th}$ through i^{th} surgeries in the schedule in the given order. For example, $C_2(D_1, D_2)$ is the cost of having the first surgery followed by second; whereas $C_3(D_2, D_3, D_4)$ is the cost of having the second surgery followed by third, and that followed by fourth surgery. If the function involves the n^{th} surgery, then the cost is incurred for idle time and overtime; whereas if it does not involve the n^{th} surgery, the cost is incurred only for the patient waiting time.

Conjecture 4.4. *The objective function,* $O_n(\mathcal{D})$*, is jointly convex over* $\mathcal{D}, \forall n \geq 1$*, where* $\mathcal{D} = (D_1, D_2, ..., D_n)$.

Conjecture 4.5. *When all surgery durations are exponentially distributed with parameters,* $\lambda_i > 0, i = 1, 2, ..., n$.

$$C_i(\mathcal{L}_1^i; \mathcal{D}_1^i) = \frac{\lambda_1 \lambda_2}{(\lambda_1 - \lambda_2)} C_1(\lambda_1; D_1) \left[C_{i-1}(\lambda_2, \mathcal{L}_3^i; \mathcal{D}_2^i) - C_{i-1}(\lambda_1, \mathcal{L}_3^i; \mathcal{D}_2^i) \right], \forall i \geq 2. \quad (4.26)$$

where $\mathcal{L}_i^j = (\lambda_i, \lambda_{i+1}, ..., \lambda_j), \forall j \geq i$ *and* $\mathcal{L}_i^j = \varnothing, \forall i > j$

This iterative relationship between the C_i and C_{i-1} functions aids us conducting our numerical analyses for n-procedure OR scheduling problem, as we do not need to explicitly define C_i functions for all $i = 1, 2, ..., n$.

5. HEURISTICS AND APPROXIMATIONS

In the previous chapter, we presented the analytical findings for the OR scheduling problem we consider in this thesis. As mentioned previously, obtaining exact analytical solutions for the problem is very hard, if not impossible. This leads us to consider several alternative approaches for finding well-performing approximate solutions. In this chapter, we present the heuristic approaches we devised for finding approximate solutions.

The problem requires two types of decisions to be made: Optimum ordering (sequence) of the procedures and optimum durations to be assigned to procedures. As stated throughout Chapter 4, the objective function we work with is depended on the sequence of procedures; i.e., for each sequencing option, we get a different objective function. Therefore, both analytical and numerical calculations always start with determining the sequence. In Section 5.1, we provide some insights about the possible sequencing policies we will be using during our numerical analyses. Although by showing the joint convexity of the objective function we know that optimum duration assignments can be found by simultaneously solving the set of first order conditions (FOC), these FOC equations are non-linear, non-separable and involve complex expressions such as multiple integrals. Therefore, obtaining an analytical solution for this problem is very hard, and the time complexity of solving these equations numerically increases exponentially. This necessitates the search for good approximations based on heuristics. Sections 5.2 through 5.4 describe three heuristic approaches for determining the assigned durations, for a given sequence. Each of these three heuristics is motivated by a real life implementation. Flows of these sections is as follows: first we give the description of the real life problem which constitutes a bound to our problem, then we present analytical calculations to obtain optimum assigned durations for this bounding problem, and finally we discuss how we incorporate the results obtained from bounding problems into heuristics.

5.1. Heuristics for Optimum Sequence

Combinatorial nature of the problem dictates that there are $n!$ different sequencing options for n-procedure OR scheduling problem. Whereas, $n!$ is a relatively manageable number for small values of n, considering all possible sequencing options for larger values of n, say $n \geq 5$ becomes very expensive in terms of computation time. In a vast majority of the literature on OR scheduling, we observed that the sequence variables are also considered as decision variables (i.e., s_{ij}: i^{th} surgery is at the j^{th} order in the sequence). Therefore, they formulate the problem as an MIP to simultaneously solve optimum sequence and optimum duration assignment problems. However, one of our main focuses in this thesis work is to draw managerial insights about the optimum sequence. We do this by first determining candidate sequencing options and then comparing the results of these sequencing options for a wide range of parameter combinations. Our goal is to observe whether there is a sequencing option which prevails as the optimum sequence for the majority of the parameter combinations, if not all, we consider. Theorem 4.2, provides a basis for determining our candidate sequencing options. We have shown in Theorem 4.2, that for the two-procedure OR scheduling problem, where one of the procedures has a deterministic duration whereas the other one has a non-negative, continuous random duration, it is always optimal to schedule the deterministic procedure, first. We discussed, at the end of Subsection 4.1.3, that this scheduling policy is closely related to "Shortest Variance First" policy, which is commonly used for various stochastic scheduling problems. Similarly, another widely used sequencing policy is called "Longest Variance First". These two sequencing policies will be our base candidates for the optimum sequence.

In our numerical analyses, we will be considering the cases where all the surgery durations are assumed to be exponentially distributed random variables with rate parameters $\lambda_i, i = 1, 2, ..., n$. Exponential distribution is a special distribution which can be completely characterized by its rate parameter. Therefore, both variance and expectation (and all other moments) of the random variable is defined in terms of the rate parameter. This special property of exponential distribution made us consider different names for our sequencing options. For example; $(\lambda_1 > \lambda_2 > ... > \lambda_n)$ sequencing option is equivalent to both "Shortest Variance First" and "Shortest Expectation First" rules. To avoid this confusion, we refer to our two candidate sequencing options $(\lambda_1 > \lambda_2 > ... > \lambda_n)$ and $(\lambda_n > \lambda_{n-1} > ... > \lambda_1)$ as: "Increasing Order (of rate parameters)" and "Decreasing Order (of rate parameters)", respectively.

5.2. An Expectation-based Heuristic

5.2.1. Background Problem

It is known that in many hospitals, durations assigned to each surgery is determined by simply taking the average of the durations of surgeries from the same type that have been conducted in the past; i.e., Durations for each surgery type is assumed to be a different random variable, and the historical average is used as an estimator for their expected value. In this subsection, we consider a special problem of scheduling of two procedures, which is motivated by this real life implementations. Instead of trying to solve two problems (finding the optimum durations and the optimum sequence) simultaneously, we assume, now, that the durations assigned to each surgery are fixed and equal to the expected value of their distributions. We also assume that both surgery durations are distributed exponentially with different parameters $\lambda_I = \lambda + \Delta\lambda$ and $\lambda_{II} = \lambda - \Delta\lambda$, where $\Delta\lambda \in (-\lambda, \lambda)$. We would like to show that the optimum sequence in this case is to schedule the surgery with larger parameter first.

5.2.2. Analytical Solution

Let us, first, write the objective functions for two possible sequences: (I, II) and (II, I):

$$\begin{aligned} O_2(D_I, D_{II}, \lambda, \Delta\lambda) = \alpha_1 &\left[-\frac{1}{(\lambda+\Delta\lambda)} - \frac{1}{(\lambda-\Delta\lambda)} + D_I + D_{II}\right] \\ &+ \alpha_2 \frac{1}{(\lambda+\Delta\lambda)} e^{-(\lambda+\Delta\lambda)D_I} \\ &+ (\alpha_1+\alpha_3)\frac{1}{(\lambda-\Delta\lambda)} e^{-(\lambda-\Delta\lambda)D_{II}} \\ &+ \frac{(\alpha_1+\alpha_3)}{2}\frac{1}{\Delta\lambda} e^{-(\lambda+\Delta\lambda)D_I} e^{-(\lambda-\Delta\lambda)D_{II}} \\ &- \frac{(\alpha_1+\alpha_3)}{2}\frac{1}{\Delta\lambda}\frac{\lambda-\Delta\lambda}{(\lambda+\Delta\lambda)} e^{-(\lambda+\Delta\lambda)D_I} e^{-(\lambda+\Delta\lambda)D_{II}} \end{aligned} \tag{5.1}$$

$$\begin{aligned} O_2(D_{II}, D_I, \lambda, \Delta\lambda) = \alpha_1 &\left[-\frac{1}{(\lambda-\Delta\lambda)} - \frac{1}{(\lambda+\Delta\lambda)} + D_{II} + D_I\right] \\ &+ \alpha_2 \frac{1}{(\lambda-\Delta\lambda)} e^{-(\lambda-\Delta\lambda)D_{II}} \\ &+ (\alpha_1+\alpha_3)\frac{1}{(\lambda+\Delta\lambda)} e^{-(\lambda+\Delta\lambda)D_I} \\ &+ \frac{(\alpha_1+\alpha_3)}{2}\frac{1}{\Delta\lambda} e^{-(\lambda-\Delta\lambda)D_{II}} e^{-(\lambda+\Delta\lambda)D_I} \\ &- \frac{(\alpha_1+\alpha_3)}{2}\frac{1}{\Delta\lambda}\frac{(\lambda+\Delta\lambda)}{(\lambda-\Delta\lambda)} e^{-(\lambda-\Delta\lambda)D_{II}} e^{-(\lambda-\Delta\lambda)D_I} \end{aligned} \tag{5.2}$$

Proposition 5.1.

$$O_2(D_I^*, D_{II}^*, \lambda, \Delta\lambda) \leq O_2(D_{II}^*, D_I^*, \lambda, \Delta\lambda),\ \forall\lambda > 0, \Delta\lambda \in (0, \lambda) \tag{5.3}$$

where $D_I^* = E[T_I] = \dfrac{1}{\lambda+\Delta\lambda}$, $D_{II}^* = E[T_{II}] = \dfrac{1}{\lambda-\Delta\lambda}$, *are the assigned (expected) durations, for procedures* I *and* II*, respectively.*

Proof.

$$\begin{aligned} O_2(D_I^*, D_{II}^*, \lambda, \Delta\lambda) &= \alpha_2 \frac{1}{(\lambda+\Delta\lambda)} e^{-1} + (\alpha_1+\alpha_3)\frac{1}{(\lambda-\Delta\lambda)} e^{-1} \\ &+ \frac{(\alpha_1+\alpha_3)}{2}\frac{1}{\Delta\lambda} e^{-1}e^{-1} - \frac{(\alpha_1+\alpha_3)}{2}\frac{1}{\Delta\lambda}\frac{(\lambda-\Delta\lambda)}{(\lambda+\Delta\lambda)} e^{-1} e^{-\frac{\lambda+\Delta\lambda}{\lambda-\Delta\lambda}} \end{aligned} \tag{5.4}$$

$$\begin{aligned} O_2(D_{II}^*, D_I^*, \lambda, \Delta\lambda) &= \alpha_2 \frac{1}{(\lambda-\Delta\lambda)} e^{-1} + (\alpha_1+\alpha_3)\frac{1}{(\lambda+\Delta\lambda)} e^{-1} \\ &+ \frac{(\alpha_1+\alpha_3)}{2}\frac{1}{\Delta\lambda} e^{-1}e^{-1} - \frac{(\alpha_1+\alpha_3)}{2}\frac{1}{\Delta\lambda}\frac{(\lambda+\Delta\lambda)}{(\lambda-\Delta\lambda)} e^{-1} e^{-\frac{\lambda-\Delta\lambda}{\lambda+\Delta\lambda}} \end{aligned} \tag{5.5}$$

Call $u = \dfrac{\lambda+\Delta\lambda}{\lambda-\Delta\lambda}$. Then,

$$\begin{aligned} &O_2(D_I^*, D_{II}^*, \lambda, \Delta\lambda) - O_2(D_{II}^*, D_I^*, \lambda, \Delta\lambda) \\ &= \alpha_2 \left[\frac{1}{(\lambda+\Delta\lambda)} - \frac{1}{(\lambda-\Delta\lambda)}\right] + (\alpha_1+\alpha_3)\left[\frac{1}{(\lambda-\Delta\lambda)} - \frac{1}{(\lambda+\Delta\lambda)}\right] \\ &- \frac{(\alpha_1+\alpha_3)}{2}\frac{1}{\Delta\lambda}\left[u e^{-\frac{1}{u}} + \frac{1}{u}e^{-u}\right] \\ &= (\alpha_1+\alpha_3-\alpha_2)\frac{2\Delta\lambda}{(\lambda+\Delta\lambda)(\lambda-\Delta\lambda)} \\ &- (\alpha_1+\alpha_3)\frac{1}{\Delta\lambda}\left[\frac{1}{2}u e^{-\frac{1}{u}} + \frac{1}{2}\frac{1}{u}e^{-u}\right] \end{aligned} \tag{5.6}$$

where, $1 < u < \infty$.
Since, $\dfrac{2\Delta\lambda}{(\lambda+\Delta\lambda)(\lambda-\Delta\lambda)} \geq 0$, it is enough to show the following:

$$\begin{aligned} &\frac{2\Delta\lambda}{(\lambda+\Delta\lambda)(\lambda-\Delta\lambda)} \leq \frac{1}{\Delta\lambda}\left[\frac{1}{2}u e^{-\frac{1}{u}} + \frac{1}{2}\frac{1}{u}e^{-u}\right] \\ &or, equivalently: \\ &\frac{2(\Delta\lambda)^2}{\lambda^2-(\Delta\lambda)^2} \leq \frac{1}{2}u e^{-\frac{1}{u}} + \frac{1}{2}\frac{1}{u}e^{-u}. \end{aligned}$$

Observe that $O_2(D_I^*, D_{II}^*, \lambda, \Delta\lambda) = O_2(D_{II}^*, D_I^*, \lambda, \Delta\lambda)$, for $\Delta\lambda = 0$.

