LOGICISM AND GÖDEL'S INCOMPLETENESS THEOREMS

BERK POLAT

BOĞAZİÇİ UNIVERSITY

2016

LOGICISM AND GÖDEL'S INCOMPLETENESS THEOREMS

Thesis submitted to the
Institute for Graduate Studies in Social Sciences
in partial fulfillment of the requirements for the degree of

Master of Arts
in
Philosophy

by
Berk Polat

Boğaziçi University
2016

DECLARATION OF ORIGINALITY

I, Berk Polat, certify that

- I am the sole author of this thesis and that I have fully acknowledged and documented in my thesis all sources of ideas and words, including digital resources, which have been produced or published by another person or institution;
- this thesis contains no material that has been submitted or accepted for a degree or diploma in any other educational institution;
- this is a true copy of the thesis approved by my advisor and thesis committee at Boğaziçi University, including final revisions required by them.

Signature B. Po..

Date 5.9.2016..

ABSTRACT

Logicism and Gödel’s Incompleteness Theorems

The view of logicism which dates back to eighteenth century when Leibniz thought that mathematical statements are nothing more than identity statements is the view that mathematics is reducible to logic. This idea brings with it several commitments about the epistemological and ontological aspects related to the nature of mathematics. In this study, the common argument that Gödel’s incompleteness theorems destroyed logicism once and for all will be considered and in the end it will be concluded that logicism is open to many interpretations, some of which can remain compatible with Gödel’s incompleteness theorems.

ÖZET

Mantıkçılık ve Gödel’in Eksiklik Teoremleri

Leibniz’ın matematiksel önermelerin aslında basit eşitlik önermelerinden ibaret olduğunu düşündüğü on sekizinci yüzyıla dayanan mantıkçılık düşüncesi kısaca, matematiğin mantığa indirgenebileceğini ileri sürer ve matematiğin doğası hakkındaki epistemolojik ve ontolojik sorulara cevaplar arar. Bu çalışma, Gödel’in eksiklik teoremlerinin mantıkçılık akımını tamamen yıktığını ileri süren genel görüşü ele alacak ve sonunda şu sonuca varmaya çalışacak: Mantıkçılık görüşü birçok yoruma açıktır ve mantıkçılığın bazı türleri, Gödel’in eksiklik teoremleri ile bağdaşır durumda olabilir.

ACKNOWLEDGMENTS

I would like to thank each one of the committee members without whose corporation and effort this current work could not have been actualized. I must also show my gratitude to my friends who have supported me to finish this thesis. Hande Tekin deserves the most: Her skills in English language made a great contribution on this written text. I have to thank the whole department because without their providing me with enough courage, I would not dare to finish a thesis in philosophy. Lastly, I must thank the curious reader who cares to take a look at this very humble work of mine.

DEDICATION

To the memory of my father Halit Polat

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

With the rigorizations in certain fields of mathematics, such as real analysis, number theory, and the like, mathematicians and philosophers of mathematics in the late nineteenth century tried to establish foundations for mathematics from which some answers for questions on epistemological, metaphysical, and ontological natures of mathematical issues can be provided. To see the historical developments of providing answers for these questions, one may go back to Descartes' time where Descartes tried to establish geometry out of arithmetic. Similarly, one may see that there was a foundational effort in Georg Cantor's work where he tried to establish the study of real analysis on the grounds of number theory (Henkin, 1962, p. 789). However, those works on foundations of mathematics did not completely answer epistemological or metaphysical questions on mathematical entities because they reduced some fields of mathematics to another field, the nature of which was still questionable. Gottfried Wilhelm Leibniz (1646-1716) worked on several questions in mathematics, along with the epistemological status of mathematical equations. He claimed that equations in mathematics are nothing but tautologies, and furthermore he asserted that in order to show that mathematical statements are tautologies one needs to analyze those equations in question, some of which can be done by only God-like things (i.e. not by human-beings). His claim basically indicates that each mathematical statement can be reduced to a logical truth, but he did not show that each mathematical sentence could be reduced to a logical truth by a human being. Even though his claim is not wholly justified, still his very idea that mathematics is nothing but logical truth is groundbreaking. This reductionist attitude of Leibniz influenced German mathematician and philosopher Friedrich Ludwig Gottlob Frege (1848-1925). He tried to come up with a philosophy of mathematics which can answer epistemological and ontological questions on mathematics. His philosophy of mathematics practically suggested the same thing as Leibniz's philosophy did, but in a more rigorous and clearer way. Frege showed that mathematical notions and statements could effectively be shown to be logical notions and truths, respectively. This is a bold claim in the sense that it suggests that we can rely on mathematics because it is as true and reliable as logic. Moreover, Frege's philosophy of mathematics sheds light on the nature of mathematical concepts: Since mathematical notions are logical notions in this view, the ontological status of logical objects and mathematical notions would be the same. Also, I think most importantly, if such a reductionist view were adopted, then the epistemological gap between mathematical entities and human beings, which cannot be solved by any Platonist view in mathematics, would be resolved. It is clear why many philosophers of mathematics tried to come up with a foundational view that would eventually answer important questions on the nature of mathematics.

Unfortunately, the formal system which confirms Frege's logicist view on mathematics turned out to be contradictory. A contradiction, known as Russell's Paradox, was found by Bertrand Russell: A set that is the set of all sets that are not a member of themselves cannot be defined without a contradiction occurring. This led Russell and Whitehead to come up with another logicist project, which is very different from Frege's logicism. Russell and Whitehead published three volumes of *Principia Mathematica* with the aim of establishing a logicist system from which each ordinary mathematical statement can be logically deduced.

Apart from these logicist works, David Hilbert, one leading mathematician in the first quarter of the 20th century, posed certain problems that the world of mathematics should solve; one of which was on the consistency of analysis, and while Kurt Gödel tried to prove it, surprisingly found out results that were contrary to his intentions to support Hilbert's Program. Here, Gödel proved two of his famous theorems about formal systems.

Gödel's incompleteness theorems had an impact on many areas in philosophy of mathematics: One influence was on the view of formalism, which mainly dealt with how one identifies provability with mathematical truth within certain formal systems (Hellman, 1981, p. 451). Furthermore, there were many who tended to believe that Gödel's theorems also debunked the status of any logicist view, mainly of Russell's logicism, because at the time there were mainly two logicist views, and Frege's logicism was refuted by the aforementioned paradox. Therefore, there was a foundational crisis because certain foundational views were to be rejected by Gödel's theorems.

However, it can be claimed that it is not that easy to assert that Gödel's incompleteness results vitiate logicism entirely. In the first place, logicism should be clearly defined. The result of giving definitions will lead one to distinguish certain theses of logicism from one another. Secondly, in the literature of mathematical logic and philosophy of mathematics, there is not only one concept of completeness. In order to evaluate the relationship between Gödel's incompleteness theorems and logicism, one needs to distinguish different completeness notions from one another. Thirdly, one needs to analyze what Gödel's theorems postulates in the context of philosophy of mathematics, without focusing on mathematical technicalities of those theorems.

After all of those tasks are done, it will be argued that certain formulations of logicism might be shown to be compatible with the results of Gödel's incompleteness theorems. However, those formulations of logicism that are compatible with Gödel's theorems will be shown to have a shortcoming: Compatibility will result in lacking epistemological and ontological resolutions.

CHAPTER 2

LOGICISM

In this chapter, different types of logicism will be considered beginning from Frege's account, then passing to consider a different understanding of Russell on the issue, and towards the end of the chapter certain contemporary accounts on logicism will be taken into consideration. The main aim of this chapter is to show that logicism is not just a simple thesis which boldly asserts that mathematics is actually logic. Different understandings of the logicist thesis will result in different relationships between the results of Gödel's renowned theorems and logicism.

2.1 Frege's logicist project on arithmetic

In modern philosophical era, epistemological concerns were at their peak. While rationalists admitted that knowledge could be obtained through only rational activities, empiricists claimed that humans could know only through sense-based experiences. It is well-known that Kant coined several terms to bring in a certain sense those two schools of thought into one: He introduced the terms *analytic* and *synthetic*, and made use of notions *a priori* and *a posteriori* so that we can categorize the things that we can know with. In this categorization he claimed that arithmetic and geometry[1] should fall under synthetic *a priori* knowledge because arithmetic statements such as '5 + 7 =12' depend essentially upon intuition, so they are not known via five senses, i.e. they are not *a posteriori*. Note that the arithmetical statement '5+7=12' is synthetic in the sense that neither 5 nor 7 is contained in the notion '12' and calls for the help of intuition according to Kant, unlike the statement

---

[1] It should be reminded that Kant refers to those mathematical areas *of his time*. Back then, they were not aware of existence of non-Euclidean geometry.

‘all bachelors are unmarried’ in which the notion ‘bachelor’ contains the entity of being ‘unmarried’, according to Kant’s categorization. Frege was not satisfied with this categorization and put forward the thesis that arithmetic can be established via pure reason without appealing to any intuition unlike Kant’s claim (Heck, 2011).

