NONPERTURBATIVE ASPECTS OF QUANTUM FIELD THEORIES

by

Burak Tevfik Kaynak

B.S., Physics, İstanbul Technical University, 2000

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ABSTRACT

NONPERTURBATIVE ASPECTS OF QUANTUM FIELD THEORIES

The main focus of this thesis is to elaborate rather unorthodox methods throughout a diverse range of mathematical techniques, which allow us to study quantum field theoretical models in a nonperturbative context. The thesis is divided into two parts: In the first part, we use semigroup integral to evaluate zeta–function regularized determinants. This is especially powerful for nonpositive operators such as the Dirac operator. In order to understand fully the quantum effective action, one should know not only the potential term but also the leading kinetic term. For this purpose, we use the Weyl type of symbol calculus to evaluate the determinant as a derivative expansion. The technique is applied both to a spin–0 bosonic operator and to the Dirac operator coupled to a scalar field. The latter is important for the Yukawa model. In the second part, we study the relativistic Lee model on static Riemannian manifolds. The relativistic Lee model is an overly simplified version of the Yukawa model, which is amenable to a nonperturbative treatment. Understanding of it could shed light on the Yukawa model. The model is constructed nonperturbatively through its resolvent, which is based on the so–called principal operator and the heat kernel techniques. It is shown that making the principal operator well defined dictates how to renormalize the parameters of the model. The renormalization of the parameters is the same in the light–front coordinates as in the instant form. Moreover, the renormalization of the model on Riemannian manifolds agrees with the flat case. The asymptotic behaviour of the renormalized principal operator in the large number of bosons limit implies that the ground state energy is positive. In 2 + 1 dimensions, the model requires only a mass renormalization. We obtain rigorous bounds on the ground state energy for the n–particle sector of the (2 + 1)–dimensional model.

ÖZET

KUANTUM ALAN KURAMININ PERTÜRBASYON DIŞI YÖNLERİ

Kuantum alan kuramı modellerini pertürbasyon dışı içeriklerde incelemeye izin veren çeşitli gelenek dışı metotların, geniş bir yelpazeye yayılmış birbirinden farklı matematiksel teknikleri kullanarak geliştirilmesi, bu tezin ana odağını oluşturmaktadır. Tez iki bölümden oluşmaktadır: Tezin ilk bölümünde, zeta fonksiyonu yardımı ile regülarize edilen determinantlar, kısmi grup integral temsili ile hesaplanmaktadır. Bu yöntem özellikle Dirac operatörü gibi pozitif olmayan operatörler için işlevseldir. Kuantum efektif eylem fonksiyonelini tamamıyla anlayabilmek için bu fonksiyonelin alanlar ve alanların türevleri cinsinden verilebilen açılımındaki potansiyel terimin tek başına değil, ana kinetik terimin de aynı zamanda bilinmesi gerekir. Bu amaçla, türev açılımı olarak determinant hesabını yapabilmek için Weyl tipi sembol hesabı kullanılmıştır. Bu teknik hem spin–0 boson operatörlerine hem de skalar alanlara kenetlenmiş Dirac operatörüne uygulanmıştır. Tezin ikinci kısmında, durağan Riemann katlı uzaylarında göreli Lee modeli çalışılmıştır. Lee modeli Yukawa modelinin pertürbasyon dışı ele alınışa uygun olan basitleştirilmiş halidir. Bu modeli anlamak Yukawa modeline ışık tutacaktır. Model, esas operatör ve ısı çekirdeği teknikleri ile çözeni kullanılarak pertürbasyon dışı kurulmuştur. Modelin parametrelerinin nasıl renormalize edileceğini, esas operatörün iyi tanımli hale getirilmesinin belirlediği gösterilmiştir. Parametrelerin renormalizasyonu, ışık önyüzü koordinatlarında ve an biçimli koordinatlarda aynıdır. Riemann uzaylarındaki modelin renormalizasyonu, uzay zamanın düz olduğu durumla uyuşmaktadır. Esas operatörün renormalizasyonunun boson sayısının çok olduğu limitteki asimptotik davranışı taban durumu enerjisinin pozitif olduğunu kastetmektedir. 2 + 1 boyutta model sadece kütle renormalizasyonu gerektirmektedir. 2 + 1 boyutlu modelin n parçacık sektörünün taban durumu enerjisi için sıkı bir sınır bulunmuştur.

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LIST OF SYMBOLS/ABBREVIATIONS

$a_k(x)$	Seeley–DeWitt coefficients
$\tilde{A}$	Symbol of a generic operator A
$\mathbb{C}$	Complex numbers
D^α	Multi–derivative
$\mathcal{F}_{\mathcal{B}}^{(n)}$	Bosonic Fock space of dimension n
${}_2F_1(a,b;c;z)$	Gauss' hypergeometric function
${}_3F_2(a,b,c;d,e;z)$	Generalized hypergeometric function
$g^{\mu\nu}$	Metric tensor
$\hbar$	Reduced Planck constant
$\mathbb{H}^d$	Hyperbolic space of dimension d
H	Hamiltonian
$\mathcal{H}$	Hilbert space
$\Im(z)$	Imaginary part of $z \in \mathbb{C}$
$K(u\|x,y)$	Heat kernel
$K_s(z)$	Modified Bessel function
$\mathcal{L}^2(\Omega)$	Space of square integrable functions
$\mathcal{M}$	Manifold
P_0	Projector onto zero modes
Q	Conserved charge
R	Curvature scalar
$R(\lambda)$	Resolvent matrix
$\tilde{R}$	Resolvent symbol
$\mathbb{R}^d$	Euclidean space/spacetime of dimension d
$\Re(z)$	Real part of $z \in \mathbb{C}$
S	Action functional
tr	Ordinary matrix trace
Tr	Operator trace
$U(N)$	Unitary group of dimension N^2
$W[\rho]$	Free energy, a functional of a generic source ρ

$\|\alpha\|$	Multi-index magnitude
γ^μ	Gamma matrices
$\Gamma(s)$	Gamma function
$\Gamma[\varphi]$	Quantum effective action, a functional of a generic field φ
$\delta_{\mu_1\mu_2\cdots\mu_n}$	Generalized Krocneker delta
Δ	Laplacian operator
$\Delta_{\mathcal{M}}$	Laplacian operator on $\mathcal{M}$
ϵ	Cut-off parameter
$\zeta(s\|A)$	Zeta function of a generic operator A
$\lambda_1(\mathcal{M})$	Spectral gap/radius of the Laplacian operator $\Delta_{\mathcal{M}}$
σ_i,σ_+,σ_-, $i = 1, 2, 3$	Pauli matrices
$\sigma(A)$	Symbol of a generic operator A
Σ	Submanifold
$\Phi(E)$	Principal operator
χ_+, χ_-	Spin states
$\partial\Omega$	Boundary of the domain Ω
$\overline{\Omega}$	Closure of the domain Ω

