

GREEN BISET FUNCTORS AND REPRESENTATION RINGS

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B.S., Mathematics, Boğaziçi University, 2014

Submitted to the Institute for Graduate Studies in
Science and Engineering in partial fulfillment of
the requirements for the degree of
Master of Science

Graduate Program in Mathematics Department
Boğaziçi University

2017

ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to my thesis supervisor Assist. Prof. Olcay Coşkun for his support and guidance in preparation of this study. Without the encouragement and motivation he has given me during the past two years, it would not be possible to complete this thesis.

I am grateful to my family for their endless encouragement and perpetual support throughout my life.

I would like to thank to TÜBİTAK for providing me with scholarship.

ABSTRACT**GREEN BISET FUNCTORS AND REPRESENTATION
RINGS**

In this thesis, we investigate Green biset functors and classification of simple modules over a Green biset functors in some special cases. Our aim is to see what conditions on the Green functor makes simple modules over it classifiable. We also study the relations between classification of simple modules over Green biset functors and classification of simple modules over Biset functors .

ÖZET

GREEN İKİ ETKİ İZLEÇLERİ VE TEMSİL HALKALARI

Bu savda, Green iki etki izleçlerini ve onlar üzerine basit modüllerin özel durumlar için sınıflandırılması araştırılmıştır. Amacımız, A Green iki etki izleçi üzerine hangi koşulları eklediğimizde, basit A-modüllerini sınıflandıra bileceğimizi öğrenmektir. Ayrıca, Green iki etki izleçleri üzerine basit modüllerin sınıflandırılması ile, iki etki izleçleri üzerine basit modüllerin sınıflandırılması arasındaki ilişkiyi de öğrenmeye çalıştık.

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1. BISET

We begin this chapter by reminding the reader basic definition of a G-set. We continue with introducing bisets and their isomorphism classes. Finally we investigate biset Burnside Group $B(H, G)$ and Mackey formula for bisets.

By a group we always mean a finite group. If G is a group, by a G-set, we always mean a finite G-set.

Let us start with reminding the reader the structure which is called a G-action on the set S. Let G be a group and S be a set. If we can construct a map φ from $G \times S$ to S such that it satisfies the conditions

$$\varphi(g_1, \varphi(g_2, s)) = \varphi(g_1 g_2, s) \tag{1.1}$$

$$\varphi(1_G, s) = s \tag{1.2}$$

for all $s \in S$ and all $g_1, g_2 \in G$. Here, 1_G stands for the identity element in G. Then, we call S a left G-set and the map φ the left G-action on the set S.

Right G-sets and right G-actions are defined analogously except elements of G are multiplied by elements of S to the right instead of to the left.

Before we proceed to the definition of a biset, let us give a description of opposite group, which makes it simpler to introduce bisets.

For a given group G , the opposite group denoted by G^{op} , is a group with the set of elements same as G but for group law we reverse the place of action i.e.

$$\text{For all } g, h \in G, gh \text{ (in } G^{op}\text{)} = hg \text{ (in } G\text{)} \quad (1.3)$$

So, knowing G -set and opposite group structures let us define bisets, which we use throughout the entire thesis.

Let G and H be groups. Then an (H,G) -biset is a left $(H \times G^{op})$ -set.

In other words, an (H,G) -biset U is both a left H -set and a right G -set, satisfying the equation

$$\text{For all } h \in H, \text{ for all } u \in U, \text{ for all } g \in G \quad (h \cdot u) \cdot g = h \cdot (u \cdot g). \quad (1.4)$$

We denote this element simply by $h \cdot u \cdot g$ or hug .

Those, who are familiar with Burnside group, remember that rewriting all transitive G -sets for a given finite group G is enough to define all G -sets.

Hence, for a biset structure we also start with defining transitive ones.

A given (H,G) -biset X , it is called transitive if for any $x, y \in X$ there exist an element $h \in H$ and an element $g \in G$ such that hxg is equal to y .

Since we defined an (H, G) -biset as a left $H \times G^{op}$ -set we can figure out that there is a bijective correspondance between the transitive (H, G) bisets and transitive left $H \times G^{op}$ -sets.

To express it in a simpler way, observe that with the definition given above (H,G) -biset X can also be regarded as a right $H \times G$ -set equipped with the action given by

$$x \cdot (h, g) = h^{-1} \cdot x \cdot g \tag{1.5}$$

for all $x \in X$ and $h \in H$, and $g \in G$.

Clearly, X is a transitive (H, G) -biset if and only if X is a transitive $H \times G$ -set.