Thus, it is enough to show that:

$$\frac{\partial}{\partial(\Delta\lambda)}\left[\frac{2(\Delta\lambda)^2}{\lambda^2-(\Delta\lambda)^2}\right] \le \frac{\partial}{\partial(\Delta\lambda)}\left[\frac{1}{2}ue^{-\frac{1}{u}} + \frac{1}{2}\frac{1}{u}e^{-u}\right],$$

where

$$\begin{aligned}
&\frac{\partial}{\partial(\Delta\lambda)}\left[\frac{2(\Delta\lambda)^2}{\lambda^2-(\Delta\lambda)^2}\right] = \frac{4\Delta\lambda\lambda^2}{(\lambda^2-(\Delta\lambda)^2)^2}, \text{ and}\\
&\frac{\partial}{\partial(\Delta\lambda)}\left[\frac{1}{2}ue^{-\frac{1}{u}} + \frac{1}{2}\frac{1}{u}e^{-u}\right] = \frac{2\lambda^2}{(\lambda^2-(\Delta\lambda)^2)^2}\left[(\lambda+\Delta\lambda)e^{-\frac{1}{u}} - (\lambda-\Delta\lambda)e^{-u}\right].
\end{aligned}$$

This simplifies the inequality to the following:

$$\begin{aligned}
&2\Delta\lambda \le (\lambda+\Delta\lambda)e^{-\frac{1}{u}} - (\lambda-\Delta\lambda)e^{-u},\\
&or, equivalently:\\
&(\lambda+\Delta\lambda) - (\lambda-\Delta\lambda) \le (\lambda+\Delta\lambda)e^{-\frac{1}{u}} - (\lambda-\Delta\lambda)e^{-u}
\end{aligned}$$

Rearranging the terms, we get:

$$\begin{aligned}
&(\lambda+\Delta\lambda)(1-e^{-\frac{1}{u}}) \le (\lambda-\Delta\lambda)(1-e^{-u})\\
&OR, equivalently:\\
&u \le \frac{1-e^{-u}}{1-e^{-\frac{1}{u}}}\\
&\Rightarrow \frac{1-e^{-u}}{1-e^{-\frac{1}{u}}} - u \ge 0
\end{aligned}$$

Define a function $f(u) = \dfrac{1-e^{-u}}{1-e^{-\frac{1}{u}}} - u$. It is enough to show that $f(u)$ is always positive in the interval $1 < u < \infty$. It is straight-forward (proof by figure) to show that this function is always positive in the given interval. See, figure below (Note that $f(1) = 0$): $\square$

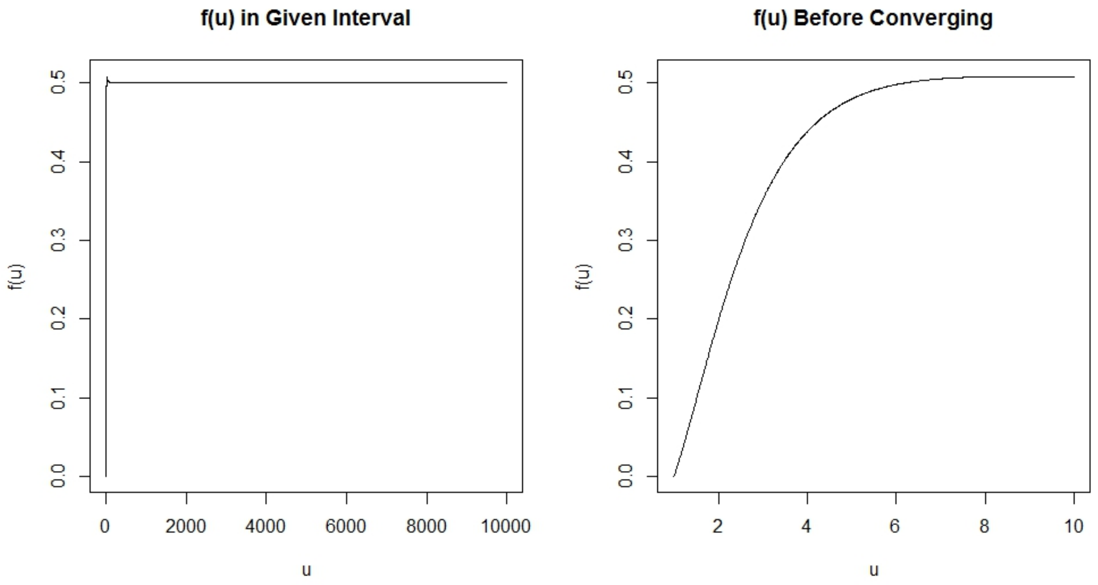

Figure 5.1. Change of f(u) function within given the interval.

5.2.3. Heuristic Approach to Original Problem

In subsection 5.2.2, we showed that for the two-procedure OR scheduling problem, when the assigned durations are predetermined as the expected values of the random variables, T_i - the actual durations -, the optimum sequence is always “Decreasing Order of rate parameters”. Although, we have not proved this result explicitly for the higher values of n, we conjecture that “Decreasing Order” is always the optimum sequence for this background problem, where the assigned durations for each surgery are determined as the expected value of their respective random variables. In numerical results, we show that the conjecture holds for $n = 2, 3, 4$ for a wide range of parameters. Our first heuristic for the assigned durations is motivated by this background problem. Instead of solving n non-linear equations simultaneously, we simply set the assigned durations to the expected values of the random variables representing the actual durations. This is, in fact, quite a common practice in reality. Determining surgery durations for scheduling purposes, schedulers usually take the mean of the durations in historical data. Nevertheless, this heuristic has a fundamental flaw, since it does not take the cost coefficients into account.

5.3. A Myopic Heuristic

5.3.1. Background Problem

The second problem we consider has a slight difference from the original problem. In the original problem, the scheduling process takes place prior to the start of the day, and all assigned durations must be determined simultaneously. However, in this bounding problem, we determine the duration of each surgery right after the preceding surgery ends. Therefore, we have n "scheduling epoches", instead of a single one. This approach can be considered as dynamic scheduling. We define a state variable, x, which represents the difference between the scheduled end and actual end of the preceding surgery. Unlike the original problem, we know the value of x at each scheduling epoch. A function of x, $V_i(x)$, is defined to represent the minimum expected cost of the surgeries that remain to be scheduled at the epoch when $i-1$ surgeries are completed, and we try to minimize this function at each scheduling epoch. The solution to this problem is given in the following subsection.

5.3.2. Analytical Solution

We have n procedures with random durations $T_1, T_2, ..., T_n$. Let $F_i(x) = Pr\{T_i \leq x\}$, and suppose that $F_i(0) = 0, i = 1, 2, ..., n$.

Without loss of generality the first operation starts at time 0. Then, $SE_1 = D_1$ and $AE_1 = T_1$. The key observation that facilitates the dynamic solution is: for $i = 1, 2, ..., n$

$$\begin{aligned} SE_i - AE_i &= SE_{i-1} + D_i - (AS_i + T_i) \\ &= D_i - T_i + SE_{i-1} - max\{SE_{i-1}, AE_{i-1}\} \\ &= D_i - T_i - (AE_{i-1} - SE_{i-1})^+ \end{aligned} \tag{5.7}$$

We can rewrite the objective function as:

$$\sum_{i=1}^{n} \alpha_1 E[(SE_i - AE_i)^+] + \sum_{i=1}^{n} \alpha_2 E[(AE_{i-1} - SE_{i-1})^+] + \alpha_3 E[(AE_n - SE_n)^+]. \tag{5.8}$$

Specifically, suppose that we are right at the start of the ith operation. Therefore, $SE_{i-1} - AE_{i-1}$ is known, and we will determine D_i. Suppose $SE_{i-1} - AE_{i-1} = x$. Define $V_i(x)$ as the minimum expected cost of the system through procedures $i, i+1, ..., n$ (until all procedures are completed), when $D_i, D_{i+1}, ..., D_n$ are optimally selected and when $PE_{i-1} - AE_{i-1} = x$. By (5.7) and (5.8),

$$V_i(x) = \underset{D_i \geq 0}{min}\{\alpha_1 E[(D_i - T_i + min(0,x)] + \alpha_2 max(0,-x) + E[V_{i+1}(D_i - T_i + min(0,x))]\} \tag{5.9}$$

for $i = 1, 2, ..., n$, and with the termination condition $V_{n+1}(x) = \alpha_3 max(0,-x)$.
Define the auxiliary function $G_i(a)$:

$$G_i(a) = \alpha_1 E[(a - T_i)^+] + E[V_{i+1}(a - T_i)].$$

By letting $a_i = D_i + min(0,x)$ we can rewrite (5.9) as:

$$V_i(x) = \underset{a_i \geq min(0,x)}{min}\{G_i(a_i) + \alpha_2 max(0,-x)\}.$$

The solution is characterized in the following Proposition.

Proposition 5.2. *a.* $V_i(x)$ *is convex in* x *for all i=1,2,...,n+1.*
b. $G_i(a)$ *is convex in* a *for all i=1,2,...,n.*
c. The optimal assigned surgery duration is given by:
$D_i^* = a_i^* - min(0,x)$,
where $a_i^* = F_i^{-1}\left(\frac{\alpha_2}{\alpha_1+\alpha_2}\right)$ *for i=1,2,...,n-1, and* $a_n^* = F_n^{-1}\left(\frac{\alpha_3}{\alpha_1+\alpha_3}\right)$.
d. $V_i(x) = G_i(a_i^*) + \alpha_2 max(0,-x)$ *for all i=1,2,...,n.*

Proof. Proof by induction. $V_{n+1}(x) = \alpha_3 max(0, -x)$ is clearly convex in x (this proves part (a) for $n+1$). Then,

$$\begin{aligned} G_n(a) &= \alpha_1 E[(a - T_n)^+] + E[V_{n+1}(a - T_n)] \\ &= \alpha_1 E[(a - T_n)^+] + \alpha_3 E[(T_n - a)^+], \end{aligned}$$

and $G_n(a)$ is convex (the sum of two convex functions) in a (proves part (b) for n). Since $G_n(a)$ is convex in a, its minimizer is found by using the first order condition as $a_n^* = F_n\left(\frac{\alpha_3}{\alpha_1+\alpha_3}\right)$.
Note that $a_n^* \geq 0$. Now, since $min(0, x) \leq 0 \leq a_n^*$,

$$\begin{aligned} \min_{a_n \geq min(0,x)} G_n(a_n) &= G_n(a_n^*), \text{ and} \\ V_n(x) &= G_n(a_n^*) + \alpha_2 max(0, -x). \end{aligned}$$

Therefore, $D_n^* = a_n^* - min(0, x)$, proving parts (c) and (d) for $i = n$. Suppose these assertions are true for some $1 < k \leq n$. Then,

$$G_{k-1}(a) = \alpha_1 E[(a - T_{k-1})^+] + E[V_k(a - T_{k-1})]$$

is convex as it is the sum of two convex functions (proves part (b)). Also,

$$G_{k-1}(a) = \alpha_1 E[(a - T_{k-1})^+] + G_k(a_k^*) + \alpha_2 E[(T_{k-1} - a)^+],$$

by the induction hypothesis. Hence, a_{k-1}^* is given by $a_{k-1}^* = F_{k-1}^{-1}\left(\frac{\alpha_2}{\alpha_1+\alpha_2}\right)$ as claimed. Since, $a_{k-1}^* \geq 0$, we have

$$V_{k-1}(x) = G_{k-1}(a_{k-1}^*) + \alpha_2 max(0, -x),$$

and the optimality policy $D_{k-1}^* = a_{k-1}^* - min(0,x)$ follows (proving parts (a), (c) and (d)). □

5.3.3. Heuristic Approach to Original Problem

As stated in Proposition 5.2-c, the optimal assigned durations are given by:

$$D_i^* = a_i^* - min(0,x),$$

where $a_i^* = F_i^{-1}\left(\frac{\alpha_2}{\alpha_1+\alpha_2}\right)$ for i=1,2,...,n-1, and $a_n^* = F_n^{-1}\left(\frac{\alpha_3}{\alpha_1+\alpha_3}\right)$.

One modification is necessary here, to be able to incorporate this result in our next heuristic: Myopic heuristic. In the background problem we have n decision epochs at which the value of x is always known. However, this is not the case when all the assigned durations are to be determined at once. Therefore, we have to assume a fixed value for x. The most reasonable value would be 0. The interpretation for this assumption would be that, we do not, knowingly, determine our scheduled end times with a bias. With this assumption, the function, $min(0,x)$ always has the value 0. Therefore, for Myopic heuristic, we define the approximate assigned durations as:

$$D_i^* = F_i^{-1}\left(\frac{\alpha_2}{\alpha_1+\alpha_2}\right) \text{ for } i = 1,2,...,n-1, \text{ and } D_n^* = F_n^{-1}\left(\frac{\alpha_3}{\alpha_1+\alpha_3}\right).$$

5.4. Veteran's Hospital Heuristic

5.4.1. Background Problem

The final problem we consider can be thought of as the closest to the original problem among the three bounding problems we considered so far. In the original problem, a procedure can not start before its scheduled time, even if the preceding procedure ends earlier than its scheduled time. However, in this bounding, problem we allow a procedure to start before its scheduled time should the preceding procedure ends before its scheduled ending time. This type of scheduling happens very commonly in types of hospitals referred to as "Veteran's Hospital", where the patients arrive to the hospital at the beginning of the day of surgery, and prepared for surgery at any time of the day. Mathematically, this arrangement is advantageous as it makes the non-separable expressions of the objective function to be separable. More specifically, with this arrangement, not only the scheduled start times can be expressed as the sum of assigned durations of the prior surgeries, but also the actual start times can be expressed as the sum of the actual durations of the prior surgeries. As can be seen from expression for the objective function in (3.1), this modification makes the objective function separable in terms of S_i and Y_i variables defined as follows, respectively:

$$S_i = \sum_{j=1}^{i} D_j \tag{5.10}$$

$$Y_i = \sum_{j=1}^{i} T_j \tag{5.11}$$

Furthermore, even if we allow procedures to start earlier than their scheduled starting times, we incur an auxiliary idle time cost for the difference between the scheduled and actual starting times, as in our original problem. This auxiliary cost helps keeping the idle time cost still effective. The analytical solution for this bounding problem is given in Subsection 5.4.2.

5.4.2. Analytical Solution

We have n procedures with random durations $\mathcal{T} = (T_1, T_2, ..., T_n)$. Let $F_i(x) = Pr\{T_i \leq x\}$, and suppose that $F_i(0) = 0, i = 1, 2, ..., n$.
Now, define $S_i = \sum_{j=1}^{i} D_i$ (so that $S_i = SE_i$), and rewrite the objective function in terms of $\mathcal{S} = (S_1, S_2, ..., S_n)$.

$$\begin{aligned}
O(\mathcal{S}) &= \sum_{i=1}^{n} \alpha_1 E[(S_i - AE_i)^+] + \sum_{i=1}^{n} \alpha_2 E[(AE_{i-1} - S_{i-1})^+] + \alpha_3 E[(AE_n - S_n)^+] \\
&= \sum_{i=1}^{n} \alpha_1 E[(S_i - AE_i)^+] + \sum_{i=1}^{n-1} \alpha_2 E[(AE_i - S_i)^+] + \alpha_3 E[(AE_n - S_n)^+] \\
&= \sum_{i=1}^{n} \{\alpha_1 E[(S_i - AE_i)^+] + \alpha_2 E[(AE_i - S_i)^+]\} + \alpha_1 E[(S_n - AE_n)^+] + \alpha_3 E[(AE_n - S_n)^+]
\end{aligned}$$

One observation is the following:

Proposition 5.3. $AE_i \geq Y_i := \sum_{j=1}^{i} T_i, \forall i = 1, 2, ..., n.$

Proof. Proof by induction. $AE_1 = T_1$. Suppose the assertion is true for $i - 1$, that is, $AE_{i-1} \geq Y_{i-1}$. Then, $AE_i = T_i + max\{AE_{i-1}, S_{i-1}\} \geq T_i + AE_{i-1} \geq T_i + Y_{i-1} = Y_i$. □

The approximation is simply replacing AE_i with its lower bound Y_i.

Intuitively, suppose that all the patients are available in the morning, but for scheduling purposes we assign scheduled ending times of procedures $1, 2, ..., n$ as $S_1, S_2, ..., S_n$.