CZO	Calderon–Zygmund operators
ΨDO	Pseudodifferential operators

1. INTRODUCTION

Nature provides us with an enormous amount of phenomena, whose underlying physics can be explained through various theories. These theories naturally depend on the energy scale at which the physical processes in question typically occur, such as the energy scale of a scattering cross section to be calculated or of a transition amplitude. It is obvious that there is a different set of parameters for each energy scale. Needless to say, the issue is basically to develop some approximation schemes so as to consistently determine these parameters. However, the number of exactly solvable models mimicking the theories which help us to understand the fundamental interactions that occur in nature is very few. These schemes should thereby be mathematically powerful and rigorous enough to eradicate the possible technical problems to be confronted with. Moreover, they should allow us to gain a good degree of physical intuition, as well. It is a widely held view that parameters of a theory under study change under interactions. This phenomenon appears in all areas of physics and calls attention to the role of effective quantities in physics. By effective quantities, ones which average microscopic effects on a macroscopic scale is meant. That is to say, physical quantities, measured in the laboratory, is dressed by fluctuations. This leads to the quest of formulating theories in such a way that whose interactions allow the typical feature of the theories remain unchanged even though the parameters of them change their values. This point of view opens the door to the concept of renormalization, which is of unquestionable importance in modern physics, in spite of its notoriously complicated nature. A well–known classical example is the difference between the propagation of light in a vacuum and the one in a medium. Due to the occurrence of complicated interactions between the electromagnetic field and the molecules of the material, both the permitivity and the permeability constant change on a macroscopic level, which happens to lead to a change in the speed of light in the material. The Maxwell equations still take the same form as in vacua, yet with a slight modification, which is the replacement of the permitivity and the permeability constant with their effective values, which typically depends on the frequency. These effective values differ from the ones measured in vacuum by a finite amount. These finite shifts are actually nothing but renormalization factors, and are called electric and magnetic susceptibility. However, in quantum field theory we are not so lucky as in classical electromagnetic theory. It turns out that the renormalization factors encountered in quantum field theory are generally found to be infinite. In field theories, renormalization factors mentioned above are generally called renormalization counterterms. They are added to the classical Lagrangian density. Not only can they be computed perturbatively, but also can be obtained nonperturbatively as long as the theory permits such a treatment. It may be said that they represent quantum fluctuations. From a perturbative diagrammatic standpoint (note that diagrammatic representations are widely used in nonperturbative approaches, as well), adding counterterms to the classical Lagrangian density leads to additional Feynman diagrams or redefines the Feynman rules.

It may be said that quantum and statistical field theory are two sides of the same medal. Historically, renormalization was introduced as a tool not only to explain the physics behind phase transition but also to predict physical features in it. It is often stated as: renormalization allows us to study the evolution of a system with respect to the scale of observation. This improves our understanding of the macroscopic physics in terms of the fundamental interactions occurring on a microscopic level. Physics at different scales are basically connected by quantum or statistical fluctuations on all the scales in between. Historically, mathematical inconsistencies emerging when computing physical quantities in high energy physics brought renormalization concept to high energy physicists' notice, independently of its use in statistical field theory. Yet, it was then well understood that they do not have different physical content. To be more specific, in high energy physics, quantum corrections involve not only the exchange of elementary particles but also creation and annihilation of the particle–antiparticle pairs. If this process is depicted by a Feynman diagram, it basically corresponds a loop placed in a propagator. While the external momenta are being kept fixed, the integration over all momentum values in the loop integrals results in a divergent expression. Now, we come to the point: whatever the regularization prescription is chosen, it is inevitable that regularization of a divergent expression involves the introduction of an energy scale. Moreover, if one wants to compute a low energy effective action of a theory at some scale of observation, then he/she should integrate out the high energy modes bigger than this scale, which amounts to computation of determinants of various operators. It turns out that regularization generally happens to be necessary in these computations. These effective actions take into account all the quantum effects which occur at the scales bigger than the specified observation scale a priori. Therefore, it is apparent that effective actions are also scale–dependent objects. Recall that renormalization describes the evolution of the system with reference to the observation scales. On that account, we actually end up with a flow. This is the so–called renormalization group flow [1]. Similar renormalization group flows are obtained in statistical field theories. But this time, dealing with thermal fluctuations leads to scale dependence, e.g. as being in the Kondo problem [2]. Renormalization group flows, generated by renormalization group transformations, are there used to diminish the resolution in the systems, namely coarse graining procedure. To put it another way, if one wants to predict the macroscopic behaviour of a system at some scale, the behaviour of this system at smaller scales should be replaced by appropriate averages. This is the so–called scaling hypothesis [3]. Taking all these into account, it is obvious that the procedures in both fields of study result in the same renormalization flow even though the main motivation behind them are distinct. In light of those discussions above, the idea of counterterms and of scales play prominent parts in renormalization. Yet, a crucial question will immediately arise: how can one renormalize a theory? Of course, the answer depends on the features of the theory under consideration. However, whatsoever the procedure is chosen to regularize the theory as an intermediate step for renormalization, the regularized theory should meet the requirements that an acceptable or a correct theory must have as long as it is not an approximate theory. The computation of counterterms and analysis of its scale dependence can be done perturbatively or nonperturbatively. In most cases nonperturbative analysis is harder to achieve.

In a very general manner, the industry called perturbative field theory, in essence, relies upon choosing a model which is of such a nice type that the terms representing the interaction in the model can be added to the exactly solvable part of it in a perturbative expansion, which is assumed to be controlled by a small parameter. The exactly solvable part can either be a simple gaussian or a mean field model, and the interaction part is given in a polynomial form. However the convergence of the series expansion is always doubtful, and in principle one has to find out a way to improve the rate of the convergence of the perturbation expansion. If the coupling constant in the model, which represents the strength of the interaction, is chosen as the perturbation parameter, then one can only study the model as a weakly coupled theory. This kind of a choice could also make the former problem manageable. On the other hand, Planck constant can also be chosen as a small parameter. This allows one to be able to compute quantum effective action of the model under study by means of a saddle point approximation. However it is hard to go beyond two–loop or three–loop in calculations since the computations become massively complicated. Moreover, these higher order correction terms contributing the physical quantities to be predicted at the scale of observation could happen to deviate more and more from the measured quantities in the laboratory. These computations are typically asymptotic expansions, therefore the Borel summability is a natural question. It is widely assumed that the perturbation expansion admits, in principle, a reparametrization such that the scale dependence can be made less sensitive, and it can even be eliminated. Therefore, the idea behind the perturbative field theory is to make testable predictions from a model by doing the best approximation as much as possible in the sense explained above. Although the results obtained in high energy experiments confirm perturbative calculations, and perturbative treatments have been highly successful in other areas of quantum theories and statistical field theories, the analysis based on these kinds of approaches masks genuinely nonperturbative effects.