Hence there is a bijective correspondence between

- (i) isomorphism classes $[X]$ of transitive (H, G) -bisets.
- (ii) conjugacy classes $[L]$ of subgroups of $H \times G$.

where the correspondence is given by $[X] \leftrightarrow [L]$ if and only if the stabilizer of a point $x \in X$ is in $[L]$.

For an example, for the given group G , we can regard G itself as a (G, G) -biset. We call this biset the identity (G, G) -biset and denote by Id_G

Now we would like to go further and uncover more results about the bisets, and so let us define category structure on them.

We assume that the reader is familiar with the concept of categories.

We denote the category of G -sets by **G-set**. It is the category where objects are G -sets and morphisms are G -equivariant maps.

Note that we have mentioned the importance of Burnside group for rewriting G -sets of a given finite group G , lets recall the definition of it and proceed to the definition of the **biset Burnside group**.

Recall that the Burnside Group $B(G)$ of G for a finite group G is the Grothendieck group of the category **G-set**.

Which means that $B(G)$ is the free \mathbb{Z} -module with basis the set of equivalence classes of finite G -sets, quotiented by relations identifying the class of the disjoint union $X \sqcup Y$ of two G -sets X and Y to the sum of the class of X and the class of Y .

In other words, it is the quotient of the free abelian group on the set of isomorphism classes of finite G -sets, by the subgroup generated by the elements of the form

$$[X \cup Y] - [X] - [Y] \tag{1.6}$$

where X and Y are finite G -sets, and $[X]$ denotes the isomorphism class of X .

Since we have defined an (H, G) -biset as a $(H \times G^{op})$ -set, we can easily pass on **biset Burnside Group** denoted by $B(H, G)$ from the Burnside group. Define the **biset Burnside Group** $B(H, G)$ as the Burnside group $B(H \times G^{op})$. In other words, its a **Grothendieck** group of the category of finite (H, G) -bisets.

Now, like in case of G -sets we would like to construct a category structure on the bisets. In an other saying, we would like to construct a category with the finite groups being objects in it and morphisms to be (H, G) -bisets.

Hence, we need to define first the composition of bisets in such a way that satisfies category properties.

For finite groups G, H , and K , let U and V be an (H, G) -biset and (K, H) -biset respectively. So define the composition of V and U as the set of H -orbits on the cartesian product $V \times U$, where the right action of H is defined by

$$\text{For all } (v, u) \in V \times U, \text{ for all } h \in H, (v, u) \cdot h = (v \cdot h, h^{-1} \cdot u). \quad (1.7)$$

Denote it by $V \times_H U$, and the H -orbit of $(v, u) \in V \times U$ as $(v, {}_H u)$.

Thus, the set $V \times_H U$ is a (K, G) -biset, with the action defined as

$$\text{For all } k \in K, g \in G \text{ and } (v, {}_H u) \in V \times_H U, \quad k \cdot (v, {}_H u) \cdot g = (k \cdot v, {}_H u \cdot g). \quad (1.8)$$

Before finalizing this section with the Mackey formula for bisets, we need to introduce notation for some groups.

First one, for finite groups G, H and K , let L and M be the subgroups of the groups $H \times G$ and $K \times H$ respectively

Then set $M * L$ to be as described below,

$$M * L = \{(k, g) \in K \times G \mid \text{there exists } h \in H, (k, h) \in M \perp (h, g) \in L\}. \quad (1.9)$$

Observe that $M * L$ is a subgroup of $K \times G$.

Second one, for finite groups G and H , let U be an (H, G) -biset.

1. For any subgroup L of H , and for an arbitrary element $u \in U$, set

$$L^u = \{g \in G \mid \exists l \in L, l \cdot u = u \cdot g\} \quad (1.10)$$

Here, L^u is a subgroup of G with $\mathbf{1}^u$ being stabilizer of u in G .

2. For any subgroup K of G , and for an arbitrary element $u \in U$, set

$${}^u K = \{h \in H \mid \exists k \in K, h \cdot u = u \cdot k\}. \quad (1.11)$$

Similarly to previous one uK is a subgroup of H and ${}^u\mathbf{1}$ is the stabilizer of u in H .