Moreover, an operation starts right after the end of the previous one. Then, Y_i is the actual ending time for operation i. If $S_i > Y_i$, then a cost of "earliness" (equals α_1) is charged for having to start the next operation earlier than planned. Otherwise, if $Y_i > S_i$, the usual waiting time cost (perhaps not for the patients, but for the operating team) is charged. The overtime cost is not affected. This approach of scheduling is commonly implemented in hospitals known as "Veteran's Hospital".
Define,

$$\begin{aligned} L_i(x) &= \alpha_1 E[(x - Y_i)^+] + \alpha_2 E[(Y_i - x)^+], for i = 1, 2, ..., n-1, and \\ L_n(x) &= \alpha_1 E[(x - Y_n)^+] + \alpha_3 E[(Y_n - x)^+]. \end{aligned}$$

The approximate objective function, $OA(\mathcal{S})$, is then written as:

$$OA(\mathcal{S}) = \sum_{i=1}^{n} L_i(S_i).$$

We will need the following definition:

Definition 5.1. *Let X and Z be two random variables. X is said to be greater than Z in convex order (written as $X \geq_{cx} Z$ if $E[h(X)] \geq E[h(Z)]$) for all convex functions h.*

Convex ordering implies that $E[X] = E[Z]$ and $Var(X) \geq Var(Z)$. If X and Z are from the families, Normal, Lognormal, Beta, Gamma, Weibull, Uniform with $E[X] = E[Z]$ and $Var(X) \geq Var(Z)$, then $X \geq_{cx} Z$. Convex ordering is preserved under convolution.

We want to show that an ordering of the operations according to one of the definitions above will yield the optimal solution for the approximate problem.

Consider an ordering $\mathcal{T} = (T_1, T_2, ..., T_n)$. Let $\mathcal{S}^* = (S_1, S_2, ..., S_n)$ be the optimal solution to $OA(\mathcal{S})$. Since $OA(\mathcal{S})$ is separable in $S_1, S_2, ..., S_n$, S_i^* is found by minimizing $L_i(S_i)$. Let $G_i(x)$ be the cdf of Y_i. Then,

$$S_i^* = G_i^{-1}(\alpha_2/(\alpha_1+\alpha_2)) \text{ for } i = 1, 2, ..., n-1, \text{ and} S_n^* = G_n^{-1}(\alpha_3/(\alpha_1+\alpha_3)). \tag{5.12}$$

Assume that $\alpha_3 \geq \alpha_2$. This, together with observing that $Y_1 \leq Y_2 \leq ... \leq Y_n$ ensures that:

$$S_1^* \leq S_2^* \leq ... \leq S_n^*.$$

Next, let $\tilde{\mathcal{T}} = (\tilde{T}_1, \tilde{T}_2, ..., \tilde{T}_n)$ be an ordering of the procedures. It is sufficient to consider a neighborhood switch. Suppose $\tilde{\mathcal{T}}$ is such that

$$\begin{aligned}
&\tilde{T}_j = T_j, \text{for } j \neq k, k+1\\
&\tilde{T}_k = T_{k+1}\\
&\tilde{T}_{k+1} = T_k.
\end{aligned}$$

Define $\tilde{Y}_i = \sum_{j=1}^{i} \tilde{T}_j$. Then, $\tilde{Y}_i = Y_i$, for $i \neq k$ and $\tilde{Y}_k = Y_{k-1} + T_{k+1}$. Also note that $Y_k = Y_{k-1} + T_k$. Analogous to $L_i(x)$ define $\tilde{L}_i(x)$ using $\tilde{Y}_i$ and note that $\tilde{L}_i(x) = L_i(x)$ for $i \neq k$.

This leads to $\tilde{S}_i^{\,*} = S_i^*$ for $i \neq k$. Then,

$$\begin{aligned}
\widetilde{OA}(\tilde{\mathcal{S}}^*) &= \sum_{i=1}^{n} \tilde{L}_i(\tilde{S}_i^*)\\
&= \sum_{i \neq k} L_i(S_i^*) + \tilde{L}_k(\tilde{S}_k^*).
\end{aligned}$$

Therefore, $OA(\mathcal{S}^*) \leq \widetilde{OA}(\tilde{\mathcal{S}}^*)$ if and only if $L_k(S_k^*) \leq \tilde{L}_k(\tilde{S}_k^*)$.

Following results provide sufficient conditions for optimal ordering of operations.

Proposition 5.4. *If* $T_{k+1} \geq_{cx} T_k$, *then* $L_k(S_k^*) \leq \tilde{L}_k(\tilde{S}_k^*)$, *and hence it is optimal to order procedures with non-decreasing convex order of procedure durations.*

Proof. Suppose $T_{k+1} \geq_{cx} T_k$. Since $L_k(x)$ is convex in x and S_k^* is its minimizer,

$$L_k(S_k^*) \leq L_k(\tilde{S}_k^*) \leq \tilde{L}_k(\tilde{S}_k^*),$$

where the second inequality follows from the fact that $Y_k \leq_{cx} \tilde{Y}_k$ as convex order is preserved under convolution. $\square$

5.4.3. Heuristic Approach to Original Problem

One major challenge with this heuristic is to find the distributions of the sum of random variables with different parameters. One way to do this would be to use Central Limit Theorem for large values of n; say, $n > 3$. However, for our numerical analyses, we used exponentially distributed surgery durations. However, Akkouchi (2008) [1] found an explicit expression for the sum of exponentials with different rate parameters, and we used these expressions to make our calculations exact. We also conducted numerical analyses using normal approximation, to provide a comparison between the accuracies and observed that the Normal approximation does not perform well, since the number of procedures we considered in our numerical analyses (reflecting approximately the average number of procedures to be scheduled in a single OR, in a day) is not very high. Therefore, to avoid confusion, we did not include the results with Normal approximation in Chapter 6.

The cumulative distribution functions of the sum of exponentials with different rate parameters can be given by the following formula, which are obtained using the formula for pdf of sum of exponentials with different rate parameters in [1]:

$$
\begin{aligned}
&F_{Y_1}(t) = 1 - e^{-\lambda_1 t}; t \geq 0\\
&F_{Y_2}(t) = 1 - \frac{\lambda_2}{(\lambda_2 - \lambda_1)} e^{-\lambda_1 t} - \frac{\lambda_1}{(\lambda_1 - \lambda_2)} e^{-\lambda_2 t}; t \geq 0\\
&F_{Y_n}(t) = 1 - \sum_{i=1}^{n} \frac{\lambda_1 \lambda_2 ... \lambda_n}{\lambda_i \prod_{j=1, j\neq i}^{n} (\lambda_j - \lambda_i)} e^{-\lambda_i t}; t \geq 0.
\end{aligned}
$$

In order to approximate the assigned durations, D_i's, we first find the sums of assigned durations $S_i = \sum_{j=1}^{i} D_j$ from (5.12), and then obtain from the formula $D_i = S_i - S_{i-1}$.

One important remark here would be that we proved in Section 5.4.2 that the optimum sequencing policy is the “Decreasing Convex Ordering” policy. However, for the special case of exponential distribution, which we use throughout our numerical analyses, “Decreasing Order (of rate parameters)” does not imply “Decreasing Convex Ordering”, it implies “Decreasing Stochastic Ordering”. Although we have not analytically shown that our results in Subsection 5.4.2 holds also for stochastic ordering of the procedures, we observe throughout our numerical analyses that the results hold for the wide range of parameter combinations we considered; i.e., “Decreasing Stochastic Ordering”, in other words, “Decreasing Order of rate parameters” always prevails as the best sequencing policy.

6. NUMERICAL RESULTS

In this chapter, we give the numerical results we obtained throughout this thesis work. We present our results in two sections based on which type of heuristic we analyze. We present our results for each type of heuristic in three subsection: First subsection, where we consider scheduling of two procedures, is essential in order to understand how the objective function and our heuristics change with different parameters and sequencing options. Next, we consider scheduling of three procedures, which serves as understanding the combinatorial nature of the problem. Finally, we consider larger numbers of procedures to schedule in order to analyze the complexity of the problem, and how our heuristics help improving the complexity of the exact solution of original problem.

As mentioned earlier, the OR scheduling problem we consider in this thesis is very hard in its most general form. One source of the difficulty is the number of different parameters that affect the behaviour of the objective function; namely, cost coefficients and distribution parameters. Whereas, in our problem, the number of cost coefficients are fixed (α_1 for idle time, α_2 for patient waiting time, and α_3 for overtime), the number of distribution parameters increases with the number of procedures to be scheduled. Although, we obtained a compact expression for the objective function in integral terms for any distribution, calculating the $n-tiple$ integrals become an arduous work as the number of procedures increase. Therefore, throughout our numerical analysis, we consider only the case where the all procedure durations are assumed to be distributed exponentially, with different rate (λ) parameters. We use explicit expressions, obtained in Chapter 4, for objective function when all procedure durations are exponentially distributed. This assumption leads to a compromise between loss of generality (results we obtain in the following subsections may not apply to other distributions) and gain in terms of calculation complexity.

6.1. Analysis of Sequencing Heuristics

6.1.1. Scheduling of Two Procedures

We start with analyzing the scheduling of two procedure. This special case of the problem is particularly important to understand the characteristics of the general problem. In a sense, two-procedure scheduling constitutes one of the building blocks of the general problem. Furthermore, as the number of procedures is very low, complexity which can be attributed to the combinatorial nature of the problem is not much revealed in this problem, which allows us to study all possible changes due to different parameters, much easier.

First, we analyze the change of optimum objective value with respect to each cost coefficient $(\alpha_1, \alpha_2, \alpha_3)$, when all other parameters are fixed. Figure 6.1 shows the change with respect to α_1, idle time cost coefficient, for three different combinations of other cost coefficients, $(\alpha_2; \alpha_3) = (0.1; 1), (1; 1), (1; 0.1)$. In each plot, change of optimum objective value for both scheduling policies ("Decreasing Order", shown with solid line; "Increasing Order", shown with dashed line) is given. It is worth to note that, we are not much interested in the actual value of optimum objective in this analysis, since higher cost coefficients would yield higher optimum value. We are rather interested in the comparison of values (and rates of change) of optimum objective for two sequencing policies; namely, decreasing order of rate parameters $(\lambda_1 > \lambda_2)$ and increasing order of rate parameters $(\lambda_2 > \lambda_1)$, where λ_1, λ_2 are the rate parameters of the procedures scheduled first and second, respectively. (a_i represents α_i, and L_i represents λ_i).

It is clearly seen from Figure 6.1 that for all three plots, solid line ("Decreasing Order" sequencing policy) is always lower than dashed line ("Increasing Order" sequencing policy) within the given range of $0.1 < \alpha_1 < 10$. Moreover, when we look at the rate of

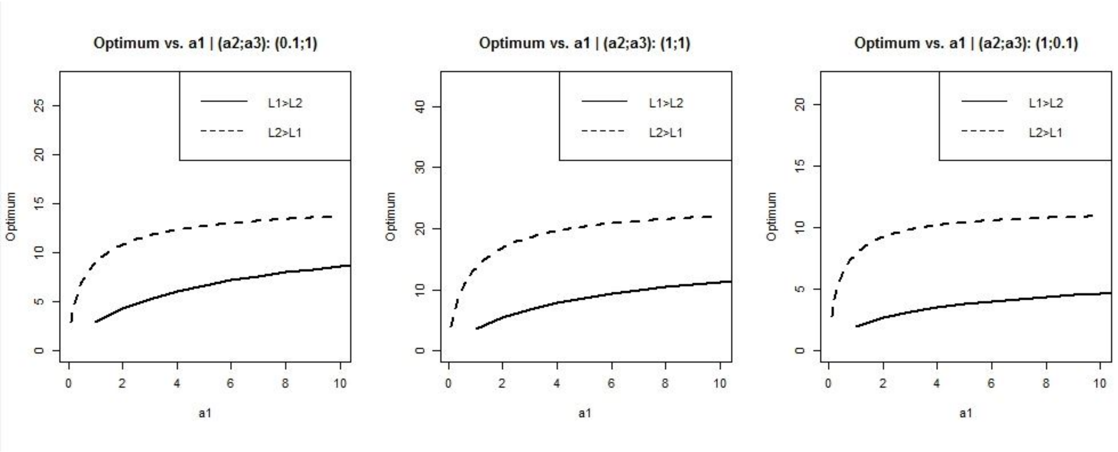

Figure 6.1. Change of Optimum Objective Value with Idle Time Cost Coefficient, $n = 2$.

change (increase) of dashed line and that of solid line, we can say that it is most likely that for given $\{\alpha_2; \alpha_3\}$ combinations, "Decreasing Order" sequencing policy will always gives a better optimum objective value than "Increasing Order" policy. However, we cannot state that with certainty without seeing the behaviour of the functions beyond values higher than 10 for α_1. Nevertheless, higher values of α_1 do not have much practical importance, since they will yield an unreasonably high weight for idle time cost.

Figures 6.2 and 6.3 let us make similar conclusions for the characteristics of optimum objective value for two sequencing policies with respect to changes in α_2 (patient waiting time cost coefficient) and α_3 (overtime cost coefficient), respectively. We can conclude that for a wide range of cost coefficients, "Decreasing Order" sequencing policy provides better optimum objective values than "Increasing Order" sequencing policy.

To conclude the analysis of individual effects of cost coefficients for two-procedure problem, it is worth to mention that although plots presented here are obtained for a unique rate parameter combination of procedures (λ=$\{0.1, 0.2\}$), we have conducted the same analysis for different fixed λ combinations which yielded similar results. We did not include them due to space constraints.

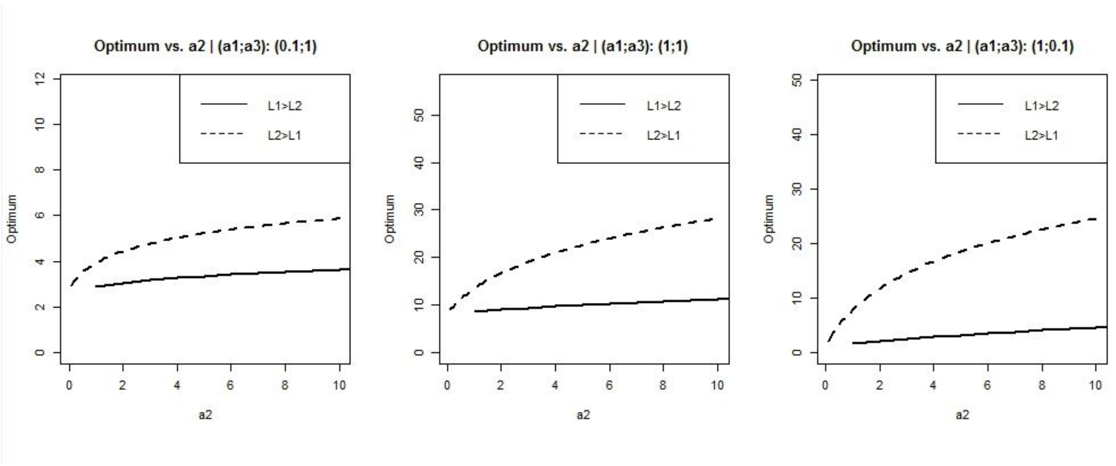

Figure 6.2. Change of Optimum Objective Value with Patient Waiting Time Cost Coefficient, $n = 2$.