This study will not be concerned with how Frege replied to Kant. Instead, the aim will be to try to understand Frege’s project in more philosophical depth. Frege thought that arithmetical notions could be based entirely upon pure logic. Of course, this claim is ambiguous in the sense that it does not specify what pure logic exactly is or what basing something upon some other thing exactly means. Therefore, Frege’s mathematical project will be briefly given so that his philosophy of mathematics will be understood better.

In order to initiate his logicism project, Frege first of all came up with an idea of ‘formal system’ of logic. There are some reasons for this attitude of his: We have seen the philosophical view on mathematical statements given by Gottfried Leibniz. Long before Frege, Leibniz had claimed that mathematical statements such as ‘2 + 2 = 4’ can be shown to be analytic by showing that ‘1 + 1 = 2’ and ‘1 + 1 + 1 + 1 = 4’. In this way finally we would have an equality statement ‘ 4 = 4’, the knowledge of which was without any doubt given by logic. However, according to Heck, “his proofs, like those of Euclid before him, rest upon assumptions that he does not make explicit” (2011). For instance, in his analysis of arithmetical statements Leibniz says nothing about the associativity law of addition, which is used in order to show that certain statements are analytic. In other words, when a mathematician derives ‘4=4’ from the statement ‘2+2=4’, in the first step she derives ‘(1+1)+(1+1)=4’ from ‘2+2=4’, and then gets ‘(1+1+1+1)=4’ by using associative law of addition, which is not justified in Leibnizian derivation. Frege, therefore, came up with an idea of ‘formal system’ in order to account for every step in proving and analyzing mathematical statements so that there will not be any implicit ideas sneaking in proofs, as associative property of addition did in Leibniz’s derivations. Also, another reason for Frege to come up with a new formal system of logic is because the equipment of working mathematicians at the time was not satisfactory to give rigorous proofs as mentioned just above, and because some statements could be expressed neither in Boolean logic nor in Aristotelian classical logic (Heck, 2011).

Frege introduced his formal system of logic in his famous book *Begriffsschrift*, published in 1879, in order to show that arithmetic can be transformed into his logical system. Once the transformation is completed, the rest is to claim that arithmetic, and real analysis, are nothing but logic, since real analysis was already arithmetized by Georg Cantor (Henkin, 1963, 789). Of course, it should be noted here that his basic project that his logical system was based on logical axioms which are true in every interpretation. Consequently, Frege’s axioms were logical truths. This idea is outstanding because once you base your system on logical truths, your claims can never be shaken since the claims in the system will stem from only logical truths and inference rules which are truth-preserving.

Note that, in the last paragraph there is a mention of ‘axioms’. It is generally accepted that before Frege, Dedekind came up with their own arithmetical axioms to give an account for their systems of arithmetic, although he published his work *Was sind und was sollen die Zahlen?* in 1888, after Frege did (Heck, 2011). In his work, Frege states his own axioms, according to Heck (2011), and shows derivations for those axioms from logical axioms (p. 13). Also, apart from axioms of arithmetic, Frege needed to give an account of arithmetical notions in terms of logical notions to carry out such an outstanding project.

Here, it is not needed to provide the full account of Frege's project of logicism but it is a good idea to give a key point in his work. The concept of cardinal number, which is a non-logical notion in the first glance, was mentioned in *Begriffsschrift* as "… the content of a statement about number is an assertion about a concept" (Tennant, 2014). Briefly put, a cardinal number was defined as the concept of certain concepts. For instance, the cardinal number 2 is a concept of all couples. Similarly, the cardinal number 0 is the concept of all things which are not self-identical. Frege himself claimed that this was nothing more than Hume's Principle given as follows:

$$Nx{:}Fx = Nx{:}Gx \ \equiv Eq_x\,(Fx, Gx)$$

where '$Nx{:}Fx$' means 'the number of x such that Fx', and where '$Eq_x\,(Fx, Gx)$' is read as 'the notions F and G are equinumerous' meaning that they both correspond with the same amount of objects. However, Frege soon realized that this definition of cardinal number was not sufficient to suggest that Julius Caesar, for instance, was not a number. Notice that Hume's Principle given above provides necessary grounds for a thing to be a cardinal number. However, Hume's Principle is not enough to show that Julius Caesar is not a cardinal number. This led Frege to give another definition for the notion of 'cardinal number' in order to avoid Julius Caesar problem. In his work *Grundlagen* he gave the following definition for cardinals: "the Number which belongs to the concept F is the extension of the concept 'equal to the concept F'"(Tennant, 2014).

I do not want to dwell on the process of this but the final part should be stated here: Russell at some point became aware of Frege's works and he found out that there was a problem in his Basic Law V which states

$$\{x \mid \Phi x\} \ = \ \{x \mid \Psi x\} \ \leftrightarrow \ \forall x(\Phi x \ \leftrightarrow \ \Psi x)$$

where Φ and Ψ express formulas. This axiom schema allows abstraction of classes, which was shown to be paradoxical by Russell. Afterwards, Frege admitted his failure in his logicist project and abandoned it.

Thankfully, Russell took up this logicist project of Frege's in his work *Principia Mathematica* with Alfred Whitehead.

2.2 Russell's logicist account

Without doubt, Russell was very influenced by the work of Frege and took up the challenge to show that logicism was a plausible project. However, there are major differences between Frege's and Russell's logicisms.

In his early writings, Russell suggests there are significant ties between logic and mathematics, which can be seen in the following passage:

> The Philosophy of Mathematics has been hitherto as controversial, obscure, and unprogressive as the other branches of philosophy. Although it was generally agreed that mathematics is in some sense true, philosophers disputed what mathematical propositions really meant: although something was true, no two people were agreed as to what it was that was known. So long, however, as this was doubtful, it could hardly be said that any certain and exact knowledge was to be obtained in mathematics. […] This state of things, it must be confessed, was thoroughly unsatisfactory. Philosophy asks Mathematics: What does it mean? Mathematics in the past was unable to answer, and Philosophy answered by introducing the totally irrelevant notion of mind. *But now Mathematics is able to answer, so far at least as to reduce the whole of its propositions to certain fundamental notions of logic*[emphasis added]. (as cited in Kraal, 2014, p.1495)

In this passage, Russell claims that once the reduction of mathematics to logic is achieved, then there will not be epistemological or ontological problems that philosophers of mathematics will not be able to answer.

Russell believed in his early writings that analyzing the set of mathematical theorems and analyzing logical statements are in a sense the same thing:

> The fact that Mathematics is Symbolic Logic is one of the greatest discoveries of our age; and when this fact has been established, the remainder

> of the principles of mathematics consists in the analysis of Symbolic logic itself. (as cited in Kraal, 2014, p. 1495)

However, if one takes a look at the revival of Russell's logicist project in *Principia Mathematica*, it can be seen that there are certain axioms accepted to be true in this work, but it is hard to see how they are logical axioms as the logicism that Frege formulated but in the end failed suggests. For instance, Axiom of Infinity suggests that there are infinitely many individuals, which is clearly not a logical truth since in a world of one object it is false, unlike a logical truth which must be true in each possible interpretation. In fact, Russell himself realized this problem. He suggests that we cannot even be sure of a world with one single object since it is perfectly conceivable to imagine a universe with empty domain (Russell, 1920, p. 203). What can be concluded from the fact that Russell's axioms are not logical truths, is that Frege and Russell have different understandings of what logicism indicates: Frege thinks that the formal system that logicism makes use of should have axioms that are logical truths, whereas Russell thinks that the axioms should not be necessarily logical truths and that axioms should be understood as principles from which each mathematical theorem can be derived[2].

On this issue, George Boolos comments on to what extent Russell is a logicist:

> It is doubtful that Russell could be considered a logicist in the fullest sense of the term while writing *Principia*, whose stated aim is to analyze the notions employed in mathematics, not to show that arithmetic to be a branch of logic. (Boolos, 1999, p.305)

Therefore, it may be concluded that Russell's logicism is not as strong as Frege's logicism where he intended to conclude that after all arithmetic, and after certain

---

[2] "[W]e have to analyse existing mathematics, with a view to discovering what premises are employed … and whether they are capable of reduction to more fundamental premises"(Russell and Whitehead, 1910, volume I, p. v).

procedures the whole mathematics is a branch of logic. However, it should be kept in mind that Russell was a logicist in the sense that mathematics can be analyzed to the fullest by reducing it to formal systems of logic and to logical languages.

In the following sections taxonomy of logicist theses will be given so that Russell's logicism will be understood better.