Third one , for finite groups G and H , take L to be an arbitrary subgroup of $H \times G$. Then, following five groups $p_1(L)$, $p_2(L)$, $k_1(L)$, $k_2(L)$, $q(L)$ are defined as below:

$$(i) \ p_1(L) = \{h \in H | \exists g \in G, (h, g) \in L\},$$

$$(ii) \ p_2(L) = \{g \in G | \exists h \in H, (h, g) \in L\},$$

$$(iii) \ k_1(L) = \{h \in H | (h, 1) \in L\},$$

$$(iv) \ k_2(L) = \{g \in G | (1, g) \in L\},$$

$$(v) \ q(L) = L / (k_1(L) \times k_2(L)).$$

Theorem. (Mackey formula for bisets) *Let G , H and K be groups. If L is a subgroup of $H \times G$, and if M is a subgroup of $K \times H$, then there is an isomorphism of (K, G) -bisets*

$$((K \times H)/M) \times_H ((H \times G)/L) \cong \bigsqcup_{h \in [p_2(M) \setminus H/p_1(L)]} (K \times G)/(M *^{(h,1)} L) \quad (1.12)$$

where $[p_2(M) \setminus H/p_1(L)]$ is a set of representatives of double cosets.

Proof. Set $V = (K \times H)/M$ and $U = (H \times G)/L$. Its easy to check that the map

$$K((k, h)M, _H (h', g)L)G \mapsto p_2(M)h^{-1}h'p_1(L) \quad (1.13)$$

is a bijection $K \setminus (V \times_H U)/G \rightarrow p_2(M) \setminus H/p_1(L)$, the inverse bijection being the map

$$p_2(M)hp_1(L) \mapsto K((1, 1)M, _H (h, 1)L)G. \quad (1.14)$$

Moreover the stabilizer of $((1, 1)M, _H (h, 1)L) \in K \times G$ is equal to $M *^{(h,1)} L$. \square

2. BISET FUNCTORS

With the motivation given in the section one let us rewrite the Biset Category we have described in section 1.

Let Z be a class of finite groups.

The biset category of finite groups is defined to be the category with objects in Z and morphism between objects to be bisets. More precisely, if we have two groups G and H in Z , then $Hom(G, H) = B(H, G)$.

For the composition of morphisms, as we have defined before, we use the composition of bisets. In formal way, for groups G, H , and K from Z and $v \in Hom(H, K)$, $u \in Hom(G, H)$ being morphisms from K to H and H to G respectively, we have seen in section 1 that $v \times_H u$ is a (K, G) -biset, hence define $v \circ u$ to be $v \times_H u$.

And take identity morphism $Hom(G, G)$ to be $[Id_G]$ for the group G in Z .

Thus we constructed a biset category of finite groups, and let denote it with \mathcal{C} .

Let R be a commutative unitary ring. Adding R -linearity to the category \mathcal{C} we constructed above make it a more usefull tool. So define a new category, which we denote by $R\mathcal{C}$.

The objects of $R\mathcal{C}$ is same with objects of \mathcal{C} that are groups in Z .

But, for morphisms we will take tensor product of R with $B(H, G)$. That is, $Hom_{R\mathcal{C}}(G, H) = R \otimes_Z B(H, G)$.

Hence, observe that the composition of morphisms in the category $R\mathcal{C}$ is the R -linear extension of the composition in \mathcal{C} .

And for identity morphism of G in \mathcal{RC} we have $R \otimes_Z Id_G$.

Then, observe that the category \mathcal{RC} is an R -linear category. In other words, the sets of morphisms in \mathcal{RC} are R -modules, and the composition in \mathcal{RC} is R -bilinear.

Now, before defining biset functors, which are our main objective in this section, let us review some properties about functors, and, definition of a Mackey functor.

A **Mackey functor** over a commutative ring R is a mapping

$$M : \{\text{subgroups of } G\} \rightarrow R\text{-mod}$$

with morphisms

- (i) $I_K^H : M(K) \rightarrow M(H)$,
- (ii) $R_K^H : M(H) \rightarrow M(K)$,
- (iii) $c_g : M(H) \rightarrow M({}^gH)$,

whenever $J \leq K \leq H$ and $g \in G$, such that

- (i) $I_H^H, R_H^H, c_h : M(H) \rightarrow M(H)$ are the identity morphisms for all subgroups H and $h \in H$,
- (ii) $R_J^K R_K^H = R_J^H$
- (iii) $I_K^H I_J^K = I_J^H$
- (iv) $c_g c_h = c_{gh}$ for all $g, h \in G$
- (v) $R_K^{{}^gH} c_g = c_g R_K^H$
- (vi) $I_K^{{}^gH} c_g = c_g I_K^H$
- (vii) $R_J^H I_K^H = \sum_{x \in [J \setminus H/K]} I_{J \cap xK}^J c_x R_{J^x \cap K}^K$ for all subgroups $J, K \leq H$

The morphism R_K^H is called restriction, c_g is called conjugation, while I_K^H is called induction. Axiom (vii) is called the **Mackey** axiom.