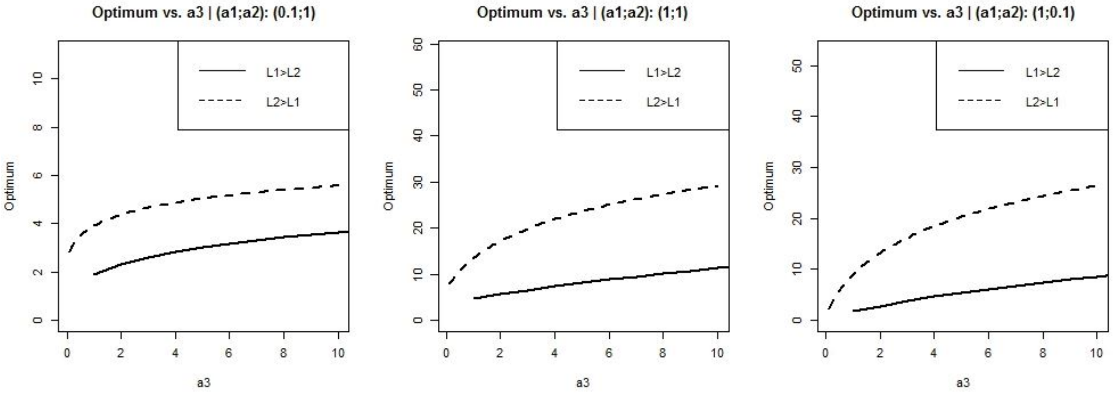

Figure 6.3. Change of Optimum Objective Value with Overtime Cost Coefficient, $n = 2$.

Finally, we try to explore the absolute deviations of sequencing heuristics from the optimum sequence for a wider range of rate parameter combinations. One important remark about the possible values for λ parameters would be that the reasonable range for these parameters is much smaller, as they are related to durations of the procedures. For example, if we take the durations of procedures in hours, then, $\lambda < 0.1$ would mean an expected procedure duration of more than 10 hours. Similarly a $\lambda > 2$ value would mean an expected procedure duration shorter than 30 minutes, which is not very likely, as one can understand. Thus, we confine our λ values to $0.1 < \lambda_i < 2$ throughout the remainder of numerical analyses. Before we move on to results of our numerical analysis, it is worth to mention the purpose of the following analysis. We would like to see how the two sequencing policies, "Decreasing Order" and "Increasing Order", change when parameters are simultaneously changed. Our expectation, based on the results provided above, was that "Decreasing Order" sequencing policy would perform better in general. Figure 6.4 shows the optimum objective values of two sequencing policies (for two-procedure case), solid line for "Decreasing Order"; dashed line for "Increasing Order", for each parameter combination.

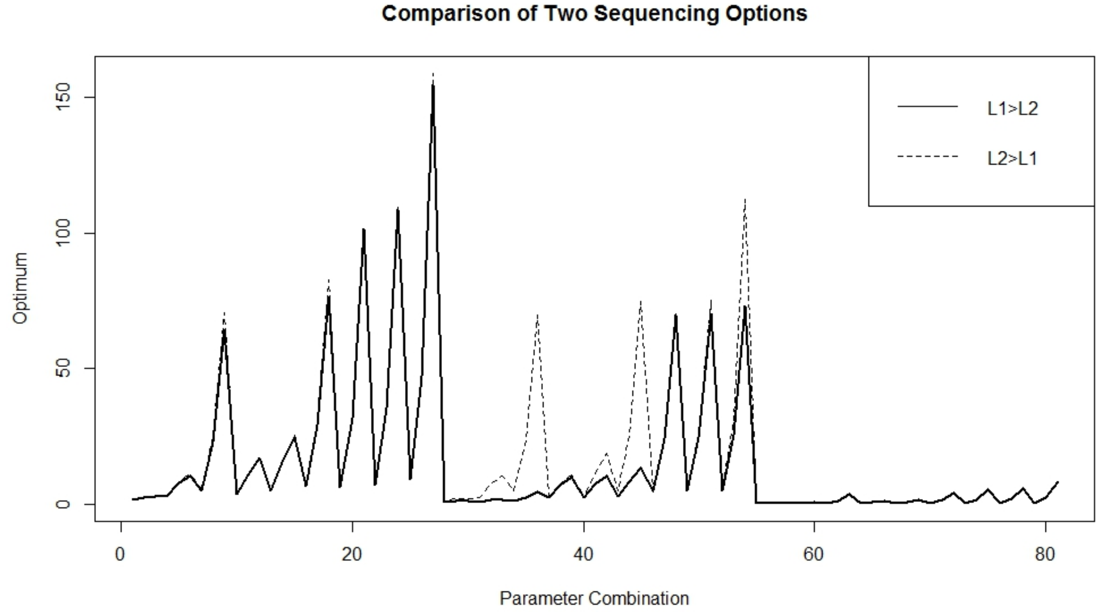

Figure 6.4. Comparison of Two Different Sequencing Options, $n = 2$.

As expected, "Decreasing Order" sequencing policy performs better across all parameter combinations. This result has a very practical importance, since it gives us an idea of the optimal sequencing policy. However, Figure 6.4 is not nearly sufficient to assert such a conclusion for all possible cases (parameter combinations). First of all, this conclusion can only be stated for two procedure OR scheduling problem, based on the plot, as we do not yet have an idea of how these sequencing policies would perform when larger numbers of procedures are considered. Secondly, even for two procedure OR scheduling problem, one needs to provide analytical proof for such a conclusion. Nevertheless, as stated before, we are interested in practically reasonable parameter combinations, for which Figure 6.4 provides a good overview. We run similar analyses for larger values of n ($n = 3, 4, 5, 6$), where we consider three different values (1,5,10) for each cost coefficient and 100 different random $\mathcal{L}$ sets. Selection of α values is done with the consideration that in practice it is very unlikely that one of the cost parameters is more than 10 times larger than another one. Our results showed that in all parameter combinations, "Decreasing Order" sequencing policy performs that than the "Increasing Order" and the average deviation of "Increasing Order" policy from "Decreasing Order" is %40. Our conclusion based on the results in this subsection is that although "Decreasing Order" policy is not always the best sequencing policy, in a very wide range of reasonable parameters it outperforms the "Increasing Order" policy, which makes it a good candidate as a sequencing heuristic.

6.1.2. Scheduling of Three Procedures

Next, we analyze the scheduling of three procedure. This special case of the problem is particularly important to understand the combinatorial nature of the general problem. Even if the number of procedures is still very low, complexity which can be attributed to the combinatorial nature of the problem starts to become evident in this special case, which is still manageable. Similar to previous subsection, we first analyze the change of optimum objective value with respect to each cost coefficient ($\alpha_1, \alpha_2, \alpha_3$), when all other parameters are fixed. Figure 6.5 shows the change with respect to α_1, idle time cost coefficient, for three different combinations of other cost coefficients, $(\alpha_2; \alpha_3) = (0.1; 1), (1; 1), (1; 0.1)$. In each plot, change of optimum objective value for 2 sequencing policies (solid line for "Decreasing Order" policy, dashed line for "Increasing Order" policy) is given. What we seek to see is that whether change of α_1, idle time cost coefficient changes the optimum sequencing policy: In this particular plot λ parameter set is fixed as $\{0.1; 0.2; 0.3\}$. However, we have produced similar plots for different λ parameter sets which yielded in similar results.

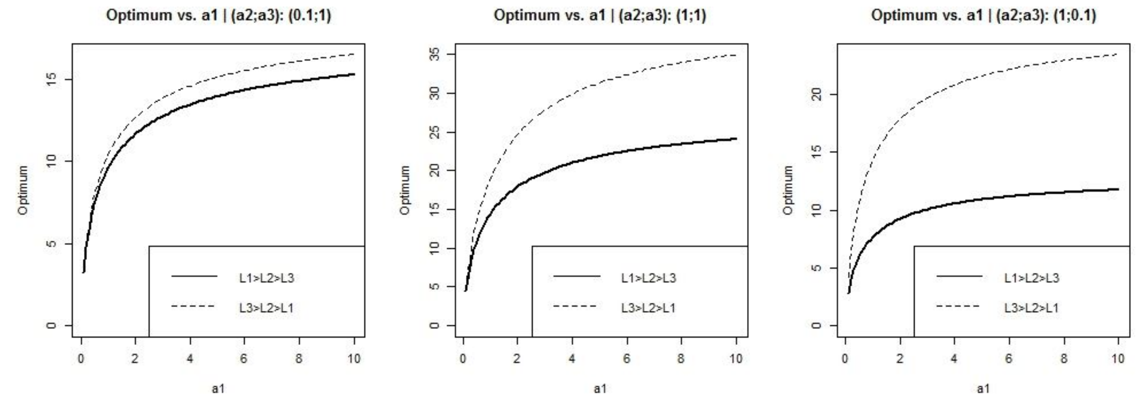

Figure 6.5. Change of Optimum Objective Value with Idle Time Cost Coefficient, $n = 3$.

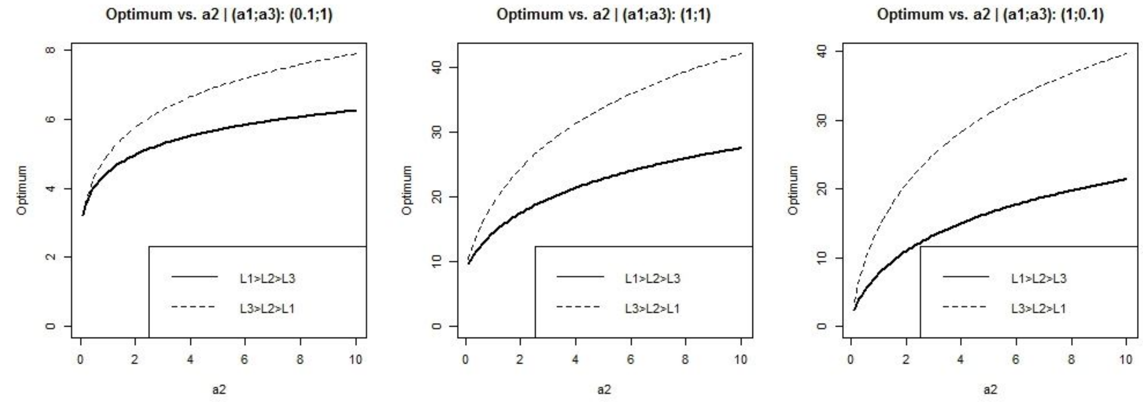

Figure 6.6. Change of Optimum Objective Value with Patient Waiting Time Cost Coefficient, $n = 3$.

We see that, in Figure 6.5, that for all three $\{\alpha_2; \alpha_3\}$ combinations $\{0.1; 1\}$, $\{1; 1\}$, $\{1; 0.1\}$, the optimum sequence remains the same “Decreasing Order”, irrespective of the value α_1 takes within given range. We observe similar results for the change of optimum objective value with respect to α_2, patient waiting time cost coefficient, in Figure 6.6. An important conclusion from these plots would be that as α_1, α_2 decreases, the performances of two sequencing options get closer to each other.

The last, and most interesting, of the plots showing the effect of cost coefficients is

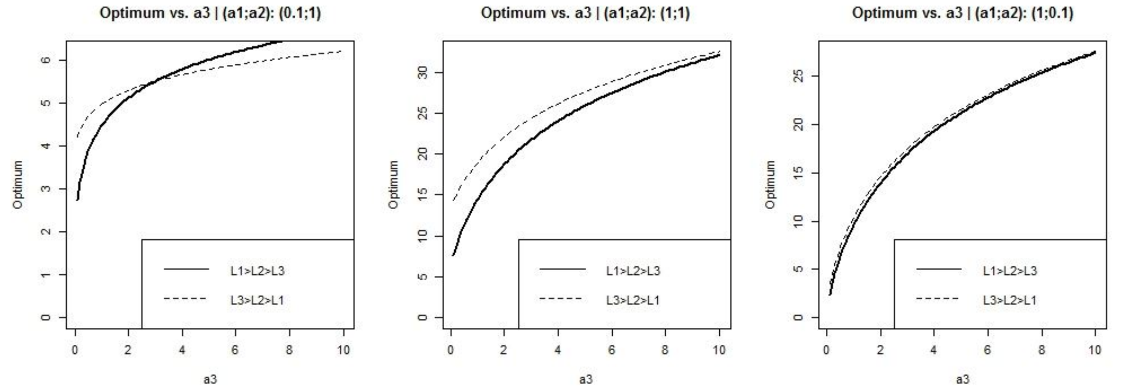

Figure 6.7. Change of Optimum Objective Value with Overtime Cost Coefficient, $n = 3$.

given in Figure 6.7. We observe from the left-most plot that there are parameter combinations for which "Decreasing Order" sequencing policy is not the best option. For the other plots, the difference between the two sequencing policies gets smaller as the α_3 value increases, and somewhere outside the range we considered for α_3, "Increasing Order" policy would start to perform better than "Decreasing Order". Therefore, we can conclude that as the overtime cost coefficient increases, performance of the sequencing policy "Decreasing Order" gets poorer. This can be intuitively justified as follows: The highest impact on the overtime cost is from the last procedure, therefore, it would make sense to schedule the procedure with smallest variance last to decrease the impact on overtime. This interpretation allows us to justify the results seen on Figures 6.5 and 6.6 As one of the two cost coefficients α_1, α_3 gets smaller, the relative impact of α_3 increases. Therefore, scheduling the procedure with smallest variance last becomes more favorable. This result is particularly important as it refutes the assumption that "decreasing order of rate parameter sequencing policy is always the best policy", which means that we have to consider all sequencing options for making the best decision about scheduling, thus combinatorial nature of the problem still stands.

6.2. Analysis of Assigned Duration Heuristics

6.2.1. Scheduling of Two Procedures

Having an idea about behaviour of optimum objective value for different parameter combinations, our next purpose is to how well the three heuristics we introduced in Chapter 5 approximate the assigned durations. We first consider Expectation-based Heuristic described in Section 5.2. As shown analytically, this heuristic always provides better result for the ($\lambda_1 > \lambda_2$), "Decreasing Order" sequencing option, which makes this heuristic a sensible candidate for an approximate solution.

In parameter selection our purpose is to cover all possible cases of rate parameter sets. We consider 3 different values for α parameters (0.1,1,10), as in the previous subsection. Table 6.1 gives the set of values we considered for the parameters for two-procedure problem. It is worth to mention that while determining the $\mathcal{L}$ sets, we consider all possible cases of proximity of rate parameters to each other. As mentioned earlier, we confined our analyses to only reasonable values of parameters. It would be accurate to say that these sets of parameters are sufficient to have an overall picture of the possible cases in various real life implementations.

Table 6.1. Parameter sets for two-procedure numerical analyses.