The problem of how to analyze field theories in a nonperturbative framework is still elusive and improvements are made in various directions. One way of capturing nonperturbative effects is to study semiclassical descriptions of various quantum field theoretical models via the large N limit. Even though some quantum fluctuations become smaller, as the order of the expansion parameter N becomes bigger, this limit still allows many of the features beyond the reach of perturbative settings to survive. As a result one finds a different classical limit of the same theory. In path integral language, this limit corresponds to a quantum effective action calculation as a large N expansion, and it can be reduced into a determinant computation at the end. Calculation of functional determinants is very important in quantum field theories. From the one-loop effective action to instanton calculations, the main tool is the evaluation of such an infinite dimensional determinant [4, 5, 6]. In this work, we present a derivative expansion for such regularized determinants which is especially suitable for nonpositive definite operators, such as the Dirac operator. The literature on regularized determinants is vast: we will not be able to do justice to all who has contributed to this area. The main tool is the introduction of a zeta function for the operator [7, 8, 9, 10]. In quantum field theory, the calculation of the zeta function through the use of heat kernel (or, its similar version, proper time regularization) is favored [4, 11, 12, 13, 14]. The advantage is that there is a systematic short time expansion of the heat kernel, the coefficients of which are all related to geometric invariants and especially suitable for theories which involve gauge fields. The disadvantage of this approach is that the operator under consideration should be positive definite or its determinant should be related to the determinant of its square without any correction terms (i.e., without a multiplicative anomaly). In general, the regularized determinants do not meet this last criterion; there is, for example, by now the well known Kontsevich-Vishik multiplicative correction [15, 16, 17]. An alternative path is to evaluate the zeta function through the semigroup integral, which is used to define complex powers of elliptic operators [7, 18, 19]. In general, for higher dimensional determinants, there is no analog of the Gelfand-Yaglom formula [20] (see, however, the recent attempts [21, 22, 23]). For such determinants, one should resort to an approximation method. It is physically reasonable to assume that the contributions coming from the derivatives of the fields are becoming smaller as the order of the derivative increases. Thus, it is natural to look for a kind of derivative expansion. The proper mathematical tool for this is the symbol calculus for pseudodifferential operators [18, 24, 25, 26, 27, 28]. In this work, we will apply this expansion to the zeta function via the semigroup integral representation. In the case of Laplace operators defined over a ball or over a generalized cone, one may actually evaluate the semigroup integral and find an exact result for the determinant [29, 30]. There are also other cases such as torus $\mathbb{T}^N$, sphere $\mathbb{S}^N$, and hyperbolic space $\mathbb{H}^N$ in which it is possible to find exact solutions of the heat kernel equation and to give the zeta function for the Laplace-Beltrami operator in closed form [13]. Other exact solutions on homogeneous spaces can be found [12] as well. It is also possible to give a complete description of the zeta determinants for Dirac- and Laplace-type operators over finite cylinders using the contour integration method equipped with different boundary conditions [31]. An important example of the evaluation of chiral Jacobians via the zeta function method and the symbol calculus is given by Muschietti *et al.* [32]. There are other ways of applying the symbol calculus essentially exploiting Wigner-type transformations [33] or utilizing a suitable representation of the logarithm as integral of a resolvent [34]; however, they are harder to generalize to manifolds.

As long as nonperturbative aspects of renormalization is concerned, even simple quantum field theory models allow us to enhance our abilities to combat against quantum theories defined on manifolds in a nonperturbative context. A profound understanding of quantum field theories in gravity backgrounds requires various toolbox of not only field theoretical but also mathematical methods. The Lee model is a simple field theory model, which requires a mass, coupling constant and wave function renormalization [35]. What is so special about the model is that the renormalizations can be carried out nonperturbatively. This is, therefore, a good testing ground for various new ideas and methods on interacting quantum field theories. In the original Lee model there are two fermion fields called N and V, assumed to be so heavy that their energies are independent of the momentum, and a single relativistic real bosonic field named usually as θ. The Lee model is amenable to exact analysis because there are two rather restricting conserved quantities. One of which is the conservation of the total number of the fermion species. Furthermore, the sum of bosons and V type fermions is conserved. These highly constrain the theory allowing only a finite number of particles interacting at any given time. If we work with a complex scalar field, the situation changes drastically and the model becomes rather difficult [36]. Although the renormalization is performed exactly, it is done so in a small number of particle sectors, and it is believed that the same prescriptions will continue to cure the divergences in all sectors. This is, of course, plausible since there are no other parameters in the theory. Once the physical V particle is determined as a composite, the coupling constant is determined in such a way to make the scattering of N and θ finite, thereby all the other physical processes should be well-defined (see a source theory approach to the model [37]). The model is asymptotically free for $d < 4$, and this point has been analysed from the modern point of view [38, 39]. There is one subtle point, beyond a certain value of the renormalized coupling there appears a ghost state. This problem has been analysed recently in the virtue of PT symmetry by Bender et al in [40]. They show that by an appropriate redefinition of the norm, the ghost state can be turned into a physical state. Moreover, an equivalent hermitian Hamiltonian through a similarity transformation is constructed in the context of quasi-hermitian quantum mechanics [41]. Even though the model is defined nonperturbatively, to understand the resulting spectrum remains largely as a challenge [42].