For preadditive categories \mathbf{A} and \mathbf{B} , let F be a functor from \mathbf{A} to \mathbf{B} , $F : \mathbf{A} \rightarrow \mathbf{B}$. Then we call F **additive** if for any objects X and Y of \mathbf{A} , the map $f \mapsto F(f)$ from $\text{Hom}_{\mathbf{A}}(X, Y)$ to $\text{Hom}_{\mathbf{B}}(F(X), F(Y))$ is a group homomorphism.

We call a subcategory a **preadditive subcategory** if the inclusion functor between a subcategory and a category is additive, i.e; \mathbf{A} is a preadditive subcategory of \mathbf{B} if the inclusion functor $\mathbf{A} \rightarrow \mathbf{B}$ is additive.

For a commutative and unitary ring R , and for some preadditive subcategory of \mathcal{C} let say \mathbf{D} , define a **biset functor** on \mathbf{D} , with values in $R\text{-Mod}$ to be an R -linear functor from $R\mathbf{D}$ to $R\text{-Mod}$.

At the end of this section we define the category of biset functors, i.e; a category considering biset functors as its objects.

Denote this new category by $\mathcal{F}_{D,R}$, where D and R are being as given above. This new category is a category with objects are biset functors over D with values in $R\text{-mod}$, and morphisms are natural transformations of functors, and composition of morphisms is the composition of natural transformations.

Example. Let \mathcal{M} denote the following subcategory of \mathcal{C} : the objects of \mathcal{M} are all finite groups. If G and H are finite groups, then $\text{Hom}_{\mathcal{M}}(G, H)$ is the subgroup of $\text{Hom}_{\mathcal{C}}(G, H) = B(H, G)$ generated by the classes of finite left and right free (H, G) -bisets. Then the biset functors on \mathcal{M} are the global Mackey functors.

3. GREEN BISET FUNCTORS

Adding ring structure to a biset functor give us a new structure which is called \mathfrak{q} Green biset functor. Let \mathcal{Z} here be a class of finite groups closed under subquotients and direct products.

Denote the Burnside functor defined in \mathcal{RC} as RB .

Proposition (3.1).

1. The Burnside functor RB is a projective object of $\mathcal{F}_{D,R}$. More precisely, if F is an object of $\mathcal{F}_{D,R}$, then

$$\text{Hom}_{\mathcal{F}_{D,R}}(RB, F) \cong F(\mathbf{1}).$$

2. In particular there is an isomorphism of R -algebras

$$\text{End}_{\mathcal{F}_{D,R}}(RB) \cong R.$$

Definition (3.1). Let M and N be objects of $\mathcal{F}_{D,R}$. Denote by $\mathcal{H}(M, N)$ the object of $\mathcal{F}_{D,R}$ defined by

$$\begin{aligned} \mathcal{H}(M, N)(G) &= \text{Hom}_{\mathcal{F}(D,R)}(M, N_G), \\ \mathcal{H}(M, N)(f) &: \mathcal{H}(M, N)(G) \rightarrow \mathcal{H}(M, N)(H), \\ \varphi \in \text{Hom}_{\mathcal{F}_{D,R}}(M, N_G) &\mapsto N_f \circ \varphi \in \text{Hom}_{\mathcal{F}_{D,R}}(M, N_H), \end{aligned}$$

for G and H are objects of \mathcal{D} and $f \in RB(H, G)$.

Proposition (3.2). 1. The correspondence

$$- \otimes - : (M, N) \rightarrow M \otimes N$$

is a bilinear functor from $\mathcal{F}_{D,R} \times \mathcal{F}_{D,R}$ to $\mathcal{F}_{D,R}$, called the tensor product of biset functors.

2. There are functorial isomorphisms

$$\text{Hom}_{\mathcal{F}_{D,R}}(M \otimes N, P) \cong \text{Hom}_{\mathcal{F}_{D,R}}(N, \mathcal{H}(M, P))$$

for any objects M, N and P of \mathcal{F}_{DR} , the functor $M \otimes -$ is left adjoint to the functor $\mathcal{H}(M, -)$

Proof. The proof is classical, and not specific to the particular monoidal category RD : set $L = \text{Hom}_{\mathcal{F}_{DR}}(M \otimes N, P)$.