α_1	{0.1, 1, 10 }
α_2	{0.1, 1, 10 }
α_3	{0.1, 1, 10 }
$\mathcal{L}$	{(0.10; 0.11), (0.1; 2.00), (1.99; 2.00) }

Figure 6.8 shows the change of optimum objective value for original problem (solid

line) and expectation-based heuristic (dashed line) for the parameter combinations shown in Table 6.1 Table 6.2 provides results on the Average and Maximum Relative Deviations (AD and MD, respectively) of our heuristic from the real optimum. We will use these two performance measures throughout the numerical analyses of assigned duration heuristics.

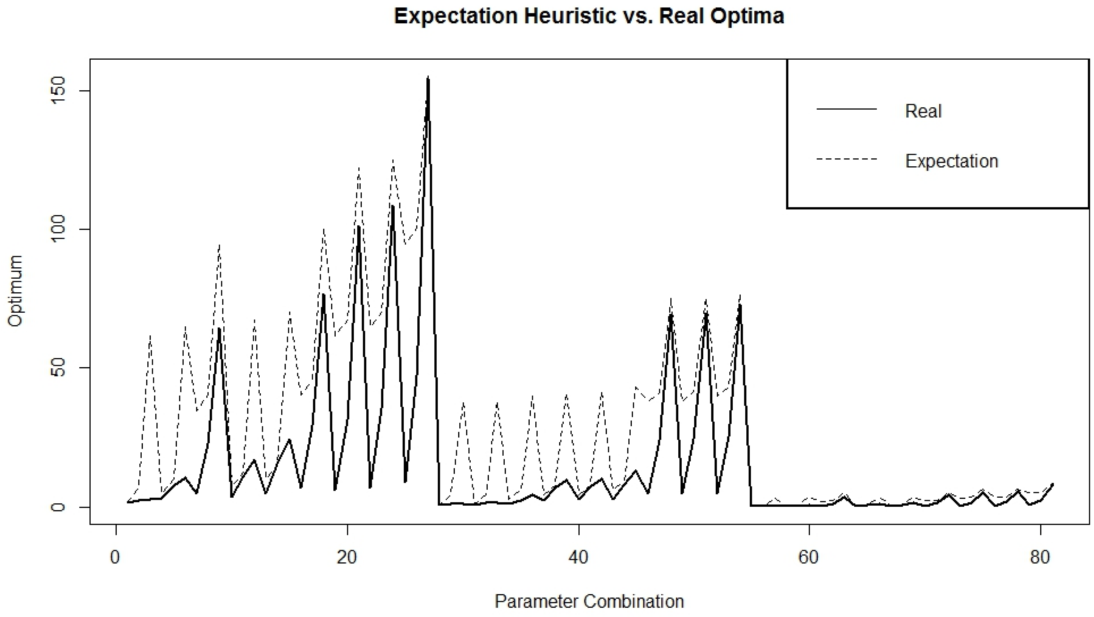

Figure 6.8. Comparison of Expectation-based Heuristic with Real Optimum, $n = 2$.

As seen from the figure, although the heuristic generally captures the behaviour of the original problem, it has significant deviations from the original objective value across all parameter combinations. From Table 6.2, it is evident that the Expectation Heuristic performs worst when α_2 and α_3 attains their minimum value, whereas α_1 attains its maximum value. We also observe that the heuristic performs best, when all the cost coefficients have equal values. The average deviation across all parameter combinations is: 336%. One can see that, this is a very high deviation, which disqualifies expectation-based heuristic from being a good approximation heuristic.

The high average deviation from the real optimum made us look for more accurate heuristics. The fact that, expectation-based heuristic was generally able to capture the

Table 6.2. Relative deviations of Expectation heuristic from real optima, $n = 2$.

		$\alpha_1 = 0.1$		$\alpha_1 = 1$		$\alpha_1 = 10$	
		AD	**MD**	**AD**	**MD**	**AD**	**MD**
$\alpha_2 = 0.1$	$\alpha_3 = 0.1$	0.02	0.05	2.26	3.09	25.76	34.19
	$\alpha_3 = 1$	0.89	0.99	0.13	0.15	3.06	3.26
	$\alpha_3 = 10$	8.68	9.43	1.00	1.14	0.16	0.21
$\alpha_2 = 1$	$\alpha_3 = 0.1$	0.43	0.57	0.96	2.29	11.50	24.32
	$\alpha_3 = 1$	0.99	1.15	0.02	0.05	2.26	3.09
	$\alpha_3 = 10$	7.88	8.31	0.89	0.99	0.13	0.15
$\alpha_2 = 10$	$\alpha_3 = 0.1$	4.68	6.16	1.09	1.77	2.92	7.86
	$\alpha_3 = 1$	3.85	5.22	0.43	0.57	0.96	2.29
	$\alpha_3 = 10$	8.94	9.91	0.99	1.15	0.02	0.05

behaviour of the real optimum was a starting point for finding our next heuristic. It was stated earlier that the expectation-based heuristic performs poorer for specific values of α parameters. An immediate question from this fact arises: "Is it possible to adjust/modify the expectation-based heuristic using cost coefficients in a way to improve the accuracy of the approximate solutions?" To this end, we developed our next heuristic: Myopic Heuristic. Section 5.3 provides information about the background problem for myopic heuristic which constitutes an upper bound to our original problem. Analytical proof of how the optimum duration assignments are calculated for this bounding problem are also given in Section 5.2 As seen from these analytical calculations, the resulting duration assignments are, indeed, a modification of the expectation-based heuristic using the cost coefficients. We expect this heuristic to perform better than expectation-based heuristic for the given parameter combinations. Figure 6.9 shows how Myopic heuristic performs compared to real optimum, and Table 6.3 presents the results on relative deviations for different α parameter combinations.

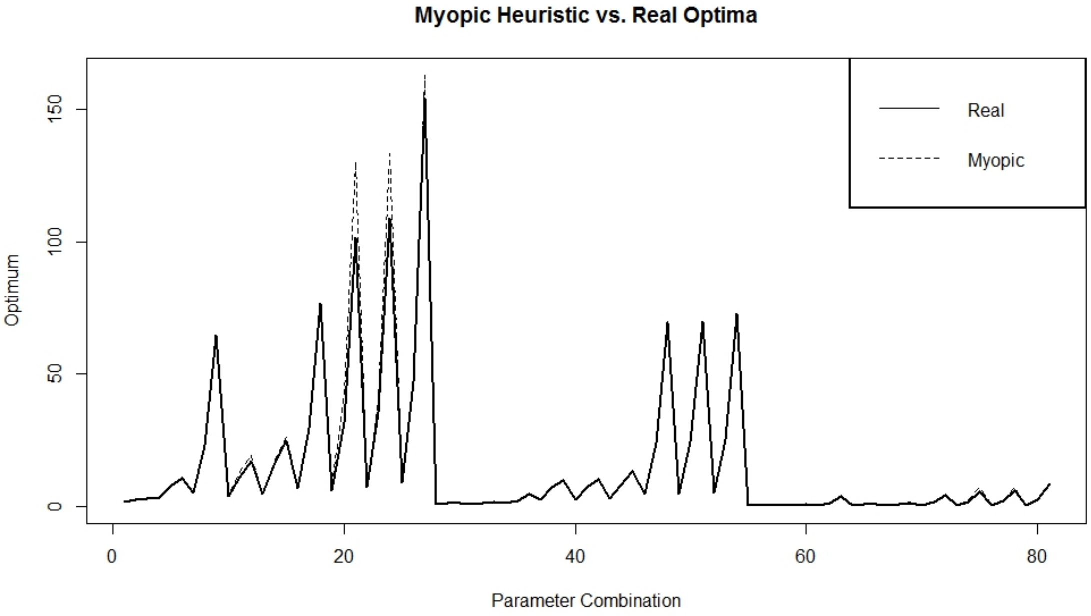

Figure 6.9. Comparison of Myopic Heuristic with Real Optimum, $n = 2$.

Table 6.3. Relative deviations of Myopic heuristic from real optima, $n = 2$.

		$\alpha_1 = 0.1$		$\alpha_1 = 1$		$\alpha_1 = 10$	
		AD	**MD**	**AD**	**MD**	**AD**	**MD**
$\alpha_2 = 0.1$	$\alpha_3 = 0.1$	0.04	0.06	0.04	0.06	0.02	0.03
	$\alpha_3 = 1$	0.10	0.16	0.16	0.24	0.08	0.12
	$\alpha_3 = 10$	0.13	0.22	0.26	0.42	0.20	0.31
$\alpha_2 = 1$	$\alpha_3 = 0.1$	0.00	0.00	0.00	0.00	0.00	0.01
	$\alpha_3 = 1$	0.00	0.01	0.04	0.06	0.04	0.06
	$\alpha_3 = 10$	0.01	0.02	0.10	0.16	0.16	0.24
$\alpha_2 = 10$	$\alpha_3 = 0.1$	0.00	0.00	0.00	0.00	0.00	0.00
	$\alpha_3 = 1$	0.00	0.00	0.00	0.00	0.00	0.00
	$\alpha_3 = 10$	0.00	0.00	0.00	0.01	0.04	0.06

One can see from Figure 6.9 that Myopic heuristic captures the behaviour of the real optimum quite efficiently, except for a few parameter sets. Table 6.3 reveals the change of relative deviation with respect to cost coefficients. We observe that relative deviation of Myopic heuristic increases as α_3, overtime cost coefficient, increases. Furthermore, relative deviation decreases as the α_2, patient waiting time cost coefficient, increases. There is no obvious relation between α_1, idle time cost coefficient, and relative deviation. We also find that the average percent deviation of approximate solutions obtained by Myopic heuristic for the given parameter combinations is 5.3%, which is a significant improvement from the expectation-based heuristic.

These results motivates our next question: "Are these relatively good results due to considering only two procedures, and is it possible that this heuristic performs much worse when larger numbers of procedures are considered in this OR scheduling problem?". This question is quite sensible, because when we revisit the results obtained in Section 5.2, we see that when approximating the optimum assigned durations, the modification on expected value is identical for the first $n-1$ procedures, and only difference is for the last procedure on the sequence. This question led us develop our last heuristic which we called Veteran's Hospital Heuristic, due to its background problem which constitutes another upper bound for our original problem. Section 5.4 provides necessary information and analytical calculations for the bounding problem, OR Scheduling in Veteran's Hospital. Figure 6.10 shows how Veteran's Hospital heuristic performs compared to real optimum, and Table 6.4 presents the results on relative deviations for different α parameter combinations.

As seen from Figure 6.10, Veteran's Hospital heuristic also catches the behaviour of the real optimum very efficiently for the given parameter combinations. However, Table 6.4 makes it evident that for some parameter combinations Veteran's heuristic performs significantly worse than Myopic heuristic. The worst performance of the Veteran's heuristic happens when α_1 and α_2 attain their minimum possible value, whereas α_3 attains its

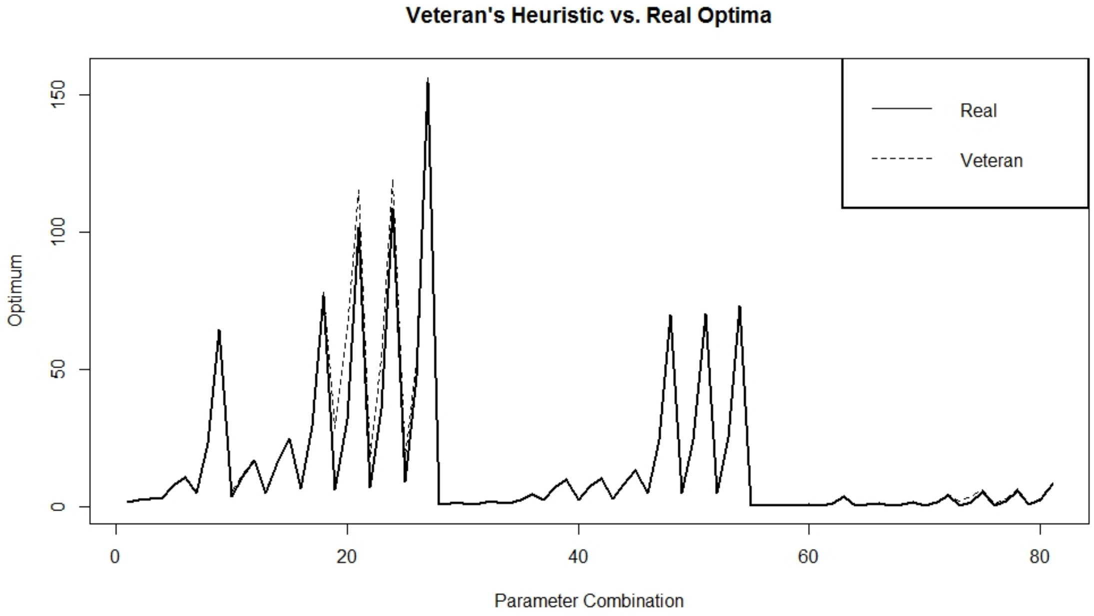

Figure 6.10. Comparison of Veteran's Hospital Heuristic with Real Optimum, $n = 2$.

Table 6.4. Relative deviations of Veteran heuristic from real optima, $n = 2$.

		$\alpha_1 = 0.1$		$\alpha_1 = 1$		$\alpha_1 = 10$	
		AD	**MD**	**AD**	**MD**	**AD**	**MD**
$\alpha_2 = 0.1$	$\alpha_3 = 0.1$	0.01	0.02	0.00	0.00	0.00	0.00
	$\alpha_3 = 1$	0.37	0.58	0.07	0.11	0.00	0.01
	$\alpha_3 = 10$	2.78	4.51	0.74	1.18	0.09	0.15
$\alpha_2 = 1$	$\alpha_3 = 0.1$	0.00	0.01	0.01	0.02	0.00	0.00
	$\alpha_3 = 1$	0.08	0.13	0.01	0.02	0.00	0.00
	$\alpha_3 = 10$	1.10	1.75	0.37	0.58	0.07	0.11
$\alpha_2 = 10$	$\alpha_3 = 0.1$	0.00	0.01	0.01	0.02	0.00	0.00
	$\alpha_3 = 1$	0.04	0.06	0.00	0.01	0.01	0.02
	$\alpha_3 = 10$	0.68	1.07	0.08	0.13	0.01	0.02

maximum possible value. A stronger comment about this result can be made from Table 6.4 that the relative deviation of Veteran's heuristic is directly proportional to α_3, and inversely proportional to α_1 and α_2. The overall relative deviation of Veteran's heuristic is approximately 24%, which is much higher than Myopic heuristic's. Although Myopic heuristic seems to perform better for these parameter combinations, it is quite evident from the Table 6.4 that this relatively high deviation of Veteran's heuristic can mostly be attributed to extreme cases ($\alpha_1 = 0.1, \alpha_2 = 0.1, \alpha_3 = 10$), which are not practically relevant. In the Subsection 6.2.3, we will consider more reasonable α parameters for larger values of n, and observe the changes in performances.

6.2.2. Scheduling of Three Procedures

Our next analysis focuses on the performances of three different heuristics for three-procedure OR scheduling problem. Table 6.5 gives the list of parameters we consider in our numerical analyses throughout this subsection. We selected the $\mathcal{L}$ sets in a way to represent all possible proximities of λ parameters with respect to each other.