The model is sufficiently rich, by restricting the total number of fermions to one, we can still get most of the interesting features. Moreover, one can assume that these fermions carry no momentum so that they have no recoil, and hence assumed to be fixed at the origin. This becomes equivalent to a two states system sitting at the origin interacting with relativistic real bosons [43, 44]. This is the version we will be working with so as to extend the model to Riemannian manifolds. The nonrelativistic version of this model, which is worked out beautifully in the book by Thirring and Henley [45], is still an interesting case study, yet in this case the coupling constant and wave function renormalizations are not needed. The study of the spectrum is still a nontrivial problem. To address these issues, there are some attempts in the literature [46, 47]. While looking at some nonrelativistic problems which require nontrivial renormalizations, Rajeev introduced a new perspective [48]. In this approach one attempts to renormalize the theory by working out the full resolvent in the Fock space of the system. The resolvent contains essentially all the information about the model. More interestingly, the bound states can be found through the zero eigenvalues of an operator, the so-called principal operator, which is parametrized by the energy in a nonlinear manner. Although one can write the resolvent, it is not possible to write down the quantum Hamiltonian of the renormalized theory. In the restricted Lee model, since the interaction is at a point, the renormalized model can be considered as a singular extension of the free bosonic Hamiltonian. This is analogous to the attractive delta function potential in two dimensions, which requires a coupling constant renormalization [49]. There one could also write down the resolvent but not the corresponding Hamiltonian. The interaction appears as a kind of boundary condition, this point of view originates from ideas of M. G. Krein on operators (see Albeverio and Kutasov [50] for a modern exposition). This point of view is extended to the nonrelativistic Lee model, inspired from this work we develop the relativistic Lee model along the same lines [48]. Having found the principal operator, and thus the resolvent, we can in principle work out all the physically important questions for all particle sectors.

Following the heat kernel based methods [51], we extend these ideas to the case of manifolds. The renormalizations are the same, of course, the resolvent contains information about the geometry through the heat kernel. The spectrum of the model is an interesting problem, we only attempt to partially understand it for large number of particles and show that the ground state energy remains positive in this limit in $3+1$ dimensions. In $2+1$ dimensions, one can make more progress, and give a rigorous bound on the energy. Developing new approximation methods for estimating the energy levels and scattering amplitudes remains as a challenge.

The organization of this thesis is as follows: Chapter 2 is mainly based on work by Kaynak *et al.* [52]. In Section 2.1, we first summarize the well known zeta-function prescription for the evaluation of determinants of an operator, the pseudodifferential operator techniques, and symbol calculus. Afterward the definition of complex powers of elliptic operators via the so-called semigroup integral representation is defined. At the end, we conclude that section with how a regularized determinant of an elliptic operator can be evaluated by means of introducing a semiclassical expansion for the symbol of complex powers of the operator under consideration.

In Section 2.5, the application of this method throughout for a spin-0 bosonic operator in $4d$ space-time is explicitly shown. In Section 2.6, we elaborate on how a zeta-function regularized determinant for a Dirac operator with a scalar field can be calculated. The ways of dealing with some of the technical difficulties encountered during the calculations are explained in detail.

In Section 2.7, the result of Section 2.6 is applied to the large-N Yukawa theory and some comments about the form of the terms involving nonlocal functional of the scalar field in the $N = \infty$ quantum effective action of the theory are made.

Chapter 3 is mainly based on recent work by Kaynak *et al.* [53]. In Section 3.1, we will, first, construct the model in flat space-time through the approach, introduced by Rajeev [48] without reviewing it. We show that the principal operator has a well-defined limit when the cut-off is removed, and the renormalized operator can be given by the renormalized mass and the renormalized coupling constant. Moreover, this limit determines the wave function renormalization constant. Afterwards, we specify how to impose the renormalization condition in this approach such that we convert the renormalized mass difference into the physical one by fixing the finite arbitrariness, which is left after renormalizing the parameters.

In Section 3.4, we will apply the ideas, presented by Erman *et al.* [51], to the relativistic Lee model on a general static Riemannian manifold. It is shown that the regularization of the ultra-violet divergence in the theory can be established through the short-time expansion of the heat kernel if the point interaction is introduced by a convolution of the bosonic field with a heat kernel. It is found that the divergence structure of the model in the manifold case is exactly the same as the flat case.

In Section 3.6, we study the asymptotic behaviours of the renormalized principal operator in the large number of bosons limit in both flat and manifold cases. In this limit, it is shown that the leading behaviour of the theory changes substantially. The ultra-static space-time $\mathbb{R} \times \mathbb{H}^3$ is given as an example.

In Section 3.7, we study the model in $2+1$ dimensions. The advantage is that it is simple and requires only a mass renormalization. This allows us to find rigorous bounds on the ground state energy for n-particle sector, thus illustrating the power of this method.

In Appendix A, an alternative approach for the computation of the determinant of a bosonic operator throughout the so–called star–exponential is studied. In Appendix B, the same techniques are tested in an oblique light-front coordinates.

2. SYMBOL CALCULUS AND ZETA–FUNCTION REGULARIZED DETERMINANTS

2.1. Zeta–Function Regularization

The zeta function of an operator A is defined by the sum over its eigenvalues,

$$\zeta(s|A) = \sum_n \frac{1}{\lambda_n^s}. \tag{2.1}$$

This sum converges only for sufficiently large values of $\Re(s) > 0$. We introduce a local zeta function by

$$\zeta(s|A)(x) = \left\langle x|A^{-s}|x\right\rangle \,, \tag{2.2}$$

which is a regular analytic function on the complex s-plane; otherwise, it is possible to define it by means of analytic continuation into the complex s-plane. It is known that it is a holomorphic function of s for $\Re(s) > \dim(M)/m = s_0$, in which m is the order of the elliptic operator under consideration, and s_0 is the abscissa of convergence of $\zeta(s|A)$. It has a meromorphic extension to the whole complex plane with merely simple poles; $\zeta(s|A)$ and its derivative are regular at $s = 0$ [7]. We can calculate the derivative of $\zeta(s|A)$ at $s = 0$ as,

$$-\frac{\partial}{\partial s}\zeta(s|A)\bigg|_{s=0} = \sum_n \ln\lambda_n e^{-s\ln\lambda_n}\bigg|_{s=0} = \sum_n \ln\lambda_n \tag{2.3}$$

$$= \ln\prod_n \lambda_n = \ln\det A\,, \tag{2.4}$$

which allows us to define the regularized determinant. It is assumed that the operator has no eigenvalues near the point zero in order to avoid some infrared divergences. However, it is possible to introduce a restricted determinant by removing these zero modes if the number of zero modes is finite [13, 21, 22, 54].