Then:

$$\begin{aligned}
L &= \int_G \text{Hom}_R((M \otimes N)(G), P(G)) \\
&\cong \int_G \text{Hom}_R(\int^{D, D'} M(D) \otimes_R N(D') \otimes_R \text{Hom}_R(D \times D', G), P(G)) \\
&\cong \int_G \int_{D, D'} \text{Hom}_R(M(D) \otimes_R N(D') \otimes_R \text{Hom}_R(D \times D', G), P(G)) \\
L &\cong \int_{D, D'} \int_G \text{Hom}_R(M(D) \otimes_R N(D') \otimes_R \text{Hom}_R(D \times D', G), P(G)) \\
&\text{(by Fubini's theorem see [39] IX.8)} \\
&\cong \int_{D, D'} \text{Hom}_R(M(D) \otimes_R N(D'), P(D \times D')) \text{ (by Yoneda's lemma see [213212] 8.2)} \\
&\cong \int_{D'} \text{Hom}_R(N(D'), \int_D \text{Hom}_R(M(D), P(D \times D'))) \\
&\cong \int_{D'} \text{Hom}_R(N(D'), \mathcal{H}(M, P)(D')) \\
&\cong \text{Hom}_{\mathcal{F}_{D, R}}(N, \mathcal{H}(M, P)).
\end{aligned}$$

□

Proposition (3.3).

$$\text{Hom}_{\mathcal{F}_{D, R}}(M \otimes N, P) = \int_{D, D'} \text{Hom}_R(M(D) \otimes_R N(D'), P(D \times D')).$$

Definition (3.2). Let \mathcal{RC} be a category defined as above with objects in Z . A biset functor A defined in \mathcal{RC} , is a Green biset functor if there exist maps of biset functors

$$\mu : A \otimes A \rightarrow A, \text{ and } e : \mathcal{RB} \rightarrow A$$

such that the following diagrams commute

$$\begin{array}{ccc}
A \otimes (A \otimes A) & \xrightarrow{Id \otimes \mu} & A \otimes A \\
\cong \downarrow & & \downarrow \mu \\
(A \otimes A) \otimes A & \xrightarrow{\mu \otimes Id} & A \otimes A
\end{array}
\begin{array}{c}
\searrow \\
\searrow \\
A
\end{array}$$

$$\begin{array}{ccccc}
\mathcal{RB} \otimes A & \xrightarrow{e \otimes Id} & A \otimes A & \xleftarrow{Id \otimes e} & A \otimes \mathcal{RB} \\
& & \downarrow \mu & & \\
& & A & &
\end{array}$$

By propositions (3.1) and (3.3) the definition given above is equivalent to the following:

Definition (3.3). *A is a Green biset functor if it is a biset functor equipped with bilinear products $A(G) \times A(H) \rightarrow A(G \times H)$ denoted by $(a, b) \mapsto a \times b$, for groups $G, H \in Z$, and an element $\varepsilon_A \in A(1)$, satisfying the following conditions:*

- *Associativity.* Let G, H and K be groups in Z . If $\alpha_{G,H,K}$ is the canonical group isomorphism from $G \times (H \times K)$ to $(G \times H) \times K$, then for any $a \in A(G)$, $b \in A(H)$ and $c \in A(K)$

$$(a \times b) \times c = A(\text{Iso}(\alpha_{G,H,K}))(a \times (b \times c)).$$

- *Identity element.* Let G be a group in Z . Let $\lambda_G : 1 \times G \rightarrow G$ and $\rho_G : G \times 1 \rightarrow G$ denote the canonical group isomorphism. Then for any $a \in A(G)$

$$a = A(\text{Iso}(\lambda_G))(\varepsilon_A \times a) = A(\text{Iso}(\rho_G))(a \times \varepsilon_A).$$

- *Functoriality.* If $\varphi : G \rightarrow G'$ and $\psi : H \rightarrow H'$ are morphisms in \mathfrak{CR} , then for any $a \in A(G)$ and $b \in A(H)$

$$A(\varphi \times \psi)(a \times b) = A(\varphi)(a) \times A(\psi)(b).$$

Example (3.1). Let \mathcal{F} be a field of characteristic zero and Z the class of finite groups. Assigning to each group G in Z the Grothendieck group of finitely generated $\mathcal{F}G$ -modules, $R_{\mathcal{F}}(G)$, defines a biset functor $R_{\mathcal{F}}$. It also has a structure of Green biset functor given in the following way: Let $s \in R_{\mathcal{F}}(G)$ and $t \in R_{\mathcal{F}}(K)$, then $R_{\mathcal{F}}(\text{Inf}_G^{G \times K})(s)$ and $R_{\mathcal{F}}(\text{Inf}_K^{G \times K})(t)$ are both in $R_{\mathcal{F}}(G \times K)$ and so the natural product in $R_{\mathcal{F}}(G \times K)$, given by $\otimes_{\mathcal{F}}$, gives us the product $s \times t$.