2.3 Epistemic logicism

Logicism as Frege suggested it in the beginning has two distinct philosophical consequences: On the one hand, by telling that mathematics can be reduced to logic it indicates that mathematical notions and objects are in the same ontological realm. This is the ontological version of the logicist thesis. In the current study I do not need to deal with this ontological version of logicism. On the other hand, logicism points out that how we know mathematics is directly related to how we know logic. This is the epistemological version of the logicist outlook: How we know about mathematics can be understood in terms of how we know about logical sentences, notions, and the like. It is the epistemological claim that is substantially related to Gödel's incompleteness theorems allegedly refuting the logicist project.

Also, the epistemological version of logicism is important in the following way: Let us grant that logicism means that mathematics can be reduced to logic. Which mathematics? Which parts or branches of it? Those are important questions since one could not assert something about the parts of mathematics that she did not know of. As Mark Steiner (1975) suggests, logicism vaguely indicates that "mathematical knowledge is reducible to logical knowledge" (pp. 25-26). To realize such a logicist project one needs a specific type of formal system, as given below:

> There is some formal system of logic such that mathematics can be effectively generated from it. (Steiner, 1975, p. 25)

This, however, would not be distinct from the very logicist claim of Frege. Now, as this study is more interested in the reduction of what one *can know* in mathematics to logic, one may come to the following type of logicist thesis which is called in the related literature as 'Epistemic Logicism':

> EL: There is a formal system K such that
>
> (i) For any knowable mathematical claim, P, there is a sentence S in the language of K such that S represents P and S is a theorem of K; and
>
> (ii) Any theorem S of K represents some knowable mathematical claim. (Hellman, 1981, p.452)

Note that this type of logicism is very distinct from Frege's and Russell's logicist theses. Epistemic logicism does not claim anything about the nature of axioms of the formal system K, whereas Frege explicitly tries to design his axioms so that they will be true under any interpretation. Furthermore, Russell says that his axioms will be the very principles that each mathematical theorem ultimately relies upon, and he does not specify that the formal system of logicism must aim at giving account for *knowable* mathematical claims, but he sees that a logicism project should direct itself to account for ordinary mathematical claims. It is needed to make such a distinction between the old-style[3] logicism, Russell's logicism, and epistemic logicism, since the difference in impact of Gödel's theorems on those types of logicism will turn out to be immense.

Of course, one must clarify the formulation of epistemic logicism given above. What is a knowable mathematical claim? Who knows it? And what does 'S in the language of K represents P in our everyday mathematical language' mean? For the aim of the current study those questions are not of primary importance, but still I will try to answer them briefly. Firstly, knowable mathematical claims will be taken

---

[3] Or Fregean type of logicism. However, I rather tend to use old-style logicism as Alan Musgrave suggests. See references.

as those claims that a human mathematician can know. Also, ‘representation’ can be understood as ‘expressibility’, the term that is used in the literature of mathematical logic.

In the following section I would like to continue considering other possible types of logicism that will help us evaluate the philosophical stance of the view ‘logicism’ under the criticism that will come from Gödel’s Incompleteness Theorems.

2.4 Neologicism

Frege used his Basic Law V as an axiom in his formal system to reduce arithmetic to pure logic but at the end his project failed due to Russell’s paradox. After this progress, Frege tried to come up with solutions so that his logicist program could be reinitiated. He could not manage that and left his program entirely.

In the second half of nineteenth century, Charles Parsons made use of Hume’s Principle to derive the basic laws of arithmetic, i.e. Peano axioms, which is nowadays called as ‘Frege’s Theorem’(Tennant, 2013). Neologicists think that Frege was wrong not to make use of Hume’s Principle to derive primitive notions for arithmetic in his formal system of logic. Although Hume’s Principle cannot be given as a definition for the primitive notion of cardinal number, it can be given as an implicit definition, “introducing a sortal concept of cardinal number” (Tennant, 2014).

Note that neologicists do not have an intention to derive whole mathematics from logic. Their main aim is directed specifically at number theory. This is not Frege’s original intention in his logicist program, but Frege and neologicists have shared motives: Neologicists add Hume’s Principle to a suitable formulation of second-order logic in order to derive Peano axioms and provide a foundation for arithmetic.

Notice that neologicists need to extend their work to that of real analysis or of set theory so that they will be able to reduce mathematics to pure logic. Neologicism that is concerned only with number theory seems as an incomplete version of Frege's logicism.

2.5 Other possible types of logicism

In order to see the whole philosophical relationship between logicism and Gödel's Incompleteness Theorems, one needs to see each possible interpretation of the logicist thesis. So far, it has been argued that Frege and Russell actually had something different in their minds when they talk of logicism. In addition to this, an epistemic version of logicism, which focuses rather on the relation between knowable mathematics and logic, signifies a distinct type of logicism from those of Russell's and Frege's because it makes an assertion on knowable mathematics. Furthermore, neologicists contended that Hume's Principle could be used in such a way that in the end number theory could be reduced to pure logic. Be that as it may, there are some other possible versions of logicism as Augustin Rayo suggests. Note that some versions, or conjunctions of some versions, of logicism that Rayo asserts will be shown to be similar to the previously given logicist theses.

The general claim of logicism is that mathematics can be reduced to logic, which may be interpreted as the claim that mathematics is a specific branch of logic. However, this general claim is imprecise in the sense that it does not help us understand how exactly mathematics, logic, and reducing something to another are to be taken into consideration. Or another question comes to mind: In what context does one reduce mathematics to logic? In a semantic context? Or just syntactic reduction? To clear those confusions from our minds Rayo (2007), first of all, claims that the general thesis of logicism "fails to distinguish" three possible interpretations of the general claim of logicism given as follows (p. 203):

> 1. Language-Logicism: The language of mathematics consists of purely logical expressions.
> 2. Consequence-Logicism: There is a consistent, recursive set of axioms of which every mathematical truth is a purely logical consequence.
> 3. Truth-Logicism: Mathematical truths are true as a matter of pure logic. (Rayo, 2007, p. 203-4)

For the time being, one may think that what Frege wanted to defend was all of the three logicist theses given above. However, it can be put forward that Russell had no ambition of showing that Truth-Logicism was the case since, for example, his Axiom of Infinity in *Principia Mathematica*, assumed to be a mathematical truth, was not a logical truth.

It should be noted that when those three theses of logicism are taken into consideration literally, it would be seen that some of them are *ipso facto* false. For instance, as Rayo (2007) points out, the language of mathematics is very different from the language of logical expressions, the direct opposite of which is suggested by Language-Logicism (p. 204). For instance, mathematicians use certain symbols and language when they express the mathematical statement that the square root of a natural number is equal to itself. The language and symbols in expressing this mathematical statement are not logical. Consequently, there should be a proper interpretation in order to assume that those theses given above could be hold. Both Frege and Russell did this process in their works where they tried to build a formal system of logic in which mathematical statements can be expressed, or interpreted. Frege used an impredicative second-order logic to do so, whereas Russell did it with a special logic with theory of types so that he would be able to avoid any paradoxes, as Frege had had before in his work.

The three logicist theses that have been given above need to be modified. I will use Rayo's terminology to define five different logicist theses that will turn out to be modified versions of the aforementioned three. Rayo (2007) suggests that a paraphrase function is needed in order to translate mathematical statements in the ordinary mathematical language into a language of formal system of logic which does not include any non-logical vocabulary (p. 204).

Let us denote the paraphrase function with the symbol $*$ so that for any mathematical statement φ, which is in the domain of the paraphrase function $*$, φ^* will be the paraphrase of the statement φ in a formal system, say, S. Also, note that the following logicist theses will be given with respect to a mathematical language $\mathcal{L}$ with an intended mathematical model $\mathcal{M}$, in order to be more specific in the definitions.

1. Language-Logicism:
   There is a paraphrase function $*$ such that, for any sentence φ of $\mathcal{L}$, φ^* contains no non-logical vocabulary.
2. Consequence-Logicism (Semantic Version):
   There is a paraphrase-function * and a consistent, recursive set of sentences $\mathcal{A}$ such that, for any sentence φ of which is true (false) according to $\mathcal{M}$, $\varphi *$ (the negation of $\varphi *$) is a semantic consequence of $\mathcal{A}$.
3. Consequence-Logicism (Syntactic Version):
   Like the semantic version, except that "semantic consequence of $\mathcal{A}$ " is replaced by "derivable from $\mathcal{A}$ on the basis of purely logical axioms and rules of inference."
4. Truth-Logicism (Semantic Version):
   There is a paraphrase-function * such that, for any sentence φ of $\mathcal{L}$ which is true (false) according to $\mathcal{M}$, $\varphi *$ (the negation of $\varphi *$) is a logical truth.
5. Truth-Logicism (Syntactic Version*):*
   Like the semantic version, except that "logical truth" is replaced by "derivable from the empty set on the basis of purely logical axioms and rules of inference." (Rayo, 2007, p. 204)

Such an analysis of the vague thesis of logicism is of great philosophical importance since it is better to evaluate separately the philosophical relevance of those theses to Gödel's incompleteness theorems. For the time being, it should be reminded that Gödel's theorems are mostly syntactical theorems[4], which will naturally be related and directed to syntactic versions of logicist thesis given above.