Table 6.5. Parameter sets for three-procedure numerical analyses.

α_1	{0.1, 1, 10}
α_2	{0.1, 1, 10}
α_3	{0.1, 1, 10}
$\mathcal{L}$	{(0.10; 0.11; 0.12), (0.10; 0.11; 1.00), (0.10; 0.11; 2.00), (0.10; 1.00; 1.01), (0.10; 1.00; 2.00), (0.10; 1.99; 2.00), (1.00; 1.01; 1.02), (1.00; 1.01; 2.00), (1.00; 1.99; 2.00), (1.98; 1.99; 2.00)}

In a similar way as in the previous subsection, we first present our results related to performance of Expectation-based heuristic. Figure 6.11 shows the change of optimum objective value for original problem (solid line) and expectation-based heuristic (dashed line) for the parameter combinations shown in Table 6.5 Table 6.6 presents the results on relative deviations for different α parameter combinations.

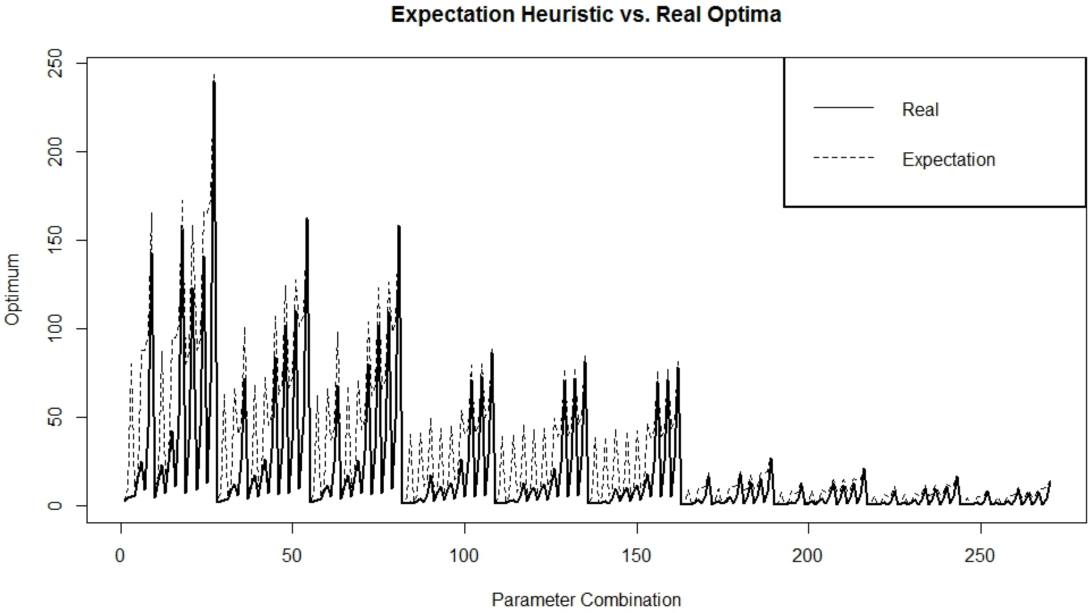

Figure 6.11. Comparison of Expectation-based Heuristic with Real Optimum, $n = 3$.

Table 6.6. Relative deviations of Expectation heuristic from real optima, $n = 3$.

		$\alpha_1 = 0.1$		$\alpha_1 = 1$		$\alpha_1 = 10$	
		AD	**MD**	**AD**	**MD**	**AD**	**MD**
$\alpha_2 = 0.1$	$\alpha_3 = 0.1$	0.02	0.04	1.81	2.83	20.82	31.10
	$\alpha_3 = 1$	0.96	1.13	0.14	0.18	2.95	3.23
	$\alpha_3 = 10$	9.31	10.60	1.13	1.39	0.21	0.29
$\alpha_2 = 1$	$\alpha_3 = 0.1$	0.73	1.05	0.52	1.62	6.18	15.49
	$\alpha_3 = 1$	1.20	1.58	0.02	0.04	1.81	2.83
	$\alpha_3 = 10$	8.11	8.76	0.96	1.13	0.14	0.18
$\alpha_2 = 10$	$\alpha_3 = 0.1$	7.07	9.25	1.17	1.67	1.01	3.71
	$\alpha_3 = 1$	5.75	8.40	0.73	1.05	0.52	1.62
	$\alpha_3 = 10$	10.14	12.45	1.20	1.58	0.02	0.04

As seen from the Figure 6.11, the Expectation heuristic has significant deviations from the optimum for most of the parameter combinations. This result is in line with what we have observed in two-procedure case. When we checked for which parameter sets these deviations are the highest and observed that, similar to two-procedure OR scheduling problem, highest deviations always occur at the largest possible value of α_3 within the parameter combinations we considered. We also calculated the average relative deviation of the expectation-based heuristic from the real optimum across all parameter combinations, which is 313%. In general, we conclude from Table 6.6 that the results on relative deviations are very similar to those of two-procedure case.

Next, we consider Myopic heuristic. Figure 6.12 shows how Myopic heuristic performs compared to real optimum and Table 6.7 presents the results on relative deviations for different α parameter combinations.

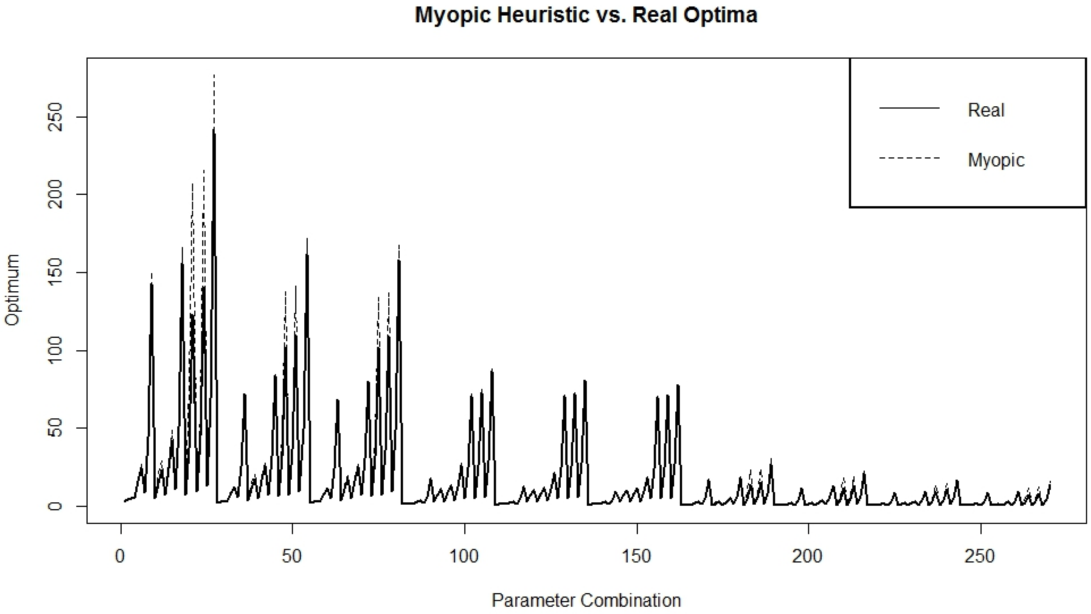

Figure 6.12. Comparison of Myopic Heuristic with Real Optimum, $n = 3$.

One can see that although Myopic heuristic captures the behaviour of the real optimum generally, there are some parameter sets for which the performance is much poorer. We observe again that relative deviation of Myopic heuristic increases as α_3, overtime cost

Table 6.7. Relative deviations of Myopic heuristic from real optima, $n = 3$.

		$\alpha_1 = 0.1$		$\alpha_1 = 1$		$\alpha_1 = 10$	
		AD	**MD**	**AD**	**MD**	**AD**	**MD**
$\alpha_2 = 0.1$	$\alpha_3 = 0.1$	0.09	0.17	0.11	0.15	0.06	0.07
	$\alpha_3 = 1$	0.19	0.43	0.31	0.57	0.17	0.25
	$\alpha_3 = 10$	0.29	0.73	0.58	1.33	0.39	0.73
$\alpha_2 = 1$	$\alpha_3 = 0.1$	0.00	0.01	0.04	0.07	0.05	0.07
	$\alpha_3 = 1$	0.01	0.02	0.09	0.17	0.11	0.15
	$\alpha_3 = 10$	0.03	0.07	0.19	0.43	0.31	0.57
$\alpha_2 = 10$	$\alpha_3 = 0.1$	0.00	0.00	0.00	0.01	0.04	0.06
	$\alpha_3 = 1$	0.00	0.00	0.00	0.01	0.04	0.07
	$\alpha_3 = 10$	0.00	0.00	0.01	0.02	0.09	0.17

coefficient, increases. Furthermore, relative deviation decreases as the α_2, patient waiting time cost coefficient, increases. There is no obvious relation between α_1, idle time cost coefficient, and relative deviation. We also find that the average percent deviation of approximate solutions obtained by Myopic heuristic for the given parameter combinations is 12%, which is significantly larger than the value we found for two-procedure case. As we suspected, this result hints us that Myopic heuristic's performance tends to decrease as the number of procedures to be scheduled increases, which justifies our initial motivation for developing the Veteran's Hospital heuristic.

Finally, we analyze the performance of our last heuristic: Veteran's Hospital heuristic. We expect this heuristic to be more robust to changes in number of procedures considered. Figure 6.13 shows how Veteran's hospital heuristic performs compared to real optimum and Table 6.8 presents the results on relative deviations for different α parameter combinations.

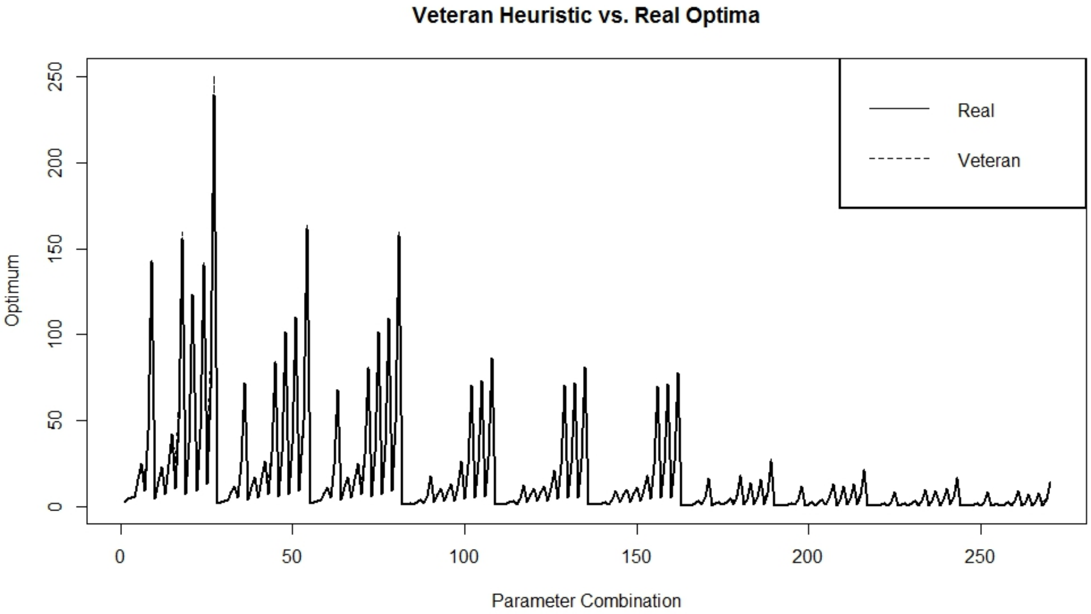

Figure 6.13. Comparison of Veteran's Hospital Heuristic with Real Optimum, $n = 3$.

Table 6.8. Relative deviations of Veteran's heuristic from real optima, $n = 3$.

		$\alpha_1 = 0.1$		$\alpha_1 = 1$		$\alpha_1 = 10$	
		AD	**MD**	**AD**	**MD**	**AD**	**MD**
$\alpha_2 = 0.1$	$\alpha_3 = 0.1$	0.03	0.05	0.00	0.01	0.00	0.00
	$\alpha_3 = 1$	0.02	0.05	0.00	0.01	0.00	0.00
	$\alpha_3 = 10$	0.04	0.06	0.00	0.01	0.00	0.00
$\alpha_2 = 1$	$\alpha_3 = 0.1$	0.14	0.23	0.02	0.03	0.00	0.00
	$\alpha_3 = 1$	0.17	0.32	0.03	0.05	0.00	0.01
	$\alpha_3 = 10$	0.12	0.20	0.02	0.05	0.00	0.01
$\alpha_2 = 10$	$\alpha_3 = 0.1$	0.57	1.03	0.09	0.13	0.01	0.02
	$\alpha_3 = 1$	0.96	1.61	0.14	0.23	0.02	0.03
	$\alpha_3 = 10$	1.06	2.11	0.17	0.32	0.03	0.05

As seen from the Figure 6.13, Veteran's Hospital heuristic performs almost perfect when three procedures are considered for scheduling. We first check the average relative deviation, which is 13.3%. We observe a significant improvement from the two-procedure case. This confirms our expectation of improvement in performance of Veteran's heuristic as the number of procedures increases. Table 6.8 reveals another interesting difference from two-procedure case, that the relative deviation of Veteran's heuristic is now directly proportional to α_2, whereas it was inversely proportional in two-procedure case.

6.2.3. Scheduling of n-Procedures

In the final subsection of our numerical analyses, we consider n-procedure OR scheduling problem, for $n = 2, 3, 4, 5, 6$. Our main objective with these analyses is to confirm our hypotheses from previous subsections about the performances of our heuristics based. The design of the numerical analyses in this subsection is in the following way: We consider three different values for α, the cost coefficient parameters. This time, to avoid extreme cases (such as one cost parameter being 100 times larger than another one) which are unlikely to happen in real life, we use the following set of values for α parameters: $\{1, 5, 10\}$. This selection still allows us observe the change of performances based on different values of parameters. Furthermore, as the number of procedures increases, the combinations necessary to cover all possible cases of $\mathcal{L}$ sets become intractable. Therefore, instead of trying to cover all possible combinations, we make random selection of λ_i values, from the interval (0.1, 2), as introduced earlier. We select 100 random sets of size n for $\mathcal{L}$ parameters. Finally, as the number of possible sequencing options also become intractable for larger values of n, we assume "Decreasing Order" sequencing heuristic as the optimum sequence. Table 6.9, provides a summary of the results for n-procedure case.

The main performance measure for different heuristics is again the relative deviations from optimum. In addition, we also observe the running times of algorithms for the heuristics and optimum objective to be able to analyze the efficiencies of our heuristics.