Definition (3.4). Let A be a Green biset functor over D , with values in $R\text{-mod}$. A left A -module M is an object of $\mathcal{F}_{\mathcal{D}, \mathcal{R}}$, equipped with a morphism of biset functors $\mu_M : A \otimes M \rightarrow M$ such that the following diagrams commute:

$$\begin{array}{ccc}
 A \otimes (A \otimes M) & \xrightarrow{\text{Id} \otimes \mu_M} & A \otimes M \\
 \cong \downarrow & & \searrow \\
 (A \otimes A) \otimes M & \xrightarrow{\mu_M \otimes \text{Id}} & A \otimes M \\
 & & \nearrow \\
 & & M
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{R}\mathcal{B} \otimes M & \xrightarrow{e \otimes \text{Id}} & A \otimes M \\
 & \searrow & \downarrow \mu_M \\
 & & M
 \end{array}$$

Equivalently, for any object G, H of D , there are product maps

$$A(G) \times M(H) \rightarrow M(G \times H),$$

denoted by $(a, m) \mapsto a \times m$, fulfilling the following conditions:

- (Associativity) Let G, H and K be objects of D . If

$$\alpha_{G, H, K} : G \times (H \times K) \rightarrow (G \times H) \times K$$

is the canonical group isomorphism, then for any $a \in A(G), b \in A(H), m \in M(K)$

$$(a \times b) \times m = \text{Iso}(\alpha_{G, H, K})(a \times (b \times m))$$

- (Identity element) Let G be an object of D . Let $\lambda_G : 1 \times G \rightarrow G$ denote the canonical group isomorphism. Then for any $m \in M(G)$

$$m = \text{Iso}(\lambda_G)(\varepsilon_A \times m)$$

- (Functoriality) If $\varphi : G \rightarrow G'$ and $\psi : H \rightarrow H'$ are morphisms in RD, then for any $a \in A(G)$ and $m \in M(H)$

$$M(\varphi \times \psi)(a \times m) = A(\varphi)(a) \times M(\psi)(m).$$

Definition (3.5). If M and N are A -modules, then a morphism of A -modules from M to N is a morphism of biset functors $f : M \rightarrow N$ such that the diagram

$$\begin{array}{ccc} A \otimes M & \xrightarrow{\mu_M} & M \\ \downarrow Id \otimes f & & \downarrow f \\ A \otimes N & \xrightarrow{\mu_N} & N \end{array}$$

is commutative.

Morphisms of A -modules can be composed, and A -modules form a category, denoted by **A -Mod**.

Definition (3.6). Let A be a Green biset functor on D , with values in R -mod. A left ideal of A is an A -submodule of the left A -module A . In other words it is a biset subfunctor I of A such that

$$A(G) \times I(H) \subseteq I(G \times H)$$

for any objects G and H of D .

The right ideal of A is defined in similar way. A two sided ideal of A is a left ideal which is also a right ideal.

In other words, a two sided ideal I of A is a biset subfunctor such that

$$A(G) \times I(H) \times A(K) \subseteq I(G \times H \times K),$$

for any objects G, H, K of D , with a slight abuse in notation, a more correct formulation being rather

$$(A(G) \times I(H)) \times A(K) \subseteq I((G \times H) \times K).$$

Observe that when I is a two sided ideal of A , then the quotient biset functor A/I is a Green biset functor, and the projection morphism from A to A/I is a morphism of Green functors.

Definition (3.7). *A Green biset functor A is called simple if its only two sided ideals are 0 and A .*

Notation (3.1). *Let X be a (G, H) -biset, then define \vec{X} to be $(G \times H, 1)$ -biset X with action $(g, h)x = gxh^{-1}$ for $g \in G$, $h \in H$, $x \in X$. Hence, for a group G , we will denote by \vec{G} the $(G \times G, 1)$ -biset. In a similar way, \overleftarrow{G} will be, a $(1, G \times G)$ -biset.*

Proposition (3.4). *Let A be a Green biset functor over RC and let P_A be the following category:*

- *The objects of P_A are the groups in Z .*
- *If G and H are in Z , then $\text{Hom}_{P_A}(H, G) = A(G \times H)$.*
- *Let H, G and K be groups in Z . The composition of $\beta \in A(H \times G)$ and $\alpha \in A(G \times K)$ in P_A is the following:*

$$\beta \circ \alpha = A(H \times \overleftarrow{G} \times K)(\beta \times \alpha).$$
- *If G is in Z , then the identity morphism of G in P_A is $\vec{A}(G)(\varepsilon_A)$.*

Then P_A is an R -linear category and $A\text{-Mod}$ is equivalent to the category of R -linear functors from P_A in $R\text{-Mod}$.