There may be questions on what type of logic a logicist is concerned with while making such a categorization of different logicist theses. There has been a long debate on the issue, but basically there are two groups of philosophers of logic, one of which tends to see logic as the first-order logic, and the other tends to regard logic as higher-order logic (i.e. second-order or more). To put it clearly, the following views can be put forth:

> The First-Order View:
> A sentence is a logical truth just in case it can be paraphrased as a first-order sentence which is true in every model of the standard first-order semantics. An axiom is logical just in case it is a logical truth. A rule of inference is logical just in case it preserves logical truth. Logical expressions are those that can be paraphrased as an expression of pure first-order logic.
>
> The Higher-Order View:
> Like the first-order view, except that "first-order" is everywhere replaced by "higher-order". (Rayo, 2007, p.207)

Moreover, it can be of use to talk about the nature of paraphrase functions whose inputs are mathematical sentences written in the ordinary mathematical language and whose outputs are mathematical statements written in a formal system of logic. Rayo (2007) argues that any paraphrase function will be recursive and will preserve truth-values, as expected (p. 207). To be more specific, let us define a minimally adequate paraphrase functions for the intended interpretations of mathematical statements written in the ordinary language of mathematics:

---

[4] See Chapter 4 for a more detailed explanation.

> Formally, let us say that a paraphrase-function * is minimally adequate for a mathematical language $\mathcal{L}$ with an intended model $\mathcal{M}$ just in case the following conditions are satisfied:
> 1. Every sentence of $\mathcal{L}$is in the domain of *.
> 2. The restriction of * to sentences of $\mathcal{L}$ is recursive.
> 3. There is a model S(intuitively, the intended model of the paraphrases) such that, for any sentence φ of $\mathcal{L}$, $\vDash_{\mathcal{M}}$φ if and only if $\vDash_{S}$ φ*. (Rayo, 2007, pp. 207-208)

Let us now try to see further results that can be obtained by considering the First-Order View at first, and then the Higher-Order View.

It is known that Gödel's Completeness Theorem concludes that the first-order logic is semantically complete and sound.[5] With that information in mind, one may deduce that the semantic and syntactic types of Truth-Logicism and Consequence-Logicism are equivalent. Moreover, it can be suggested that Truth-Logicism is a special case of Consequence-Logicism where the set $\mathcal{A}$ is empty. In other words, when there is no pre-determined set of axioms, unlike in Zermelo-Fraenkel set theory and *Principia Mathematica*, Consequence-Logicism is equivalent to Truth-Logicism because Truth-Logicism suggests that truths or theorems of mathematics are equivalent to logical truths or tautologies since they are simple derived from logical truths without appealing to extra-logical presumptions.

In addition, when the paraphrase function is minimally adequate, "Language-Logicism implies Consequence-Logicism" (Rayo, 2007, p. 208). This implication can be shown as follows: Any consistent first-order theory T has a recursive axiomatization if T has only sentences containing just logical vocabulary. Notice that such a theory is adequate for Language-Logicism. Then, if T has a finite model, then each sentence of T will follow from a sentence indicating that there are n objects in that model, for some n. Similarly, if T has an infinite model, then there will be a

---

[5] In the next chapter, I will talk about completeness in detail.

sentence expressing that there are infinitely many individuals in the corresponded model. Notice that the fact that sentences in theory T follows from a single sentence indicating the number of individuals in the model shows that the theory T is adequate for Consequence-Logicism.

Secondly, if one accepts the Higher-Order View on what counts as logic, then we will not have such nice consequences as we did in the First-Order View since the higher-order logics are not semantically complete. Yet, since they are sound according to the Higher-Order View, one may conclude that the syntactic versions of Truth-Logicism and Consequence-Logicism imply the corresponding semantic versions.

Similar to the case in the First-Order View, if one takes the set $\mathcal{A}$ empty, then syntactic version of Consequence-Logicism will imply syntactic version of Truth-Logicism. In addition to that, under such an assumption on $\mathcal{A}$, the semantic version of Consequence- Logicism will imply the semantic version of Truth-Logicism. Proof sketch of this implication is similar to that of First-Order View.

Certain analogies between logicist theses can be preceded. First of all, Frege’s logicism indicated that mathematical truths should be logical truths by deriving them from only purely logical laws. This implies that Frege’s logicist thesis indicates Truth-Logicism in syntactic and semantic ways. Furthermore, Frege wanted to symbolize mathematical statements in a formal system of logic equipped with a logical language, it can be suggested that Frege’s logicism implies Language-Logicism. It should be kept in mind that Frege used second-order logic to actualize his formal system, which results with Frege adopting logicism with respect to Higher-order view. Therefore, it can be asserted that Frege’s logicism is equivalent to the conjunction of Language-Logicism and the semantic and syntactic versions of Truth-Logicism with respect to the Higher-Order View.

Russell's logicist account can be regarded with respect to the Higher-Order View as well, since in *Principia Mathematica* there are functions of second-order. Furthermore, since Russell's logicism presumes a set of axioms, it may be suggested that Russell's logicism commits, in a certain sense, to semantic and syntactic types of Consequence-Logicism. In the formulation of Consequence-Logicism, take the set of principles given in *Principia Mathematica*, as the set $\mathcal{A}$. However, Russell and Whitehead do not take every mathematical statement into account. They are rather focused on mathematical statements that mathematicians have deduced so far.[6] Consequently, Russell and Whitehead in their logicist work do not commit to a full-fledged Consequence-Logicism. In addition to this, Russell's logicism suggests that mathematical notions and objects can be written in a logical language, which leads one to the conclusion that Russell's logicist project implies Language-Logicism. Therefore, it can be put forth that Russell's type of logicism is equivalent to the conjunction of relaxed versions of semantic and syntactic versions of Consequence-Logicism where it aims at paraphrasing only known and ordinary mathematical statements, and Language-Logicism with respect to the Higher-Order View.

Now, it is time to consider different meanings of the concept 'completeness' in order to evaluate what the essential relevance of Gödel's theorems to logicism is.

---

[6] See Chapter 3, Section 3.2 for a more detailed explanation.

CHAPTER 3

COMPLETENESS

Although in formal logic there is a concept of completeness that Gödel assumes in his studies, according to Michael Detlefsen this type of completeness is not assumed in the project of Russell and Whitehead's *Principia Mathematica*. In this chapter, I would like to briefly survey some types of the notion of completeness and in what contexts they are desirable. If one can find different aims in different types of the notion of completeness, then one may find a way to reconcile logicism and Gödel's theorems.

3.1 Gödel completeness[7]

Gödel completeness is the completeness that is with respect to provability. For instance, in a complete formal system, take any sentence and this sentence or the negation thereof will be provable according to the rules of inference and axioms of the given system. Gödel completeness may therefore be given in the following way:

> Let T be an axiomatic theory whose language is $\mathcal{L}$. T is Gödel-complete in case for every sentence σ of $\mathcal{L}$, either σ is a theorem of T or $\neg\sigma$ is a theorem T. (Detlefsen, 2014, p. 61)

Note that this type of completeness may also be called 'negation completeness' in the sense that any sentence of a formal system or its negation is provable within that system.[8] In addition to this, since this type of completeness works in syntax only, it is sometimes referred as *syntactical completeness.*

---

[7] This section will mainly follow Detlefsen's terminology.

[8] Although Gödel-completeness seems to be an invention of Gödel, it is not the case, according to Detlefsen. Hilbert in 1930, Langford in 1927, and Finsler in 1926 used this sort of completeness under different names.

3.2 Descriptive completeness

According to Detlefsen, when they were starting the project *Principia Mathematica* Russell and Whitehead had in mind a type of completeness entirely different from that of Gödel's. Russell and Whitehead in the beginning had a data set, the known mathematical propositions, theorems, and the like. And under the assumption that those groups of sentences are known, they tried to come up with a system in which the very principles are given and in which all those true mathematical sentences can be derived (Detlefsen, 2014, p. 61). This can be seen in the following passage of Russell and Whitehead:

> In constructing a deductive system such as that contained in the present work, there are two opposite tasks which have to be concurrently performed. On the one hand, we have to analyse existing mathematics, with a view to discovering what premises are employed … and whether they are capable of reduction to more fundamental premises. On the other hand, when we have decided upon our premises, we have to build up again as much as may seem necessary of the data previously analysed … [T]he chief reason in favour of any theory on the principle of mathematics must always lie in the fact that the theory in question enables us to deduce ordinary mathematics. (as cited in Detlefsen, 2014, p. 62)

Russell and Whitehead's methodology in their essential work can be regarded similar to classical scientific methodology. A scientist in the first place collects a data set, by observing and experimenting, and only after certain considerations can come up with the principles where that very data set relies upon.