If the operator under consideration is a positive-definite operator, one of the convenient ways to make sense of the ζ-determinant is to use heat equation method, i.e

$$\zeta(s|A) = \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \mathrm{Tr} e^{-tA} \,, \tag{2.5}$$

in which the heat trace, $\mathrm{Tr}e^{-tA} = \sum_n e^{-t\lambda_n}$, λ_n being the eigenvalues of the operator A. The pole structure of the ζ-function is related to the heat kernel coefficients since the poles of $\zeta(s)$ coincide with the values which comes in the asymptotic expansion of the heat trace, namely, $s = (d-n)/m$. As mentioned, if zero modes are present, one has to define a restricted determinant by subtracting them, replacing $\mathrm{Tr}e^{-tA}$ by $\mathrm{Tr}e^{-tA} - P_0$, where P_0 is the projector onto the zero modes. Therefore, the ζ-function for the operator A in the presence of zero modes is given by

$$\zeta(s|A) = \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \left(\mathrm{Tr} e^{-tA} - P_0 \right) \,. \tag{2.6}$$

For the sake of completeness, we will give the one-loop quantum effective potential calculation for the scalar field theory with a $\lambda\phi^4$ self-coupling as an example for the ζ-function regularization [4, 6, 13]. The operator of the Eucledian classical action we will deal with is now $A = -\partial^2 + m^2 + \lambda\phi_{cl}^2(x)/2$, where m is the mass of the scalar particles, λ is the coupling constant, and $\phi_{cl}(x)$ is the solution of the classical field equations driven by the source ρ. To be more clear, it is defined by the classical limit of the field expectation value in the presence of the external source. Since the classical limit corresponds to tree graphs in the loop or equivalently $\hbar$ expansion, this solution is the tree graph approximation to the vacuum expectation value of the field operator in the presence of the external source. The corresponding action functional, whose functional derivatives with respect to the source give this expectation value, is the free energy $W_{\mathrm{tree}}[\rho]$ and it is the generator of all possible connected tree graphs. If we expand the general quantum field $\phi(x)$ and the free energy $W[\rho]$ in powers of $\hbar$ as

$$\langle\phi(x)\rangle = \langle\phi(x)\rangle_{tree} + \hbar\langle\phi(x)\rangle^{(1)} + \frac{\hbar^2}{2}\langle\phi(x)\rangle^{(2)} + \cdots\,, \tag{2.7}$$

$$W[\rho] = W_{\rm tree}[\rho] + \hbar W[\rho]^{(1)} + \frac{\hbar^2}{2}W[\rho]^{(2)} + \cdots\,, \tag{2.8}$$

the lowest order terms in these expansions brings about the definition of the classical field,

$$\phi_{cl}(x)[\rho] = \langle\phi(x)\rangle_{\rm tree} = \frac{\delta W_{\rm tree}[\rho]}{\delta\rho(x)}\,. \tag{2.9}$$

The quantum effective action is the functional Legendre transform of the free energy $W[\rho]$,

$$\Gamma[\langle\phi\rangle] = \int d^4x\rho(x)\langle\phi(x)\rangle - W[\rho]\,, \tag{2.10}$$

and to one-loop order in its expansion in powers of $\hbar$, it is given by

$$\begin{aligned}\Gamma[\phi_{cl}] &= S[\phi_{cl}] + \frac{\hbar}{2}\ln\det\left(\frac{A}{\mu^2}\right) + O(\hbar^2)\\ &= S[\phi_{cl}] - \frac{\hbar}{2}\zeta'\left(0|\frac{A}{\mu^2}\right)\\ &= \int d^4x\left[V_{eff}(\phi_{cl}) + \frac{1}{2}Z_{eff}(\phi_{cl})\partial_\mu\phi_{cl}\partial_\mu\phi_{cl} + \cdots\right]\,,\end{aligned} \tag{2.11}$$

in which $A = -\partial^2 + m^2 + \lambda\phi_{cl}^2(x)/2$, μ is introduced for dimensional bookkeeping and here the replacement of $\langle\phi\rangle$ by ϕ_{cl} does not induce any error up to $O(\hbar)$. The restriction of the classical field $\phi_{cl}(x)$ to a constant field $\bar{\phi}$ gives the definition of the one-loop effective potential $V_{eff}(\bar{\phi})$ in terms of the classical potential $V_{cl}(\bar{\phi})$ and the lowest order quantum correction,

$$\begin{aligned}V_{eff}(\bar{\phi}) &= V_{cl}(\bar{\phi}) + \hbar V^{(1)}(\bar{\phi}) + O(\hbar^2)\\ &= V_{cl}(\bar{\phi}) + \frac{\hbar}{2\Omega}\ln\det\left(\frac{A}{\mu^2}\right)\,,\end{aligned} \tag{2.12}$$

Ω being the volume of spacetime.

Let's define the heat kernel of the operator A by $K(u|x,y) = e^{-uA}(x,y)$. It satisfies the heat equation,

$$\frac{\partial}{\partial u}K(u|x,y) + AK(u|x,y) = 0\,. \tag{2.13}$$

In four dimensions the heat kernel of the operator $-\partial^2$ is given by

$$K(u|x,y) = \frac{\mu^4}{16\pi^2 u^4}e^{-\mu^2(x-y)^2/4u}\,, \tag{2.14}$$

where u is dimensionless since μ, introduced earlier, has a mass dimension. For the arbitrary $\phi_{cl}(x)$, it is not easy to solve the heat equation. Since it is enough to consider constant field configuration to obtain the effective potential, this choice makes easier to solve the heat equation, and the integration of it yields the following heat kernel representation,

$$K(u|x,y) = \frac{\mu^4}{16\pi^2 u^2}e^{-\mu^2(x-y)^2/4u}e^{-(m^2+\lambda\bar{\phi}^2/2)u/\mu^2}\,. \tag{2.15}$$

So the zeta function of the operator is equal to

$$\begin{aligned}\zeta(s|A) &= \frac{1}{\Gamma(s)}\int_0^\infty du\, u^{s-1}\int d^4x \frac{\mu^4}{16\pi^2 u^2}e^{-(m^2+\lambda\bar{\phi}^2/2)u/\mu^2} \\ &= \frac{\mu^4}{16\pi^2}\left(\frac{m^2+\lambda\bar{\phi}^2/2}{\mu^2}\right)^{2-s}\frac{\Gamma(s-2)}{\Gamma(s)}\int d^4x\,. \end{aligned}\tag{2.16}$$

The one-loop correction to the classical potential, $V^{(1)}(\bar{\phi})$, is given by

$$\begin{aligned}V^{(1)}(\bar{\phi}) &= -\frac{1}{2\Omega}\frac{d}{ds}\zeta(0|A) \\ &= \frac{1}{64\pi^2}\left(m^2+\frac{\lambda}{2}\bar{\phi}^2\right)^2\left[-\frac{3}{2}+\ln\left(\frac{m^2+\lambda^2\bar{\phi}^2/2}{\mu^2}\right)\right]\,. \end{aligned}\tag{2.17}$$

Therefore, the one-loop effective potential in terms of the bare parameters of the theory takes the form

$$V_{eff}(\bar{\phi}) = \frac{1}{2}m^2\bar{\phi}^2 + \frac{\lambda}{4!}\bar{\phi}^4 + \frac{1}{64\pi^2}\left(m^2 + \frac{\lambda}{2}\bar{\phi}^2\right)^2 \times \left[-\frac{3}{2} + \ln\left(\frac{m^2 + \lambda^2\bar{\phi}^2/2}{\mu^2}\right)\right]. \quad (2.18)$$