Lemma (3.1). *Let A and C be Green functors and $f : C \rightarrow A$ a morphism of Green functors. If $\beta \in C(H \times G)$ and $\alpha \in C(G \times K)$, then*

$$f_{H \times K}(\beta \circ \alpha) = f_{H \times G}(\beta) \circ f_{G \times K}(\alpha).$$

Definition (3.8). *Let M be a biset functor defined on \mathcal{RC} . A group H in Z is called minimal for M if $M(H) \neq 0$ and $M(K) = 0$ for any $K \in Z$ with $|K| < |H|$.*

Definition (3.9). *Let A be a Green bise functor on \mathcal{RC} , and H be group in Z . We will write $I_A(H)$ for the submodule of $A(H \times H)$ generated by elements of the form $a \circ b$, where a is in $A(H \times K)$, b is in $A(K \times H)$ and K is a group in Z of order smaller than $|H|$. We will denote by $\hat{A}(H)$ the quotient $A(H \times H)/I_A(H)$.*

It is clear that $\hat{A}(H)$ is an R -algebra. If $A = RB$, it is known, that $\hat{RB}(H)$ is isomorphic to $R\text{Out}(H)$. It is also obvious that this quotient is not always differ from zero.

Thus, the classification of simple biset functor is a particular case of the following conjecture.

Conjecture. *Suppose that A is a Green biset functor. Let S_A be the set of equivalence classes of couples (H, V) , where $\hat{A}(H) \neq 0$, V is a simple $\hat{A}(H)$ -module and $(H, V) \sim (G, W)$ if there exists an isomorphism of groups $\varphi : H \rightarrow G$ such that $V \cong^\varphi W$.*

Then the isomorphism classes of simple A -modules are in one-to-one correspondence with the elements of S_A .

Lemma (3.2). *Let S be a simple A -module for a Green biset functor A defined in $\Omega_{R,Z}$. If $H \in Z$ is a minimal group for S , then $\hat{A}(H) \neq 0$ and $S(H)$ is a simple $\hat{A}(H)$ -module.*

4. SIMPLE MODULES OVER GREEN BISET FUNCTORS

Let k be a field of characteristic 0.

Let $\hat{\Omega}$ be the category with class of finite groups as the objects, and the quotient $kB(H \times G)/Ker(klin_{\mathbb{F}H \times G})$, with \mathbb{F} a field of characteristic 0 and

$$klin_{\mathbb{F}} : kB \rightarrow kR_{\mathbb{F}}$$

being the linearization morphism.

So define rhetorical biset functor to be a k -linear functor from $\hat{\Omega}$ to $k\text{-Mod}$. Hence, by Proposition (3.4), for $\mathbb{F} = \mathbb{Q}$, rhetorical biset functors coincide with $kR_{\mathbb{Q}}$ -modules.

Lemma (4.1). *Let H , G and K be groups. If $[M]$ is in $kR_{\mathbb{Q}}(H \times G)$ and $[N]$ is in $kR_{\mathbb{Q}}(G \times K)$, then*

$$kR_{\mathbb{Q}}(H \times G \times K)([M] \times [N]) = [M \otimes_{\mathbb{Q}G} N].$$

Hence, by Laurence Barkers work referred in [1], it gives a classification of $kR_{\mathbb{Q}}$ -modules.

Definition (4.1). *Let m be in \mathbb{N} . A simple $k(\mathbb{Z}/m\mathbb{Z})^{\times}$ -module V is called primitive if for n divisor of m*

$$x \cdot 1 = 1 \quad \forall x \in Ker \pi_{m,n} \Rightarrow n = m$$

where $\pi_{m,n}$ is the natural projection $(\mathbb{Z}/m\mathbb{Z})^{\times} \rightarrow (\mathbb{Z}/n\mathbb{Z})^{\times}$.

Lemma (4.2). *The simple $kR_{\mathbb{Q}}$ -modules are the simple biset functors $S_{H,V}$, where H satisfies $k\hat{R}_{\mathbb{Q}}(H) \neq 0$ and V is a simple $k\hat{R}_{\mathbb{Q}}(H)$ -module.*

Lemma (4.3). *i) Let G be any non trivial group. Then $I_{kR_{\mathbb{Q}}}(G)$ is equal to*

$$\sum_{K \text{ cyclic}} kP_{\mathbb{Q}}(G \times K) \circ kR_{\mathbb{Q}}(K \times G)$$

$|K|$ proper divisor of $|G|$

ii) If G is not a cyclic group, then $k\hat{R}_{\mathbb{Q}}(G) = 0$.