A careful reading of the above passage should focus on the usage of the statement 'ordinary mathematics'. Ordinary mathematics is a type of mathematics that does not rely upon strong axioms such as Axiom of Choice or Axiom of Infinity. Russell and Whitehead in the above quote act careful in their word choice because reducing ordinary mathematics to pure logic is a much closer objective to that of Frege's original logicist program. A successful attempt to reduce ordinary mathematics to logic will result in the epistemological explanations of nature of ordinary mathematics and in ontological status of statements of ordinary mathematics. Notice that this type of logicism is not as strong as Frege's logicism, since a full-fledged logicist would want to reduce all mathematics to logic, including large cardinals.

To turn back to the main topic, Detlefsen (2014) argues that descriptive completeness calls for *descriptive axiomatization* (p. 61). Take the example of Russell and Whitehead's *Principia* project. There was a pre-axiomatic body of mathematical theories in the beginning. Afterwards, with the aim of descriptive completeness in their minds, they tried to analyze everything they had in this pre-axiomatic body of knowledge and in the end they realized that all those mathematical theories must rely upon the principles which they asserted as the axioms in their work *Principia.* So, for instance, there must have been a choice axiom in order to account for certain mathematical theorems. Without the choice axiom, part of that pre-axiomatic body of mathematical knowledge would not have been axiomatized after all. It can be suggested that projects like *Principia Mathematica* and Zermelo-Fraenkel Set Theory with Choice would be examples where the main aim is attaining descriptive axiomatization of the pre-axiomatized mathematical body of knowledge, rather than attaining Gödel-completeness. It is arguable whether or not those projects have fulfilled their aims, which would be a discussion independent of Gödel-completeness. Note that whether or not those logicist projects which were carried out in the past were successful does not change the present status of logicism.

Here at this point, it should be mentioned that descriptive completeness aims at embodying a certain body of mathematical knowledge, i.e. ordinary mathematics.

3.3 Analytic completeness

Detlefsen points out another type of completeness as one of the main aims of the *Principia* project. It is better to see the formulation of Russell and Whitehead before explaining what analytic completeness means: “We have to analyse existing mathematics, with a view to discovering what premises are employed … and whether they are capable of reduction to more fundamental premises”(as cited in Detlefsen, 2014, p. 62). This suggests that *Principia*, as its name indicates, is aimed at stating the very principles of ordinary mathematical practice of the time. Those principles should be such that no more basic principles can be found. This is called *analytic completeness*. In other words, analytic completeness requires a search for the very basic principles “to assure rationally that no continuation of it would yield principles more basic than those identified” (Detlefsen, 2014, p.62).

3.4 Semantic completeness

In addition to those types of completeness which pertain to an axiomatized theory, there is another usage in the literature of mathematical logic that is also called ‘complete’: This type of completeness suggests that all tautologies of a given system are also theorems of that system. This means the converse of soundness, which claims that all theorems of a system are true with respect to every interpretation adequate for the given system. In this sense, first-order predicate logic is semantically complete, while it is not Gödel-complete. Gödel showed semantic completeness of first-order predicate logic in his First Completeness Theorem (Hunter, 1996, p. 256).

To clear up, although in the literature of logic there seems to be only one type of notion of completeness which also Gödel’s theorems are related to, Detlefsen points out that there can be other types of completeness, and those types, he suggests, are the types of completeness that *Principia* project aims at. Along with *Principia*, it can be asserted further that ZFC also aims at those types of completeness, i.e. descriptive and analytic completeness, to account for the whole body of mathematical knowledge. In the next section, Gödel's famous theorems will be explained to make sure that basic ingredients are at hand to understand the relationship between logicism and Gödel's theorems.

CHAPTER 4

GÖDEL’S INCOMPLETENESS THEOREMS

So far, we have considered basic accounts of logicism and completeness. The remaining ingredient related to this study is Gödel’s theorems. In this chapter, the main aim is to show enough details of Gödel’s theorems to undertake the arguments on compatibility of Gödel’s theorems and certain formulations of logicism in the next chapter. This chapter will not cover the mathematical technicalities of how Gödel proved his theorems. Rather, this chapter will cover what the theorems indicate, from both mathematical and philosophical perspectives.

4.1 Gödel’s first incompleteness theorem

To begin with, we need to work on a formal system whose language will be appropriate for expressing arithmetic statements. This point is crucial because Gödel’s first theorem results in a conclusion only for formal systems that are capable of covering elementary arithmetic, such as Peano Arithmetic.[9] This condition does not cause a problem in relating Gödel’s theorems to logicism, as various formulations of logicism are concerned with reducing the whole body, at least a part, of mathematics which includes arithmetic to a certain type of formal system. In the context of logicism, one would not be interested in a formal system which even could not express elementary arithmetic. Consequently, the formal systems that Gödel’s first incompleteness theorem is applicable to, and the formal systems that

---

[9] It will be abbreviated as ‘PA’ in the rest of the work.

logicism makes use of coincide. Now we may state the following version of Gödel's first incompleteness theorem:[10]

> First Incompleteness Theorem (Gödel-Rosser):
> Any consistent formal system S within which a certain amount of elementary arithmetic can be carried out is incomplete with regard to statements of elementary arithmetic: there are such statements which can neither be proved, nor disproved in S. (Franzén, 2005, p. 16)

So, under the assumption that S is a formal system which covers elementary arithmetic and which is consistent, there will be undecidable arithmetical sentences. It must be noted that those undecidable sentences are constructed in such a way that they must be assumed to be true. Therefore, the main result that shook the whole world of mathematics and philosophy of mathematics in the second quarter of the twentieth century is that a consistent formal system which is able to cover elementary arithmetic will turn out to be incomplete in the sense that there will be a sentence σ such that neither σ nor $\neg\sigma$ is provable with regards to the axioms and the rules of inference of that system.[11]

At first regard, it may seem absurd that one comes to know that there are *true* sentences which are unprovable. One might ask: How do we know that that sentence is true without being able to provide a proof in that system? Or, as one may recall the definability of truth within a metasystem introduced by Alfred Tarski in 1930's, it can be argued that the condition of knowing that such a Gödel sentence is true is to check its truth-value within a metasystem which covers the formal system within which that Gödel sentence is created. Another way to argue how we come to know that Gödel sentences are true may be that we know they are true in virtue of their being obviously true arithmetical statements. Those ways of arguing for the truth of

---

[10] In the literature there are plenty of versions of this theorem, and I will adopt a version named as 'Gödel-Rosser' by Torkel Franzén.

[11] From this instant on, I will simply call this type of undecided sentences created in Gödel's first incompleteness theorem as 'Gödel sentences'.

Gödel sentences are wrong, and the answer to how we come to know that Gödel sentences are true lies in the specific design of such sentences. Gödel in his proof for the first incompleteness theorem makes use of the idea of *arithmetization of syntax* so that he can talk about arithmetic statements within the formal system on which he is working. *Gödel numbering* is the basic idea to code each expression that can be written in the intended formal system of logic. With the help of Gödel numbering, he designs a sentence that is practically saying "A sentence G given in the formal system is not provable". The reason for us to assume that such a Gödel sentence is accepted to be true is explained as follows:

> Assuming that the formalized provability predicate used is normal, one can prove, even inside F, that G_F is true if and only if F is consistent, although neither side of the equivalence can be proved in F. Therefore, the truth of the sentence G_F is already implicitly assumed in the beginning of the proof, in the form of the assumption that F is consistent.[12] (Raattkainen, 2005)

Therefore, the truth of the statement 'this system is consistent' and that of the Gödel sentence have the same value. Notice that we do not want to assume that the system is inconsistent since *ex falso (sequitur) quodlibet.*[13]

Note that there has not been any arithmetical statement in the ordinary mathematics that can be shown to be 'undecidable' in ZFC (Franzén, 2005, p. 24). This result points out that a formulation of a logicist program that covers all ordinary mathematics within a formal system of logic might not have been influenced by Gödel's first theorem after all. This will be discussed in the next chapter. For now, let us see what Gödel's second theorem implies.

---

[12] Here the writer names the formal system 'F', and the Gödel sentence 'G_F'.

[13] Lat. from falsehood, anything (follows).