2.2. Pseudodifferential Operators and Symbol Calculus

Pseudodifferential operators (ΨDO) or Calderon-Zygmund operators (CZO) [7, 18, 24, 25, 26, 27, 28] can be considered as a generalization of differential operators. Let us consider the differential operator,

$$A = \sum_{|\alpha|\leq m} A_\alpha D^\alpha \quad (\text{with } D^\alpha = \prod_{i=1}^{d}(-i\partial/\partial x_i)^{\alpha_i}) \quad \text{and} \quad |\alpha| = \sum_{i=1}^{d}\alpha_i). \quad (2.19)$$

They can equivalently be defined by their symbols,

$$\sigma(A)(x,p) = \sum_{i=0}^{m} a_{m-i}(x,p)\,, \quad a_{m-i} = \sum_{|\alpha|=m-i} A_\alpha(x)p^\alpha\,, \quad (2.20)$$

m being the order of the operator. Their actions on $\mathcal{L}^2(\mathbb{R}^d)$ functions can be given by

$$Au(x) = \int \frac{d^dp}{(2\pi)^d} e^{ix\cdot p}\sigma(A)(x,p)\hat{u}(p) = \int d^dy \frac{d^dp}{(2\pi)^d} e^{i(x-y)\cdot p}\sigma(A)(x,p)u(y)\,. \quad (2.21)$$

The pseudodifferential are defined by the generalization of the concept of symbols. Symbols are basically smooth matrix valued functions on the phase space $\mathbb{R}^d \oplus \mathbb{R}^d$ and can be viewed as a generalization of the characteristic polynomial. For a classical ΨDO (e.g. $\ln|p|$ is not a classical ΨDO), the symbol of it admits the following asymptotic expansion which is valid for $|p| \to \infty$,

$$\sigma(A) \sim \sum_{i=0}^{\infty} a_{m-i}(x,p)\,, \tag{2.22}$$

in which a_{m-i} is homogeneous of degree $m-i$ in p, and a_m term is its principal symbol. This series for $\sigma(A)$ does not need to converge. Classical ΨDO's are unique operators up to an addition of an infinitely smoothing operator since two different ΨDO's may have the same asymptotic expansion although any ΨDO has a unique asymptotic expansion. Infinitely smoothing operators are the ones whose symbols approach zero faster than inverse of any polynomial in $|p|$ as $|p| \to \infty$ (e.g. $e^{-|p|^2}$).

A differential operator is said to be elliptic, if the principal symbol of the operator satisfies the following condition,

$$a_m(x,p) \neq 0 \quad \text{for} \quad (x,p) \in X \times (\mathbb{R}^d \setminus 0)\,, \tag{2.23}$$

where X is an open set in $\mathbb{R}^d$. This definition can also be considered as the invertibility condition for the principal symbol $a_m(x,p)$ of the operator A for large $|p|$, constrained by the condition that there exists a constant C such that $|a(x,p)^{-1}| \leq C(1+|p|)^{-m}$, for $|p| \geq C$. Therefore, an elliptic ΨDO is the one with an elliptic symbol.

An important property of pseudodifferential operators is the notion known as pseudolocality [55]. But we at first need to define support of a function, $\text{supp}\{\cdot\}$. In analysis, the closure set of points in the domain of a real or complex valued function $u(x)$, where $u(x) \neq 0$, this is called the support of the function $u(x)$. What we know for differential operators is that they preserve supports, which means that the support of Pu is contained in the support of u,

$$\text{supp}\{Pu\} \subset \text{supp}\{u\}\,, \tag{2.24}$$

as long as P being a differential operator so that P is said to be local. However, pseudodifferential operators do not generally have this property unlike differential operators. Instead of locality, pseudodifferential operators possess pseudolocality mentioned at the beginning of this paragraph. It is simply the following property,

$$\text{sing supp}\{Pu\} \subset \text{sing supp}\{u\}\,, \tag{2.25}$$

in which sing supp$\{u\}$ stands for the subset of supp$\{u\}$ on which $u(x)$ is singular. With reference to distributional analysis, singular support can intuitively be thought as the set of points at which a distribution fails to be a function. Therefore, it can be stated that pseudodifferential operators preserve singular supports. There can be an interesting case in which sing supp$\{Pu\} =$ sing supp$\{u\}$. Such operators are called hypoelliptic. If sing supp$\{Pu\} = \varnothing$, then it follows that u is also smooth. Although every elliptic operator is hypoelliptic, the converse is not generally true. For example, the Laplace operator is both an elliptic and a hypoelliptic operator. In particular, the heat operator is an example of a hypoelliptic operator, although it is not an elliptic operator.

The action of a ΨDO A onto square integrable vector valued functions in $\mathbb{R}^d$ can be defined by its kernel,

$$Au(x) = \int dy A(x,y)u(y)\,. \tag{2.26}$$

The symbol, $\sigma(A)$, of a ΨDO A can be defined by the Fourier transform of its kernel in the distributional sense with respect to the relative coordinate, such that it is given by

$$\tilde{A}(x,p) = \int d^d\xi A\left(x+\frac{\xi}{2}, x-\frac{\xi}{2}\right) e^{-i\xi\cdot p}\,, \tag{2.27}$$

$$A(x,y) = \int \frac{d^dp}{(2\pi)^d}\tilde{A}\left(\frac{x+y}{2}, p\right) e^{ip\cdot(x-y)}\,, \tag{2.28}$$

$\tilde{A}(x,p)$ also being the symbol of the operator A, and from now on we will use $\sigma(A)$ and $\tilde{A}(x,p)$ interchangeably. In the definitions above, so as to set up a one to one correspondence between functions on the phase space and operators acting onto the Hilbert space $\mathcal{L}^2(\mathbb{R}^d)$, we use the Weyl ordering [24, 28, 56]. Since the symbols are the functions of the coordinates, x and p, the multiplication between the operators will induce a new multiplication rule between their symbols, preserving the multiplication rule. When the multiplication of operators on the Hilbert space is translated into the multiplication of the symbols, we find a noncommutative multiplication which can be given in closed form as

$$\tilde{A} \circ \tilde{B} = \left[e^{\frac{i\hbar}{2}\left(\frac{\partial}{\partial x^\mu} \frac{\partial}{\partial p'_\mu} - \frac{\partial}{\partial p_\mu} \frac{\partial}{\partial x'^\mu} \right)} \tilde{A}(x,p) \tilde{B}(x',p') \right]_{x=x' \, ; \, p=p'} . \tag{2.29}$$