Table 6.9. Summary of n-Procedure Numerical Analyses.

		n=2	**n=3**	**n=4**	**n=5**	**n=6**
Average Deviation	Expectation	0.41	0.40	0.41	0.43	0.46
	Myopic	0.04	0.10	0.18	0.22	0.30
	Veteran	0.02	0.04	0.08	0.09	0.13
Maximum Deviation.	Expectation	3.03	2.69	1.99	2.24	2.39
	Myopic	0.25	0.61	0.94	1.25	1.43
	Veteran	0.19	0.38	0.55	0.73	0.79
Running Time (min.)	Expectation	0.004	0.008	0.018	0.036	0.089
	Myopic	0.005	0.009	0.019	0.038	0.089
	Veteran	0.050	0.100	0.170	0.270	0.474
	Optimum	0.195	0.630	2.023	6.084	18.140

Table 6.9 shows that Expectation heuristic's performance is almost steady around 40% average relative deviation. However, Myopic heuristic performs worse as the number of procedures increase. It gets even very close to the performance of Expectation heuristic, which has been shown to perform poorly, throughout the previous sections. On the other hand, Veteran's heuristic performs much better than both other heuristics. Although its performance also worsens as the number of procedures increase, the change is minimal. Similar interpretations can also be made about the max relative deviations. Lastly, in terms of running times, we see that Expectation and Myopic heuristics has almost same running times across all values of n. However, Veteran's heuristic's running time is longer than theirs, although for the given values of n, the maximum running time is 0.5 minutes, which is still quite negligible. On the other hand, finding the real optimum has a significantly higher running time. For $n = 6$, the running time is around 18 minutes. We observe that the running time for heuristics double whereas it triples for optimum at every increase of n, which means that they all have an exponential time complexity ($2^n, 3^n$, for heuristics and optimum, respectively). Based on this complexity, we can expect to find the optimum for $n = 10$, in 27 hours, whereas it would require only 8 minutes, if we use

Veteran's heuristic. One can say that around $10 - 20\%$ deviation is a good compromise for reducing the running time as significantly as shown above.

Lastly, Tables 6.10-14 show the change of average and maximum relative deviations of heuristics for different α combinations. We observe from these tables that our hypotheses about the change of relative deviations of heuristics are confirmed. For, example, Veteran's heuristic's average relative deviation is directly proportional to α_2, whereas it is inversely proportional to α_1 and α_3. On the other hand, Myopic heuristic's average relative deviation is inversely proportional to α_2, directly proportional to α_1 and α_3. As seen from these relationships, the effects of cost coefficients on the performances of Myopic and Veteran's heuristics is exactly the opposite of each other. A sensible way of handling this inverse relationships would be to devise a hybrid heuristic which determines the assigned durations as follows:

- When α_1 is very high(low) use Veteran's(Myopic) heuristic,
- When α_2 is very high(low) use Myopic(Veteran's) heuristic,
- When α_3 is very high(low) use Veteran's(Myopic) heuristic.

Such a hybrid approach would decrease the deviation, for n=6 case, from 13% to 6%. Since our main focus was to provide a comparison between alternative heuristics, we do not get into deeper analyses of hybrid heuristics in this thesis work.

Table 6.10. Relative deviations of Heuristic from real optima for $n = 2$.

			Expectation		Myopic		Veteran	
			AD	**MD**	**AD**	**MD**	**AD**	**MD**
$\alpha_1 = 1$	$\alpha_2 = 1$	$\alpha_3 = 1$	0.01	0.05	0.03	0.06	0.01	0.02
		$\alpha_3 = 5$	0.35	0.40	0.05	0.13	0.01	0.02
		$\alpha_3 = 10$	0.89	0.99	0.06	0.15	0.01	0.02
	$\alpha_2 = 5$	$\alpha_3 = 1$	0.16	0.20	0.00	0.00	0.08	0.16
		$\alpha_3 = 5$	0.39	0.48	0.01	0.02	0.03	0.06
		$\alpha_3 = 10$	0.85	0.97	0.01	0.02	0.03	0.04
	$\alpha_2 = 10$	$\alpha_3 = 1$	0.46	0.57	0.00	0.00	0.11	0.17
		$\alpha_3 = 5$	0.57	0.73	0.00	0.00	0.09	0.19
		$\alpha_3 = 10$	0.97	1.15	0.00	0.01	0.07	0.13
$\alpha_1 = 5$	$\alpha_2 = 1$	$\alpha_3 = 1$	0.90	1.33	0.06	0.07	0.00	0.00
		$\alpha_3 = 5$	0.10	0.11	0.11	0.19	0.00	0.01
		$\alpha_3 = 10$	0.10	0.13	0.12	0.25	0.00	0.01
	$\alpha_2 = 5$	$\alpha_3 = 1$	0.38	1.06	0.01	0.01	0.01	0.03
		$\alpha_3 = 5$	0.01	0.05	0.03	0.06	0.01	0.02
		$\alpha_3 = 10$	0.04	0.06	0.04	0.09	0.01	0.02
	$\alpha_2 = 10$	$\alpha_3 = 1$	0.28	0.94	0.00	0.00	0.02	0.04
		$\alpha_3 = 5$	0.02	0.05	0.01	0.02	0.02	0.04
		$\alpha_3 = 10$	0.04	0.07	0.02	0.04	0.01	0.03
$\alpha_1 = 10$	$\alpha_2 = 1$	$\alpha_3 = 1$	2.16	3.03	0.05	0.06	0.00	0.00
		$\alpha_3 = 5$	0.36	0.38	0.12	0.18	0.00	0.00
		$\alpha_3 = 10$	0.13	0.15	0.14	0.24	0.00	0.00
	$\alpha_2 = 5$	$\alpha_3 = 1$	0.98	2.42	0.01	0.01	0.01	0.01
		$\alpha_3 = 5$	0.19	0.33	0.05	0.07	0.00	0.01
		$\alpha_3 = 10$	0.05	0.06	0.06	0.11	0.00	0.01
	$\alpha_2 = 10$	$\alpha_3 = 1$	0.61	2.05	0.00	0.00	0.01	0.02
		$\alpha_3 = 5$	0.11	0.30	0.02	0.03	0.01	0.02
		$\alpha_3 = 10$	0.01	0.05	0.03	0.06	0.01	0.02

Table 6.11. Relative deviations of Heuristic from real optima for $n = 3$.

			Expectation		Myopic		Veteran	
			AD	**MD**	**AD**	**MD**	**AD**	**MD**
$\alpha_1 = 1$	$\alpha_2 = 1$	$\alpha_3 = 1$	0.01	0.04	0.09	0.16	0.02	0.05
		$\alpha_3 = 5$	0.37	0.48	0.11	0.32	0.02	0.05
		$\alpha_3 = 10$	0.93	1.12	0.12	0.40	0.02	0.05
	$\alpha_2 = 5$	$\alpha_3 = 1$	0.31	0.44	0.01	0.02	0.12	0.19
		$\alpha_3 = 5$	0.50	0.71	0.02	0.04	0.08	0.17
		$\alpha_3 = 10$	0.94	1.19	0.02	0.05	0.07	0.14
	$\alpha_2 = 10$	$\alpha_3 = 1$	0.77	1.03	0.00	0.01	0.16	0.24
		$\alpha_3 = 5$	0.81	1.16	0.01	0.01	0.17	0.38
		$\alpha_3 = 10$	1.17	1.54	0.01	0.02	0.14	0.30
$\alpha_1 = 5$	$\alpha_2 = 1$	$\alpha_3 = 1$	0.69	1.17	0.14	0.18	0.01	0.01
		$\alpha_3 = 5$	0.10	0.12	0.23	0.45	0.01	0.02
		$\alpha_3 = 10$	0.12	0.18	0.27	0.61	0.01	0.02
	$\alpha_2 = 5$	$\alpha_3 = 1$	0.24	0.79	0.06	0.08	0.03	0.05
		$\alpha_3 = 5$	0.01	0.04	0.09	0.16	0.02	0.05
		$\alpha_3 = 10$	0.06	0.11	0.10	0.22	0.02	0.05
	$\alpha_2 = 10$	$\alpha_3 = 1$	0.23	0.70	0.03	0.05	0.04	0.05
		$\alpha_3 = 5$	0.05	0.09	0.04	0.08	0.05	0.10
		$\alpha_3 = 10$	0.09	0.17	0.05	0.11	0.04	0.08
$\alpha_1 = 10$	$\alpha_2 = 1$	$\alpha_3 = 1$	1.70	2.69	0.13	0.15	0.00	0.01
		$\alpha_3 = 5$	0.33	0.37	0.24	0.39	0.00	0.01
		$\alpha_3 = 10$	0.14	0.18	0.29	0.55	0.00	0.01
	$\alpha_2 = 5$	$\alpha_3 = 1$	0.61	1.74	0.07	0.09	0.02	0.03
		$\alpha_3 = 5$	0.12	0.28	0.12	0.19	0.01	0.03
		$\alpha_3 = 10$	0.03	0.05	0.15	0.28	0.01	0.03
	$\alpha_2 = 10$	$\alpha_3 = 1$	0.37	1.35	0.05	0.07	0.02	0.03
		$\alpha_3 = 5$	0.07	0.24	0.07	0.11	0.03	0.06
		$\alpha_3 = 10$	0.01	0.04	0.09	0.16	0.02	0.05

Table 6.12. Relative deviations of Heuristic from real optima for $n = 4$.

			Expectation		Myopic		Veteran	
			AD	**MD**	**AD**	**MD**	**AD**	**MD**
$\alpha_1 = 1$	$\alpha_2 = 1$	$\alpha_3 = 1$	0.02	0.05	0.17	0.28	0.05	0.09
		$\alpha_3 = 5$	0.41	0.53	0.21	0.50	0.04	0.09
		$\alpha_3 = 10$	0.99	1.19	0.22	0.63	0.03	0.08
	$\alpha_2 = 5$	$\alpha_3 = 1$	0.48	0.67	0.03	0.05	0.19	0.26
		$\alpha_3 = 5$	0.65	0.92	0.03	0.06	0.15	0.28
		$\alpha_3 = 10$	1.07	1.37	0.03	0.07	0.13	0.24
	$\alpha_2 = 10$	$\alpha_3 = 1$	1.10	1.43	0.01	0.02	0.27	0.35
		$\alpha_3 = 5$	1.11	1.55	0.01	0.02	0.29	0.55
		$\alpha_3 = 10$	1.43	1.89	0.01	0.02	0.25	0.48
$\alpha_1 = 5$	$\alpha_2 = 1$	$\alpha_3 = 1$	0.49	0.84	0.25	0.30	0.01	0.03
		$\alpha_3 = 5$	0.09	0.11	0.41	0.67	0.01	0.04
		$\alpha_3 = 10$	0.15	0.21	0.49	0.94	0.01	0.04
	$\alpha_2 = 5$	$\alpha_3 = 1$	0.14	0.42	0.13	0.18	0.06	0.08
		$\alpha_3 = 5$	0.02	0.05	0.17	0.28	0.05	0.09
		$\alpha_3 = 10$	0.08	0.15	0.19	0.36	0.04	0.09
	$\alpha_2 = 10$	$\alpha_3 = 1$	0.22	0.43	0.08	0.11	0.07	0.10
		$\alpha_3 = 5$	0.11	0.19	0.08	0.14	0.09	0.16
		$\alpha_3 = 10$	0.15	0.26	0.09	0.18	0.07	0.14
$\alpha_1 = 10$	$\alpha_2 = 1$	$\alpha_3 = 1$	1.28	1.99	0.22	0.25	0.01	0.02
		$\alpha_3 = 5$	0.30	0.34	0.40	0.57	0.01	0.02
		$\alpha_3 = 10$	0.15	0.18	0.50	0.82	0.01	0.02
	$\alpha_2 = 5$	$\alpha_3 = 1$	0.33	0.89	0.17	0.22	0.03	0.05
		$\alpha_3 = 5$	0.07	0.18	0.23	0.33	0.03	0.06
		$\alpha_3 = 10$	0.02	0.03	0.27	0.44	0.03	0.06
	$\alpha_2 = 10$	$\alpha_3 = 1$	0.21	0.64	0.13	0.17	0.04	0.06
		$\alpha_3 = 5$	0.05	0.15	0.15	0.22	0.05	0.10
		$\alpha_3 = 10$	0.02	0.05	0.17	0.28	0.05	0.09

Table 6.13. Relative deviations of Heuristic from real optima for $n = 5$.

			Expectation		Myopic		Veteran	
			AD	**MD**	**AD**	**MD**	**AD**	**MD**
$\alpha_1 = 1$	$\alpha_2 = 1$	$\alpha_3 = 1$	0.03	0.09	0.21	0.40	0.06	0.14
		$\alpha_3 = 5$	0.40	0.58	0.24	0.69	0.04	0.13
		$\alpha_3 = 10$	0.97	1.24	0.24	0.88	0.04	0.13
	$\alpha_2 = 5$	$\alpha_3 = 1$	0.57	0.90	0.03	0.07	0.22	0.35
		$\alpha_3 = 5$	0.71	1.13	0.03	0.08	0.17	0.39
		$\alpha_3 = 10$	1.10	1.56	0.03	0.09	0.15	0.36
	$\alpha_2 = 10$	$\alpha_3 = 1$	1.25	1.82	0.01	0.02	0.33	0.52
		$\alpha_3 = 5$	1.22	1.92	0.01	0.03	0.32	0.73
		$\alpha_3 = 10$	1.51	2.24	0.01	0.03	0.29	0.65
$\alpha_1 = 5$	$\alpha_2 = 1$	$\alpha_3 = 1$	0.44	0.79	0.32	0.44	0.02	0.05
		$\alpha_3 = 5$	0.08	0.10	0.48	0.88	0.02	0.06
		$\alpha_3 = 10$	0.14	0.22	0.55	1.25	0.02	0.06
	$\alpha_2 = 5$	$\alpha_3 = 1$	0.14	0.38	0.19	0.30	0.07	0.11
		$\alpha_3 = 5$	0.03	0.09	0.21	0.40	0.06	0.14
		$\alpha_3 = 10$	0.09	0.20	0.22	0.50	0.05	0.13
	$\alpha_2 = 10$	$\alpha_3 = 1$	0.25	0.42	0.10	0.19	0.10	0.16
		$\alpha_3 = 5$	0.14	0.29	0.11	0.22	0.11	0.23
		$\alpha_3 = 10$	0.18	0.36	0.11	0.25	0.09	0.21
$\alpha_1 = 10$	$\alpha_2 = 1$	$\alpha_3 = 1$	1.16	1.88	0.29	0.36	0.01	0.03
		$\alpha_3 = 5$	0.28	0.33	0.47	0.73	0.01	0.04
		$\alpha_3 = 10$	0.14	0.18	0.57	1.05	0.01	0.04
	$\alpha_2 = 5$	$\alpha_3 = 1$	0.30	0.81	0.26	0.36	0.05	0.07
		$\alpha_3 = 5$	0.06	0.16	0.30	0.48	0.04	0.09
		$\alpha_3 = 10$	0.02	0.04	0.33	0.61	0.04	0.09
	$\alpha_2 = 10$	$\alpha_3 = 1$	0.21	0.57	0.19	0.29	0.06	0.10
		$\alpha_3 = 5$	0.06	0.14	0.20	0.34	0.07	0.15
		$\alpha_3 = 10$	0.03	0.09	0.21	0.40	0.06	0.14