Lemma (4.4). *Let K be a cyclic group with $|K|$ a proper divisor of $|H|$. Let σ be an automorphism of H . Consider a $\mathbb{Q}(H \times H)$ -module T of the form*

$$1_C \text{Inf}_C^{H \times K} \otimes_{\mathbb{Q}K} 1_D \text{Inf}_D^{K \times H}$$

with $C \leq H \times K$ and $D \leq K \times H$ cyclic subgroups. Then the coefficient of $[1_{\Delta_{\sigma}(H)} \text{Inf}_{\Delta_{\sigma}(H)}^{H \times H}]$ for $[T]$ in terms of the basis $\beta(H \times H)$ is different from 0 if and only if

$$i) \pi_H(C) = H = \pi_H(D)$$

$$ii) C = \langle (h, x) \rangle \Leftrightarrow D = \langle (x, \sigma(h)) \rangle.$$

In this case, the coefficient is $|K|/|H|$.

Proposition (4.1). *Let x_n be*

$$\sum_{[t] \in \text{Ker} \pi_{m,n}} [t]$$

The kernel of l_H is the ideal generated by $\{x_n | n \text{ proper divisor of } m\}$.

Corollary (4.1). *Let H be a group such that $\hat{R}_{\mathbb{Q}}(H) \neq 0$, and V a simple $k\text{Out}(H)$ -module. Then V is a $k\hat{R}_{\mathbb{Q}}(H)$ -module if and only if V is primitive.*

Notation (4.1). Let W be an $A(G \times G)$ -module. If $\varphi : H \rightarrow G$ is a group isomorphism, we denote by ${}^\varphi W$ the $A(H \times H)$ -module W with action of $a \in A(H \times H)$ in $w \in W$ given by

$$a \cdot w := (\alpha_1 \circ a \circ \alpha_2)w$$

using the notation of the previous paragraph.

Proposition (4.2). Suppose that A is a Green biset functor for which the minimal groups of each simple A -module form a single isomorphism class. Let S_A be the set of equivalence classes of couples (H, V) , where $\hat{A}(H) \neq 0$, V is a simple $\hat{A}(H)$ -module and $(H, V) \sim (G, W)$ if there exists an isomorphism of groups $\varphi : H \rightarrow G$ such that $V \cong^\varphi W$.

Then the isomorphism classes of simple A -modules are in one-to-one correspondence with the elements of S_A .

Proposition (4.3). $\mathbb{C}R_{\mathbb{C}}$ is the only simple $\mathbb{C}R_{\mathbb{C}}$ -module. In particular, it is a simple Green biset functor.

For a group C , denote by RB_C the Yoneda -Dress construction at C of RB . If F is a biset functor, then F_C send each group G to $F(G \times C)$. In an element $\varphi \in RB(G \times H)$ it is defined as $F(\varphi \times C)$

Lemma (4.5). If A is a Green biset functor and C is a group, then A_C is a Green biset functor.

Lemma (4.6). Let G, L, K and C be groups. If E is a subgroup of $G \times L \times C$ and D is a subgroup of $L \times K \times C$, then we have an isomorphism of $G \times K \times C$ -sets

$$(G \times L \times C)/E \times_K^d (L \times K \times C)/D \cong \bigsqcup_{(l,c)} (G \times K \times C)/(E *^{(l,1,c)} D)$$

where (l,c) is running through a set of representatives of the double cosets $p_{2,3}(E) \setminus L \times C/p_{1,3}(D)$, with $p_{2,3}(E)$ and $p_{1,3}(D)$ being the projections over $L \times C$ of E and D respectively.

Lemma (4.7). *Let C be a group of prime order and S be a simple RB_C -module. Then any two minimal groups H and K of S are isomorphic.*

Corollary (4.2). *If C is group of prime order, then the simple RB_C -modules are in one-to-one correspondance with the equivalence classes of couples (H,V) where H is a group and V is a simple $R\hat{B}_C(H)$ -module.*

Lemma (4.8). *For any groups G and C , we have $R\hat{B}_C(G) \neq 0$.*

5. CONCLUSION

In this thesis, we investigate Green biset functors and classification of simple modules over a Green biset functors in some special cases. In first two chapters, we remind the reader basic definitions of Biset and Biset functors. In third chapter, we have given the definition of Green biset functors and introduce the conjecture by Serge Bouc. In last chapter, we exemplify the conjecture with three simple modules over Green biset functors.

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