Another way of computing this multiplication is to use so-called the generalized Poisson brackets with respect to the phase space coordinates x and p [56]. After expanding the exponential, one ends up with a series consisting of these brackets

$$\tilde{A} \circ \tilde{B} = \sum_{n=0}^{\infty} \left(\frac{i\hbar}{2} \right)^n \frac{1}{n!} \left\{ \tilde{A}, \tilde{B} \right\}_{(n)} , \tag{2.30}$$

where the generalized Poisson brackets are given by

$$\left\{ \tilde{A}, \tilde{B} \right\}_{(n)} = \sum_{i=0}^{n} (-1)^n \tilde{A}^{\nu_1 \cdots \nu_i}_{\mu_1 \cdots \mu_{n-i}} \tilde{B}^{\mu_1 \cdots \mu_{n-i}}_{\nu_1 \cdots \nu_i} , \tag{2.31}$$

in which $\tilde{A}^{\mu_i} = \frac{\partial \tilde{A}}{\partial p_{\mu_i}}$ and $\tilde{A}_{\mu_i} = \frac{\partial \tilde{A}}{\partial x^{\mu_i}}$.

It is, therefore, possible to do a semiclassical expansion with the assistance of this multiplication since after the leading order, which is actually the pointwise multiplication of the operators, the next orders give the desired corrections.

The trace of an operator basically transforms into a phase space integral

$$\mathrm{Tr} A = \int d^d x \frac{d^d p}{(2\pi)^d} \tilde{A}(x,p) . \tag{2.32}$$

If the operator under consideration has discrete indices, then one should also take another trace over these indices, as in the case of Dirac operators which will be discussed in Section 2.6. Therefore, the trace takes the form,

$$\mathrm{Tr}A = \int d^d x \frac{d^d p}{(2\pi)^d} \mathrm{tr}\tilde{A}(x,p)\,, \tag{2.33}$$

in which the trace under the integral sign stands for the ordinary matrix trace. A ΨDO is called a trace class operator if it is subjected to the condition that its order m is strictly less than $-d$.

2.3. The Holomorphic Semigroup and the Complex Powers of an Elliptic Operator

In general, in order to compute the regularized determinant of an operator, heat kernel method is used. However, for this method to work, the operator under consideration should be positive definite as in the example given in Section 2.1. If the operator is not positive definite, then $AA^\dagger$ or $A^\dagger A$ is used. However, in this case, there can be an extra term coming from the eta invariance or Seeley-De Witt integral coefficients [16, 57]. There is also another way of evaluating such a determinant independent of its positive definiteness. For the complex powers of operators, there is a very powerful prescription which is called the semigroup integral representation [7, 18].

Following [7, 18], we briefly give the definition of the holomorphic semigroup, firstly. Let's assume that the operator A is elliptic of order $m > 0$ with parameter on a Hilbert space with a common spectral cut $R_\theta = \left\{\rho e^{i\theta} : \rho \geq 0\right\}, \theta \in [0, 2\pi)$. The spectrum spec$(A)$ of the operator A consists of a discrete subset of $\mathbb{C}$ [18]. The ray $L_\theta = \{\lambda \in \mathbb{C} : \arg\lambda = \theta\}$ in the complex plane is called a ray of minimal growth(of the resolvent) if there is no eigenvalue of the principal symbol $a_m(x,p)$ lying on that ray [7]. Suppose that the operator is invertible so that there are no more than a finite number of eigenvalues of spec(A) in the neighborhood Λ of this ray. Then along this ray, the $\mathcal{L}^2$ operator norm of the resolvent $\|(A-\lambda)^{-1}\|$ is $O(1/|\lambda|)$ as $\lambda \to \infty$ [18]. For

$\Re(s) > 0$, the complex powers can be defined as,

$$A^{-s} = \frac{i}{2\pi} \int_{\Gamma} \lambda^{-s}(A - \lambda I)^{-1} d\lambda \,, \tag{2.34}$$

Γ being the contour which begins at ∞, then passes clockwise about the origin, and back to ∞, enclosing the spectrum of the operator. λ^{-s} is defined as a holomorphic function of λ for $\lambda \in \mathbb{C} \setminus [0, \infty)$. The branch of λ^{-s} is defined by the spectral cut R_θ,

$$\lambda^{-s} = |\lambda|^{-s} e^{-is \arg \lambda}, \quad \theta - 2\pi \leq \arg \lambda \leq \theta \,. \tag{2.35}$$

As in the definition, it is stipulated that $\text{spec}(A) \cap L_\theta$ is either empty or consists of the point 0 only. So we assume that $0 \notin \text{spec}(A)$, i.e. A is invertable as an operator so that A^{-1} exists as an operator. However, this assumption is not very essential and one can rid of this condition by replacing A with $A + i\varepsilon$. Moreover, if the ray L is chosen such as $L = [0, \infty)$, it is possible to deal with it by studying the operator $e^{i\varphi} A$ instead of A [18].

If $\Re(s) > 0$ and $\Re(r) > 0$, then one has the semigroup property

$$A^{-s} A^{-r} = A^{-(s+r)} \,. \tag{2.36}$$

For $k \in \mathbb{Z}_+$,

$$A^{-k} = \left(A^{-1}\right)^k \,, \tag{2.37}$$

and A^{-s} coincides with $A^s A^{-s-k}$ such that $\Re(s) < -k$, $-s - k$ belongs to the domain of the definition (2.34) and is independent of the choice of integer k. Thus, the holomorphic semigroup constitutes a natural extension of the group containing $A^0 = I$, $A^1 = A$ and A^{-1} to the one containing k-th power of the operator A.

2.4. Symbol of a Resolvent and Semiclassical Expansion for the ζ–Function of an Elliptic Operator

Since the semigroup integral representation depends on the resolvent of the operator, our first task should be to find the symbol of the resolvent. In order to do this, one can use the product rule so that the resolvent itself and the corrections to it can be calculated. The symbol of the inverse complex power of an operator A is given by

$$\sigma[A^{-s}] = \frac{\sin \pi s}{\pi} \int_0^\infty d\lambda\, \lambda^{-s} \sigma[\frac{1}{\lambda + A}]\,. \tag{2.38}$$

This definition is formal since it is thought that the symbol of the resolvent fulfills the convergence requirements. If we think of this definition from the point of the action of the operator A^{-s}, its kernel A_{-s} is continuous if $\Re(ms) < d$, and whose derivative is also continuous in any local coordinate system. In this way, for $x \neq y$, $A_s(x, y)$ extends to an entire function of s in any coordinate system. However, if $x = y$, then the kernel extends to a meromorphic function such that the singularity structure is determined by the simple poles occuring at the coincidence of the power s with the numbers $(d - n)/m$ with $n = 0, 1, \cdots$. Therefore for such circumstances, as long as a meromorphic extension of the symbol family $\sigma(z, s)$ to the whole complex plane with respect to complex parameter s can be found by means of a suitable analytic continuation of this symbol family in s, the contour integral makes sense. This is the case for the zeta function regularization since the trace of the kernel is needed.