4.2 Second incompleteness theorem

Gödel’s second incompleteness theorem, published after the first one, postulates something very much related to the first one. Recall that the first incompleteness theorem assumes in the first place that the formal system covering certain amount of arithmetic is consistent. Gödel, with the use of his method of arithmetization of syntax, expresses the statement ‘This system is consistent’ within the language of that very system. The rest is the following:

> Second Incompleteness Theorem:
> For any consistent formal system *S* within which a certain amount of elementary arithmetic can be carried out, the consistency of *S* cannot be proved in *S* itself. (Franzèn, 2005, p. 34)

At this point it is a good idea to illustrate certain misunderstandings the second incompleteness theorem led to. For instance, Franzén maintains that the following commentary given by J. Kadvany on the second incompleteness theorem is fallacious:

> Gödel’s Second Theorem implies that the consistency of *Principia* can be mathematically proven only by conjecturally assuming the consistency of *Principia* outright (which is what mathematicians implicitly do in practice), or by reducing the consistency of *Principia* to that of a stronger system, thereby beginning an infinite regress. (as cited in Franzèn, 2005, p. 37)

Even though the formal systems of *Principia* or ZFC do not just assume their consistency, there are good reasons to accept that those ‘logicist’ systems are consistent: The axioms and rules of inferences accepted in those systems are in no way randomly decided. The mathematical practice, appealing to mathematical reasoning, and the regressive method that Russell and Whitehead used in their works, are some reasons to accept that those systems are consistent. Furthermore, to answer the infinite regress problem given by Kadvany, Franzén (2005) argues that we have that type of problems since antiquity and that we do not base the consistency of a theory on another theory because at some point it surely creates an infinite regress

(p. 38). Therefore, we must focus on only what Gödel’s theorem suggests: It does not say anything about another system, about a metasystem, and the like. It simply suggests that a system that covers certain amount of elementary arithmetic cannot prove the statement claiming its own consistency.

So far, we have seen what is a logicist system, logicist program, types of logicism, types of completeness, and Gödel’s theorems. The remaining task is to see whether or not Gödel’s theorems in fact refute any type of logicist project or there can be a compatibility result demonstrating that Gödel’s incompleteness theorems and a certain formulation of logicism. The next chapter will deal with this.

CHAPTER 5

LOGICISM VS. GÖDEL'S INCOMPLETENESS THEOREMS

Logicism, as expected, provides a foundation for mathematics, but in the literature it is widely held that Gödel's theorems destroyed any chance to have a foundation for mathematics, including the view of logicism. In this chapter it will be argued that this refutation was fallacious by introducing Leon Henkin's views on the issue, which represents the common view of the alleged refutation of logicism by Gödel's theorems. Afterwards, several other ways will be considered to argue that a certain formulation of logicism can indeed survive even after the damage done by Gödel's theorems on Hilbert's program.

5.1 Alleged refutation of logicism

Leon Henkin in his paper entitled "Are Logic and Mathematics Identical?" argues that both of Gödel's incompleteness theorems do not leave any room for any logicist project by focusing on the logicist project of Russell and Whitehead. In the first place, he points out that *Principia Mathematica* does not assure that the project reduces all mathematics to logic.

> The world of empirical science, of course, expects to achieve conviction on the basis of empirical evidence, but the quintessence of the mathematician's approach, especially of the mathematical logician's, is the demand always for *proof* before a thesis is accepted. Yet you see that whereas Russell was interested in establishing that a certain sense of all mathematics could be obtained from his logical axioms and concepts, he never really set out to give a proof of this fact! (Henkin, 1962, p. 789)

A couple of questions can be asked: Henkin asks for a proof from Russell and Whitehead on the issue of whether or not logicism is the case. It is possible that from Henkin's aforementioned words to conclude that Henkin asks for a mathematical proof. If this is the case, then one might ask if a philosophical thesis necessarily calls for a mathematical proof. Henkin here should not confuse Russell's identity of being mathematician with that of being a mathematical, or for that matter analytic, philosopher.

Furthermore, as one may see in the aforementioned quote, Henkin takes logicism as a vague claim throughout his paper. As mentioned in the second chapter, logicism can be made precise in several different ways, and probably we may correct Henkin's definition of logicism as follows: Logicism is the claim that all *ordinary* mathematics can be obtained from certain logical axioms and concepts. It has been considered that Russell and Whitehead work just like empirical scientists who use a regressive method, from a pre-axiomatized body of data to principles, to come up with certain axioms to secure the base of their axiomatic theory.

Now, let us see how Henkin sees the relationship between logicism and Gödel's first incompleteness theorem:

> Gödel was able to demonstrate that the system of *Principia Mathematica,* taken as a whole, was *incomplete.* That is, he showed explicitly how to construct a certain sentence, about natural numbers, which mathematicians could recognize as being true under the intended interpretation of the symbolism but which could not be proved from the axioms by the rules of inference which were part of that system. (Henkin, 1962, p. 790)

Here Henkin gives a counterexample for the system of *Principia Mathematica*, this counterexample is a sentence that can be recognized as true by mathematicians. However, it should be noted that Gödel sentences, the counterexample that Henkin provides, do not belong to the data that working mathematicians are concerned with, although it is purely arithmetical. The mentioned sentence is rather a syntactic construction of an arithmetical statement which turns out to be undecidable within the formal system under discussion.

Moreover, mathematicians accept this sentence that provides the counterexample in Henkin's argument as true only if they accept that this system is consistent, which will go against the other argument of Henkin:

> … Russell and Whitehead were concerned that no paradox should be demonstrable in their own system. And yet they themselves never attempted a *proof* that their system was consistent! … Gödel was able to show that the question of consistency and completeness were very closely linked to one another. He was able to show that *if* a system such as the *Principia* were truly consistent, then in fact it would not be possible to produce a sound proof of this fact! (Henkin, 1962, p. 791)

On the one hand he claims that the consistency of *Principia* cannot be proved with a mathematical proof, on the other hand he suggests in his previous argument that mathematicians can recognize those *true* sentences about numbers in the formal system under the intended interpretation. It seems as though Henkin uses a technique of using unjustified information to refute an argument in order to destroy Russell's logicist program. Gödel's second incompleteness theorem does not suggest that consistency of such a system as *Principia* cannot be shown. Henkin forgets to mention that the second theorem talks of a consistency proof *given within* that system, which indicates a big difference. For instance, the system of PA which includes elementary arithmetic can be shown to be consistent in another formal system although there is no consistency proof of that system within itself (Franzén, 2005, pp. 37-38). What Gödel's second theorem indicates is that it is not possible to provide a proof for the consistency statement of the system within that system.

Another issue is that even if *Principia* provides us a consistency proof of itself by only using its own axioms and inference rules, i.e. a proof which belongs to the formal system of *Principia*, how can one trust that type of proof? If *Principia* itself is inconsistent, one can prove any mathematical statement, including its own consistency. It is surely similar to the liar's case: A liar who says "I am not a liar" is in no way trustworthy. However, Henkin's point is correct in the sense that Russell and Whitehead do not provide a consistency proof for the whole work of *Principia Mathematica*. Yet, this is not relevant to the issue that Gödel's second theorem undermines logicism.

Henkin's argument is only of such alleged refutations of logicism by Gödel's theorems. Although proponents of Gödel's theorems mainly suggest that there will always be undecidable sentences in every formal system that tries to reduce all mathematics, those undecidable sentences are mostly syntactic sentences designed in a very technical manner by Gödel. Russell's logicist project of covering all known mathematics that working mathematicians of his time were interested in does not get refuted by Gödel's first incompleteness theorem. In addition to this, the second incompleteness theorem is irrelevant in the sense that we would not expect such a consistency proof within a formal system of which a mathematician is trying to show the consistency.

In the next section, certain criticisms directed at epistemic logicism will be pointed out.

5.2 Arguments against epistemic logicism

As discussed in Chapter 2, the thesis of epistemic logicism is the view that there is a formal system of logic whose theorems can represent or express knowable mathematical statements. Geoffrey Hellman analyzes the view that such a version of logicism may remain unharmed notwithstanding Gödel's incompleteness theorems.

Let a formal system K for the task of EL is given. Then:

> (X) If K is consistent, then a Gödel sentence A thus constructed for K is true and undecidable. (Hellman, 1981, p. 453)

This sentence can be expressed in the language of K and is a theorem of the system K, thanks to Gödel's first theorem. Now, the key point is that, as argued in the previous chapter, the Gödel sentence A for the system K does not cause any harm to the view of EL because actually we *do not* know that A is the case, which is the result of the fact that we cannot know that K is consistent. We could only come to know that A is true if we can apply *Modus Ponens* with the statement (X) and the consistency statement for the system K. Hellman questions if there is a system for the project of EL whose consistency can be known; because only in this case we can come to know that there would be Gödel sentences which are true and which can be given as a counterexample against EL. In that way, one would *know* that there were mathematical truths that are not provable (i.e. not representable by a theorem of the system) and that violates the second property of EL which suggests that any knowable mathematical claim can be represented by a theorem in the formal system.

As argued in the previous section, there are some axiomatic systems whose consistency can be proven, such as Peano Arithmetic. However, recall that EL, or any logicist view for that matter, is too strong to prove its own consistency within itself. So, Hellman (1981) argues, the question is whether or not there are formal systems adequate for EL and whose consistency a human can know by using the instruments of those systems (p. 454). If there are such formal systems, then EL will fail due to Gödel's second incompleteness theorem.