Table 6.14. Relative deviations of Heuristic from real optima for $n = 6$

			Expectation		Myopic		Veteran	
			AD	**MD**	**AD**	**MD**	**AD**	**MD**
$\alpha_1 = 1$	$\alpha_2 = 1$	$\alpha_3 = 1$	0.05	0.12	0.29	0.49	0.09	0.17
		$\alpha_3 = 5$	0.43	0.59	0.33	0.77	0.07	0.15
		$\alpha_3 = 10$	0.99	1.23	0.34	0.95	0.06	0.14
	$\alpha_2 = 5$	$\alpha_3 = 1$	0.72	1.06	0.04	0.09	0.29	0.47
		$\alpha_3 = 5$	0.84	1.23	0.04	0.09	0.24	0.45
		$\alpha_3 = 10$	1.21	1.62	0.04	0.09	0.21	0.41
	$\alpha_2 = 10$	$\alpha_3 = 1$	1.52	2.11	0.01	0.03	0.43	0.70
		$\alpha_3 = 5$	1.46	2.13	0.01	0.03	0.43	0.79
		$\alpha_3 = 10$	1.73	2.39	0.01	0.03	0.38	0.73
$\alpha_1 = 5$	$\alpha_2 = 1$	$\alpha_3 = 1$	0.34	0.66	0.43	0.55	0.03	0.07
		$\alpha_3 = 5$	0.07	0.10	0.63	1.02	0.03	0.07
		$\alpha_3 = 10$	0.14	0.21	0.74	1.43	0.02	0.07
	$\alpha_2 = 5$	$\alpha_3 = 1$	0.14	0.34	0.28	0.43	0.10	0.17
		$\alpha_3 = 5$	0.05	0.12	0.29	0.49	0.09	0.17
		$\alpha_3 = 10$	0.11	0.22	0.31	0.58	0.08	0.16
	$\alpha_2 = 10$	$\alpha_3 = 1$	0.30	0.47	0.15	0.26	0.14	0.23
		$\alpha_3 = 5$	0.20	0.36	0.15	0.27	0.15	0.27
		$\alpha_3 = 10$	0.24	0.42	0.15	0.29	0.13	0.25
$\alpha_1 = 10$	$\alpha_2 = 1$	$\alpha_3 = 1$	0.95	1.60	0.38	0.46	0.02	0.04
		$\alpha_3 = 5$	0.25	0.30	0.59	0.85	0.02	0.04
		$\alpha_3 = 10$	0.14	0.18	0.73	1.20	0.02	0.04
	$\alpha_2 = 5$	$\alpha_3 = 1$	0.22	0.63	0.38	0.51	0.07	0.11
		$\alpha_3 = 5$	0.04	0.13	0.42	0.60	0.06	0.12
		$\alpha_3 = 10$	0.02	0.05	0.45	0.74	0.05	0.11
	$\alpha_2 = 10$	$\alpha_3 = 1$	0.18	0.48	0.28	0.42	0.09	0.15
		$\alpha_3 = 5$	0.07	0.14	0.29	0.45	0.10	0.18
		$\alpha_3 = 10$	0.05	0.12	0.29	0.49	0.09	0.17

7. CONCLUSIONS

7.1. Future Research Directions

In this thesis work, we studied single OR scheduling problem both analytically and numerically. We observed that the most general version of the problem is very complex due to its combinatorial and non-linear nature, which has been pointed out by several studies in the literature. We adopted several assumptions which do not necessarily compromise applicability of the resulting models to real life problems. We developed some heuristics to approximate optimal solutions which was, in most cases, too challenging to find analytically. Finally, we conducted numerical analyses to understand the characteristics of optimum solution, effects of different parameters in optimum solutions, and to evaluate the performance of heuristics for a wide range of parameter combinations. Despite obtaining very significant results throughout our study, we believe that there are still many questions unanswered which point to an untapped research potential within the problem we considered. In this section, we try to present some of the potential research directions related to OR scheduling problem we studied.

The most relevant research direction to our thesis would be to provide analytical proofs for the results which we could give only as Conjectures, for n-procedure OR scheduling problem. These conjectures include the compact form of expressions for the objective function, convexity of the objective function over D_i's-the assigned surgery durations. These proofs would make the non-linear analysis of the problem much easier. Furthermore, our observations from the numerical analyses showed that for a very wide range of "reasonable" cost coefficients, the optimum sequencing policy is "to order the surgeries in decreasing rate parameter". An analytical proof showing the exact relationship between the optimum sequence and cost coefficients (or for which ranges of cost coefficients, is the optimum sequence always $\lambda_1 > \lambda_2 > ... > \lambda_n$) would moderate the computational complexity which can be attributed to combinatorial nature of the problem.

A meaningful extension of the problem we considered would be to relax the assumption that cost coefficients for each surgeries are equal; i.e., $\alpha_{11} \neq \alpha_{12} \neq ... \neq \alpha_{1n}$, $\alpha_{21} \neq \alpha_{22} \neq ... \neq \alpha_{2n}$, $\alpha_{31} \neq \alpha_{32} \neq ... \neq \alpha_{3n}$, where $\alpha_{1i}, \alpha_{2i}, \alpha_{3i}$ are the idle time, patient waiting time and overtime cost coefficients of the i^{th} surgery, respectively.

We defined overtime as the time beyond total planned duration of the surgeries, which is different than what is generally considered in literature. A widely accepted perception is that if the last surgery is completed before the working hours end (before 5 pm, for example), then there is no overtime cost incurred, because there is nothing to be paid to staff for their overtime. However, overtime cost in the OR scheduling context is not necessarily a financial cost. It is also undesirable to go beyond the scheduled time, even if it is still within the working hours. Our formulation allows this kind of cost to be incorporated into objective function, as well. However, consider a problem where scheduler has to choose a set of surgeries from m surgeries (not necessarily a fixed number $n < m$) which yields minimum expected cost. For this problem overtime (as well as undertime) costs, as widely used in literature, would become very important since it will restrict the maximum (minimum) number of surgeries to be selected. Our results from this thesis work could prove to be very useful as a baseline to conduct such a study.

Throughout our numerical analyses, we assumed that the surgery durations are exponentially distributed, which was a reasonable assumption to make a compromise between the applicability of the results and the complexity of calculations to obtain optimum solutions. One extension would be to obtain similar results for other known distributions (such as log-normal, Weibull) which are shown in literature (Reference) to be more accurate fits for surgery durations.

Khaniyev, Kayis, Suermondt and Sylvester [36] showed time of the day (and order in the sequence) a surgery is conducted is one of the most significant factors influencing the surgery duration. This result suggests that the characteristics (parameters) of uncertainty (distribution) in surgery durations are also variable depending on the operating room schedule. This reveals that there is a mutual relationship between the surgery duration and sequencing policies; namely, not only surgery durations affect the optimum schedule, but also scheduling policies affect the optimum surgery duration assignments. We observed that although OR scheduling problems with uncertain durations have been studied in the literature previously, to our knowledge the problem of scheduling with sequence dependent uncertain durations has never been studied before. A mutual relationship between surgery durations and scheduling policies makes the problem much more realistic as well as much more complex. We started to study this mutual relationship, however, no significant results have yet been obtained to include in this thesis work.

Another extension to the problem discussed in previous paragraph would be to incorporate the effects of other factors (such as number of staff, joint past experience of staff), which were found to be among significant factors determining the surgery duration in aforementioned paper by Khaniyev et. al [36] into the problem. This would turn the problem into a scheduling and resource planning problem. Several other such extensions could be considered, for which the results obtained in this thesis work would constitute a baseline.

Several other extensions of the problem can be listed as introducing constraints mentioned in literature review, Multiple OR scheduling, staff (resource, in general) allocation, incorporation of the effects of OR schedule on post/pre-operative processes and minimizing the variance of objective function, rather than the expected value. All of these problems have been studied in literature, previously. However, we have not come across with any study which used the same approach as we adopted in this thesis work. Thus, we believe one of the main contributions of our work to future research directions would be adoption of this new approach in OR scheduling problems with different assumptions, constraints, variables, and objectives.

7.2. Conclusions

This thesis focuses on developing analytical and numerical solutions to the multi-objective, unconstrained, single-OR scheduling problem which is one of the common and most important problems in many health care providers. The majority of earlier studies consider either deterministic surgery durations, or use simulation/numerical methods when durations are assumed to be uncertain. We first gave a description of the problem and notations we used throughout this thesis in Chapter 3. From the nature of the problem it was evident that even formulating the objective function in a compact way was not a trivial job.

In relevant subsections of Chapter 4, we presented the most compact form of objective function applicable to any probability distribution. The simplifications significantly improved the complexity of the problem, since in the original expression for the objective function, number of integral terms increased exponentially with the number of surgeries to be scheduled, whereas in simplified version number of integral terms increased polynomially (quadratically, to be precise). One of the most important contributions of this thesis would be to prove the convexity of the objective function over assigned durations (D_i's), which ensured the existence of a global optima, therefore making the results of numerical analysis more reliable (removing the possibility of finding a local optima instead of global one). In Chapter 4, we also investigated several special cases of the problem; such as when one of the surgery durations is assumed to be deterministic, or when the surgery durations are exponentially distributed. Obtaining exact solutions for these special cases played an essential role in letting us understand the nature and complexity of the problem. We mainly included results for the cases where small number of surgeries were considered (2 and 3 surgeries to be scheduled), however, the approach we adopted in working with small number of instances can easily be extended to the cases where larger numbers of instances are considered, only with more arduous mathematical calculations. We preferred to adopt heuristic approaches to find approximate solutions to the problem at hand, since whether or not a solution is exact does not have much of a practical importance when it comes to implementation. In Chapter 5, we described backgrounds and derivations of several heuristic approaches to obtain approximate solutions to our original problem using different bounding problems. First we considered the bounding problem where the assigned durations are pre-determined as the expected value of respective surgery's duration. The motivation for this heuristic comes from the real life implementation. In many hospital environments, decision about how much time to allocate to each surgery is made simply by looking at the average durations obtained from historical data of the surgeries of same type. Although being a widely implemented approach, we doubted that this approach would always yield good approximations to the original problem's optimum solution. Therefore, we developed another heuristic which can be interpreted as an improvement on the first heuristic. In the second heuristic, we simply tuned the base estimation for surgery durations, i.e., expected values, with the cost coefficients. This approach is in a way connected to the myopic version of the problem, where the decisions about the surgery durations and which surgery should be scheduled the next is made at the time when the previously scheduled surgery ends. As one can see, this is a dynamic programming problem. Numerical analysis of the first two heuristics lead us search for a better heuristic, since in some combinations of the cost coefficients, both heuristics performed poorly. Finally, we developed the Veterans's Hospital Heuristic, which as obvious from the name, is based on the

OR scheduling problem for the type of hospitals which is commonly known as Veterans' Hospitals. The only difference in this type of hospitals is that patients are assumed to be ready for operation from the start of the day, therefore as soon as the previous procedure ends, the next one can start, even if the scheduled time for the next procedure is still ahead.

Chapter 6 was dedicated to numerical results. Since we had already obtained simplified expressions in the form of integrals, and proved convexity for the objective function over D_i's in Chapter 4, we were able to numerically calculate the optimum duration assignments. However, as the order of integrals increased with the number of surgeries considered, computational complexity of finding even the approximate values with discrete summation (as an approximation to integral) would be prohibitive. Thus, we decided to consider only the case when the surgery durations are assumed to be distributed exponentially. We were able to find explicit expressions for this case, earlier in Chapter 4. Using explicit expressions instead of integral expressions (or Monte-Carlo simulation) made computations 30,000 times faster when only 3 procedures were to be scheduled. One can imagine how this improvement would project to higher number of surgeries. Our numerical results showed that in all heuristics we considered the optimum sequence is always given by "Decreasing Order of lambda parameter" policy. Results for the original problem also showed that for reasonable ranges of cost coefficients the optimum sequence is also given by the same policy. However, for some extreme cases, the optimum sequence changes. Examining how much the deviation from optimum value would be had we always ordered the surgeries with "decreasing order of lambda parameter" policy, we observed a negligible deviation for a wide range of parameters (approximately 0.1% deviation, on average).

We can list our most significant results within this thesis work as follows:

- We proved the joint convexity of objective function over the decision variables, for two and three procedure scheduling problems, and we conjectured that joint convexity holds also for any number of procedures. This result is practically very important, since it ensures a unique optimum, which makes numerical analyses much more efficient.

- Explicit expressions of the objective functions in terms of decision variables - assigned durations - were essential for both analytical and numerical optimizations. We first found initial expressions for the objective functions, which we thought could be further simplified. Simplifications to reduce the number of distinct expressions in the objective functions were also essential since the number of distinct expressions were increasing exponentially. Analytical calculations lead to much more compact forms of expressions (we called them auxiliary cost functions) which could be recursively expressed in terms of each other. These results significantly improved the time complexity of the numerical analyses.
- In our numerical analyses, our first goal was to determine the best sequencing policy. Calculations conducted for a wide range of parameter sets revealed that the "Decreasing Order of rate parameters" sequencing policy outperforms other possible sequencing options for approximately 95% of the parameter combinations. And even for the cases where it is not the best sequencing option, the deviation from the optimum sequence is very negligible. Therefore, one could use this sequencing policy as a heuristic for optimum sequence in order to eliminate the combinatorial nature of the problem, for larger numbers of procedures to be scheduled.
- Our second goal for the numerical analyses was to find heuristic approaches which accurately approximates the optimum decision variables - assigned durations. Without the presence of such heuristics, the optimum decision variables could still be found by numerical methods for solving non-linear systems of equations. However, as one can guess, the time required for solving such equations increases exponentially as the number of variables increases. Of the three heuristics we considered, we observed that the Veteran's heuristic is the best performing one with an average deviation of 10% from the optimum, for a wide range of parameter combinations. We considered

such relatively small deviation as a reasonable compromise to gain from the running times.

- Our final significant result is about the effects of cost coefficients. Our numerical analyses showed that both for optimum sequence heuristic and optimum assigned durations heuristics, cost coefficients are the most prominent factors influencing the performances of heuristics. These observations give us an idea about how well a heuristic will perform given a set of cost coefficients. We further observed that for the two best performing assigned duration heuristics (Myopic and Veteran's), the effects of the cost coefficients are exactly the opposite of each other. This points a sensible managerial conclusion that a cleverly constructed hybrid heuristic as a combination of Myopic and Veteran's heuristics would perform even better than the best of those heuristics.

We encountered several obstacles mostly related to the complex nature of the problem, throughout our study. To overcome these obstacles we adopted several assumptions, heuristics and numerical approaches whenever needed, which let us have a clear picture of the solutions, even if they were not always exact. Very often, health providers are simply interested in an improvement compared to their status quo implementation, rather than finding the optimum solutions to their problems. Therefore, we believe that the results of our study, with some extensions, could prove to be very useful for health care providers.

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