In order to look for such formal systems, Hellman proves a metametatheorem with respect to finitely axiomatized systems:

> Metametatheorem I:
> There is no formal system that is finitely axiomatizable and that meets the conditions of the epistemological logicist thesis. (Hellman, 1981, p. 456)

Hellman’s result shows that there will not be any hope to revive an EL project with respect to finitely axiomatized formal systems. Now, one wonders if there can be any hope for infinitely axiomatizable systems. Before giving Hellman’s argumentation, some preliminary facts should be given: In Hellman’s notation ‘$\lozenge Kn$ ‘ means ‘is knowable’; given a formal system K adequate for EL that contains infinitely many axioms in the recursive set $\Gamma = \{A_1, A_2, \dots\}$, $\{B_i\}$ is the sequence of finite conjuctions of the first I members of the set Γ (Hellman, 1981, pp. 455-457). Now, he argues that there can be such infinitely axiomatizable systems adequate for EL only when we can infer (ii) from (i):

(i) $\forall i \ \lozenge Kn \ \Rsh B_i \ is\ true \Lsh$
(ii) $\lozenge Kn \ \Rsh \forall i \ B_i \ is\ true \Lsh$ (Hellman, 1981, p. 457)

From then on Hellman comes up with another theorem, which will in a sense ends the argumentation:

> Metametatheorem II:
> No recursively axiomatizable formal system for which (material) conditional “if (i) then (ii)” holds can meet conditions of the epistemological logicist thesis (EL). (Hellman, 1981, p. 459)

That is, there is a hope for EL thesis if there are infinitely axiomatizable formal systems which are adequate for the task of EL and for which the material conditional from (i) to (ii) does not hold. However, he argues that we are in no position to know such a system exists:

> Although there is a window left open through which the EL thesis can pass, *it can never be known of any recursively axiomatizable system* (to which the Gödel theorems apply) *that it passes through!* (Hellman, 1981, p. 459)

But the question is, that type of system will admit its own consistency as a theorem of itself, which will contradict with Gödel’s second incompleteness theorem.

Hellman’s suggestion here is the following:

> The escape route left open for EL exists only in principle: there may possibly be in fact a system satisfying EL, containing infinitely many axioms such that it is not knowable that all of them are true, and it may conceivably even be

> knowable that this is the case; but literally could not be known of any individual system that it is such a system. In the epistemic sense, codifying the limits of mathematical knowledge must transcend those limits. (Hellman, 1981, p. 459)

At the end, Hellman leaves us unable to decide whether or not the EL thesis holds. The important result is that the common argument that Gödel's theorems refute logicism does not hold for the Epistemic Logicism since it remains a plausible thesis even though we cannot know if it is true or not. In principle epistemic logicism can remain unharmed even after the impact of Gödel's incompleteness theorems because one is not able to know if there are formal theories adequate for the EL project. In other words, one cannot be sure that EL does not hold due to Gödel's incompleteness theorems. This result can be interpreted in the following way: One cannot be certain of the existence of any logical foundations for knowable mathematics. Thus, even in principle a certain formulation of logicism, namely EL, can be compatible with Gödel's theorems.

In the next section, Rayo's five formulations of logicism will be considered with respect to First-Order and Higher-Order Views in order to investigate if there are formulations that can be compatible with Gödel's theorems.

5.3 Reviewing other types of logicism

In Section 2.4 it has been suggested that the very vague view of logicism that mathematics is a branch of logic could be interpreted in different ways. It was proposed there that there are mainly five different formulations of the logicist thesis, and that they are open to a further interpretation on the basis of how logic is understood, either as first-order logic or as higher-order logic. Next subsection will assess those distinct interpretations of logicist thesis with the help of Gödel's incompleteness theorems so that in the end one will see how many of those interpretations can survive Gödel’s impact.

5.3.1 Evaluating the first-order view

Recall that the first-order view is that the paraphrase function which will give outputs with no non-logical vocabulary has the range of first-order sentences. Assuming that the paraphrase function is minimally adequate, it follows that the syntactic version of Consequence-Logicism is refuted by Gödel’s first incompleteness theorem, since there are true Gödel sentences which are not derivable with the logical apparatus of the corresponding formal system. Moreover, it has been argued in Section 2.4 that semantic and syntactic versions of Consequence-Logicism imply one another, because first-order predicate logic is semantically complete and sound by Gödel’s first completeness theorem. Thus, since the syntactic version of Consequence-Logicism is refuted by Gödel’s first incompleteness theorem, and since the syntactic version implies the semantic version, the latter is also refuted by the first incompleteness theorem.

Furthermore, it has been suggested in Section 2.4 that syntactic version of Truth-Logicism implies the syntactic version of Consequence-Logicism, which implies that the syntactic version of Truth-Logicism is false, too. A similar implication holds for the semantic version of Truth-Logicism. Thus, both versions of Truth-Logicism are ruled out by Gödel’s first incompleteness theorem.

Lastly, it has been argued in Chapter 2 that Language-Logicism implies semantic version of Consequence-Logicism, which shows the falsity of the semantic version of Truth-Logicism according to the first-order view, as the semantic version of Consequence-Logicism has been proven to be false.

Generalizing the above conclusions, I claim that if five different theses of logicism could be ever realized in the first-order logic, they would not be able to survive the mathematical impact of Gödel’s first incompleteness theorem. However, it does not seem that first-order logic is not powerful enough to carry out a logicist program since certain notions such as infinity and countability, in mathematics can be expressed by second-order sentences, not by first-order sentences due to the results of the compactness and Skolem-Löwenheim theorems (Boolos, 1999, p.48).

5.3.2 Evaluating the second-order view

I claim that the syntactic version of Consequence-Logicism is also refuted by Gödel’s first incompleteness theorems, according to the higher-order view. Since, as argued in Section 2.4, the syntactic version of Truth-Logicism implies the syntactic version of Consequence-Logicism, it follows that Truth-Logicism is also false.

On the other hand, Gödel’s theorems have no impact over those three logicist theses (Rayo, 2007, pp. 210-212). Nevertheless, it should be noted that although Rayo thinks that the syntactic version of truth logicism must be false, he claims that Language-Logicism and the semantic version of Consequence-Logicism can remain to be true.

Language-Logicism by using higher-order logic immediately reminds one of Russell and Whitehead’s logicist project. It can be claimed further that *Principia Mathematica* can be suggested to be a relaxed realization of Consequence-Logicism. Consider the following modifications on the definitions of the semantic version of Consequence-Logicism and the Language-Logicism.

> Consequence-Logicism (Semantic Version):
> There is a paraphrase-function * and a consistent, recursive set of sentences $\mathcal{A}$ such that, for any ordinary-mathematical sentence φ of which is true

> (false) according to $\mathcal{M}$, φ^* (the negation of φ^*) is a semantic consequence of $\mathcal{A}$.
>
> Consequence-Logicism (Syntactic Version):
> Like the semantic version, except that "semantic consequence of $\mathcal{A}$ " is replaced by "derivable from $\mathcal{A}$ on the basis of purely logical axioms and rules of inference.
>
> Language-Logicism:
> There is a paraphrase function $*$ such that, for any ordinary-mathematical sentence φ of $\mathcal{L}$, φ^* contains no nonlogical vocabulary.

Those formulations of logicism can be compatible with Gödel's incompleteness theorems. However, the price to pay would be that the ontological and epistemological implications of those types of logicist views would not be as strong as the original formulation of logicism, i.e. Frege's logicism, provides. Giving a logicist view only with respect to ordinary mathematics would not answer the ontological status of mathematical objects. Recall that one implication of the full-fledged logicism is that the ontological status of mathematical objects is the same as that of logical objects. Furthermore, from an epistemological point of view, the full-fledged logicism implies that we know mathematics just as we know logical truths. In other words, mathematical knowledge is *a priori* and certain, just as logical truths are. The three modified versions of logicism can only claim that once we base our logic on certain set of principles, or axioms, then one would be sure that ordinary mathematics is epistemologically justified. Those three theses fall short of giving any strong ontological respond as the original version of logicism does.

In the following section it shall be argued that once we distinguish the two meanings of the term 'completeness', there may be a way to hold that logicism, especially Russell's logicism, is compatible with Gödel's theorems.

5.4 Compatibility claim

Thus far, it seems as though it had been assumed that Gödel's theorems were easily applicable to the logicist views. Michael Detlefsen thinks on the contrary. In this chapter, I would like to point out that it is maybe too quick to assume that Gödel's incompleteness theorems are applicable to every formal system that logicist projects make use of.

First of all, what Gödel thinks about *Principia Mathematica* should be given explicitly so that we can distinguish the completeness projects separately.

> Whitehead and Russell … constructed logic and mathematics by initially taking certain evident propositions (*evidente Sätze*) as axioms and deriving the theorems (*die Sätze*) of logic and mathematics from these by means of some precisely formulated principles of inference in a purely formal way (that is, [w]ithout making further use of the meaning of the symbols). … when such a procedure is followed the question naturally arises at once (*erhebt sich natürlish sofort*) whether the initially postulated system of axioms and principies of inference is complete (*vollständig*), that is whether it actually suffices to deduce (*deduzierèn*) every logico-mathematical proposition (*jeden logisch-mathematischen Satz*), or whether, perhaps, it is conceivable that there are true (wahre) propositions … that cannot be derived in the system under consideration. (Detlefsen, 2014, p. 67)

Russell and Whitehead do not consider axioms as evident propositions. This has been explained in the sections "Russell's Logicism" and "Analytic Completeness". In addition to this, Russell and Whitehead did not aim at deducing every logico-mathematical proposition from their axioms and rules of inference. Instead, they tried to come up with a formal system so that it will help provide a basis for the *ordinary* mathematics of their time. From this point, it can be deduced that Gödel did not entirely understand the nature of the logicist project of the *Principia Mathematica*, which resulted in his criticisms claiming that his incompleteness theorems damaged logicism (Detlefsen, 2014, p. 68).

The main issue here is whether Gödel completeness is a concern for a project like *Principia*. Detlefsen (2014) asserts that Gödel completeness at best can be a second concern for such a project, since the main aim of *Principia* is to state the most basic principles for all of the ordinary mathematics, as analytic completeness requires, and to provide a formal system, axioms and rules of inference which can deduce all of true mathematical statements of ordinary mathematics (p. 68). At this point, let me clarify whether Gödel completeness is necessary for analytic and descriptive completeness.

I agree that the formal system of *Principia* is Gödel-incomplete, which is to say, if assumed consistent, then there are number-theoretical undecidable Gödel sentences. Does that mean that *Principia* will turn out to be descriptive incomplete? Not necessarily, since descriptive completeness takes a basis of a given pre-axiomatized set of statements, which is the ordinary mathematics of all branches, whereas Gödel-completeness is only about syntactic completeness. Gödel does not care about ordinary mathematics, or a pre-axiomatic body of statements in general. "Gödel incompleteness does not generally impede realization of descriptive completeness, which is a central goal of descriptive axiomatization" (Detlefsen, 2014, p. 70).

On the other hand, Gödel-completeness is not concerned with what nature fundamental or basic axioms the related formal system has or how they are derived. Yet, the central aim of employing analytic completeness, as argued in Chapter 3, is to determine the most basic axioms so that no more basic principles can be found out. The areas of application of Gödel-completeness and of analytic completeness are different. As Detlefsen (2014) suggests too, Gödel-completeness is not required for analytic completeness (p. 71). This result implies that Gödel's incompleteness theorems are not adequate to criticize logicist projects such as *Principia Mathematica* and ZFC.

One more consideration given by Carlo Celluci should be taken into account. He argues that there are typically two types of formal systems: Open and closed ones, and Russell's logicism works within an open system whereas Gödel's theorems apply to closed formal systems. Open systems are those "whose main feature is that they are always subject to unanticipated outcomes in their operation and can receive new information from outside at any time" (Celluci, 1992, p. 103). Working with open systems are in a certain sense like an empirical investigation where there is always a chance to admit one more evidence into the system. Russell's descriptive completeness goal, I think, is related to that type of open formal systems.

Celluci argues that an open system like *Principia* concerns only empirical data, i.e. ordinary mathematics which does not account for, for instance, infinite cardinals (Celluci, 1992, p. 105). Hence, one may conclude that metasystematical claims like Gödel sentences are not any concern of open systems such as *Principia*. Of course, logicism would be concerned with the undecidable sentences in the ordinary mathematical practice, because in that case logicism would fail to provide a formal proof for such undecidable sentences that are in the scope of interest of logicism. As an open system, a formal system as in *Principia* would allow a modification of its axioms, which is the main advantage of open systems. Therefore, with the distinction of open and closed formal systems in mind, one might argue that logicist projects which make use of open formal systems like *Principia Mathematica* are not affected by the results of Gödel's incompleteness theorems.

To sum up, by differentiating various types of completeness, by determining which types of completeness logicist projects aim at, and by considering different types of formal systems, one can conclude that Gödel's incompleteness theorems are compatible with certain logicist projects, such as those in *Principia* and ZFC.

CHAPTER 6

CONCLUSION

Mathematics is fascinating but at the same time subject to ambiguous philosophical interpretations. This work has focused on one of those interpretations, namely logicism. In mathematical logic and philosophy of mathematics, many philosophical accounts of mathematics were influenced from top to bottom by Gödel's works, especially by his incompleteness theorems. Many philosophers have suggested that Gödel once and for all refuted the view of logicism. Yet, one should not be hasty. This study has provided detailed accounts for what logicism might mean, how completeness differs in different contexts, and how Gödel's theorems may be linked to logicism in order to give a proper evaluation of the relationship between logicism and Gödel's theorems.

Certain proponents, as Henkin, of the view that Gödel's theorems refute logicism suggest that logicist projects like *Principia Mathematica* are not complete and therefore cannot provide an extensive logicist account for all mathematical statements. This type of evaluations was fallacious in the sense that certain logicist accounts are not to be 'complete' in certain contexts. Furthermore, one should be careful on the link where Gödel's theorems and logicism interacts; otherwise fallacious accounts may be made, as Henkin did.

As mentioned, logicism is a view that is open to different interpretations. According to epistemic logicist account, it cannot be known that logicism is a realizable project or not, even if Gödel's theorems are taken into account. This suggests that epistemic logicism and Gödel's theorems might be compatible with each other, though we cannot know it.

Rayo provides a detailed account for what logicism might mean, depending on what type of logic is taken into consideration. Evaluation of those accounts show that the versions of logicism that are implied by Frege's type of logicism are doomed to be left behind due to Gödel's theorems, but there are certain types of logicist accounts, such as Russell's account of logicism, with which Gödel's theorems do not interact. In the last part of philosophical evaluations, regarding different accounts of completeness, thanks to Detlefsen's work on Gödel and axiomatization, one can conclude that Russell's type of logicist account can be regarded compatible with Gödel's theorems because Gödel's theorems are aimed at a different type of completeness from that of Russell's logicism and furthermore it can be asserted that Gödel's and Russell's formal systems are of distinct types. Therefore, it is possible to provide a careful formulation of logicism compatible with Gödel's incompleteness theorems.

REFERENCES

Boolos, G. (1999). *Logic, logic, and logic*. Cambridge, MA: Harvard University Press.

Celluci, C. (1992). Gödel's incompleteness theorem and the philosophy of open systems. In D. Miéville (ed.), *Kurt Gödel: Actes du Colloque, Neuchâtel 13-14 Juin 1991* (pp. 103-127). Neuchâtel Université de Neuchâtel Centre de Recherches Sémiologiques.

Detlefsen, M. (2014). Completeness and the end of axiomatization. In Juliette Kennedy (ed.), *Interpreting Gödel: Critical essays* (pp. 59-77). Cambridge, England: Cambridge University Press.

Franzén, T. (2005). *Gödel's theorem: An incomplete guide to its use and abuse*. Wellesley, MA: A K Peters.

Heck, R. G. Jr. (2011). Frege's theorem: an introduction. Retrieved from http://rgheck.frege.org/pdf/published/IntroToFregesTheorem.pdf

Hellman, G. (1981). How to Gödel a Frege-Russell: Gödel's incompleteness theorems and logicism. *Noûs, 15*(4), 451-468.

Henkin, L. (1961). Are mathematics and logic identical? *Science, 138*(3542), 788-794.

Hunter, G. (1996). *Metalogic: An introduction to the metatheory of standard first order logic*. Berkeley and Los Angeles, California: University of California Press.

Kraal, A. (2014). The aim of Russell's early logicism: A reinterpretation. *Synthese*, *191*(7), 1493-1510.

Musgrave, A.(1977). Logicism revisited. *British Journal for Philosophy of Science, 28*, 99-127.

Raattkainen, P. (2005). On the philosophical relevance of Godel's incompleteness theorems. *Revue Internationale de Philosophie*, *4*(214), 513,534.

Raatikainen, P. (2015). Gödel's Incompleteness Theorems. In E. N. Zalta (Ed.), *The Stanford encyclopedia of philosophy* (Spring 2015 Ed.). Retrieved from http://plato.stanford.edu/archives/spr2015/entries/goedel-incompleteness/

Rayo, A. (2007) Logicism Reconsidered. In Steward Shapiro (ed.), *The Oxford handbook of philosophy of mathematics and logic* (pp. 203-235). New York: Oxford University Press.

Russell, B. (1920). *Introduction to mathematical philosophy*. London: George Allen & Unwin, Ltd.

Steiner, M. (1975). *Mathematical Knowledge*. Ithaca: Cornell University Press.

Tennant, N. (2014). Logicism and Neologicism. In E. N. Zalta (Ed.), *The Stanford encyclopedia of philosophy* (Fall 2014 Ed.). Retrieved from http://plato.stanford.edu/entries/logicism/