The next step is now to find the symbol of the resolvent as a semiclassical expansion [7, 56]. If the resolvent symbol is defined as

$$\tilde{R}(\lambda) = \sigma\left[\frac{1}{\lambda + A}\right]\,, \tag{2.39}$$

then the symbol should satisfy

$$\tilde{R}(\lambda) \circ \sigma[\lambda + A] = 1\,, \tag{2.40}$$

where the resolvent symbol can be expanded in a power series in $\hbar$,

$$\tilde{R}(\lambda) = \sum_{n=0}^{\infty} \hbar^n \tilde{R}_{(n)}(\lambda)\,. \tag{2.41}$$

This expansion is actually a derivative expansion. Equation (2.40) can be rewritten in terms of the generalized Poisson brackets

$$\begin{aligned}
\tilde{R}(\lambda) \circ \left(\lambda + \tilde{A}\right) &= \sum_{n=0}^{\infty} \left(\frac{i\hbar}{2}\right)^n \frac{1}{n!} \left\{\tilde{R}(\lambda), \lambda + \tilde{A}\right\}_{(n)} \\
&= \tilde{R}(\lambda) \left(\lambda + \tilde{A}\right) + \left(\frac{i\hbar}{2}\right) \left\{\tilde{R}(\lambda), \lambda + \tilde{A}\right\}_{(1)} \\
&\quad + \left(\frac{i\hbar}{2}\right)^2 \frac{1}{2!} \left\{\tilde{R}(\lambda), \lambda + \tilde{A}\right\}_{(2)} \\
&\quad + \left(\frac{i\hbar}{2}\right)^3 \frac{1}{3!} \left\{\tilde{R}(\lambda), \lambda + \tilde{A}\right\}_{(3)} + \cdots ,
\end{aligned} \tag{2.42}$$

where $\tilde{A}$ is just the symbol of the operator A. Equation (2.40) results in a set of recursion relations after collecting the terms with the same order in $\hbar$ and we get

$$\hbar^0 : \tilde{R}_{(0)} = \frac{1}{\left(\lambda + \tilde{A}\right)}\,, \tag{2.43}$$

$$\hbar^1 : \tilde{R}_{(1)} = -\frac{i}{2} \left\{\tilde{R}_{(0)}(\lambda), \lambda + \tilde{A}\right\}_{(1)} \frac{1}{\left(\lambda + \tilde{A}\right)}\,, \tag{2.44}$$

$$\hbar^2 : \tilde{R}_{(2)} = -\frac{i}{2} \left\{\tilde{R}_{(1)}(\lambda), \lambda + \tilde{A}\right\}_{(1)} \frac{1}{\left(\lambda + \tilde{A}\right)} + \frac{1}{8} \left\{\tilde{R}_{(0)}(\lambda), \lambda + \tilde{A}\right\}_{(2)} \frac{1}{\left(\lambda + \tilde{A}\right)}\,, \tag{2.45}$$

$$\vdots : \qquad \vdots$$

By means of this expansion, one becomes ready to evaluate the symbol of any inverse complex power of the desired operator order by order in $\hbar$ as follows:

$$
\begin{aligned}
\sigma[A^{-s}] = {} & \frac{\sin \pi s}{\pi} \int_0^\infty d\lambda\, \lambda^{-s} \frac{1}{\lambda + \tilde{A}} \\
& - \frac{i\hbar}{2} \frac{\sin \pi s}{\pi} \int_0^\infty d\lambda\, \lambda^{-s} \left\{ \tilde{R}_{(0)}(\lambda), \lambda + \tilde{A} \right\}_{(1)} \frac{1}{\left(\lambda + \tilde{A}\right)} \\
& - \frac{i\hbar^2}{2} \frac{\sin \pi s}{\pi} \int_0^\infty d\lambda\, \lambda^{-s} \left\{ \tilde{R}_{(1)}(\lambda), \lambda + \tilde{A} \right\}_{(1)} \frac{1}{\left(\lambda + \tilde{A}\right)} \\
& + \frac{\hbar^2}{8} \frac{\sin \pi s}{\pi} \int_0^\infty d\lambda\, \lambda^{-s} \left\{ \tilde{R}_{(0)}(\lambda), \lambda + \tilde{A} \right\}_{(2)} \frac{1}{\left(\lambda + \tilde{A}\right)} \\
& + \cdots .
\end{aligned}
\tag{2.46}
$$

The next step is to take the phase space integral in order to find the zeta function of the operator,

$$
\zeta(s|A) = \int d^d x \frac{d^d p}{(2\pi)^d} \sigma[A^{-s}] . \tag{2.47}
$$

As mentioned at the beginning of this section, the regularized determinant of the operator is just minus the derivative of the zeta function with respect to the complex parameter s at $s = 0$; therefore, one finds

$$
\ln \det(A) = - \left.\frac{\partial}{\partial s}\right|_{s=0} \int d^d x \frac{d^d p}{(2\pi)^d} \sigma[A^{-s}] . \tag{2.48}
$$

2.5. The Determinant of a Bosonic Operator

For the bosonic case with zero spin, the operator which we would like to evaluate its zeta function is $A = -\partial^2 + V(x)$, where the term $V(x)$ may actually be a functional of the field $\phi(x)$. For example, in the massless ϕ^4–theory, the potential term is merely

$\lambda\phi^2(x)/2$. The first Poisson bracket is

$$\left\{\tilde{R}_{(0)}(\lambda), \lambda + \tilde{A}\right\}_{(1)} = \frac{\partial}{\partial x^\mu}\frac{1}{\left(\lambda + \tilde{A}\right)}\frac{\partial}{\partial p_\mu}\left(\lambda + \tilde{A}\right) - \frac{\partial}{\partial p_\mu}\frac{1}{\left(\lambda + \tilde{A}\right)}\frac{\partial}{\partial x^\mu}\left(\lambda + \tilde{A}\right)$$
$$= 0 \tag{2.49}$$

The first correction to the resolvent is, thus, zero,

$$\tilde{R}_{(1)}(\lambda) = 0\,. \tag{2.50}$$

The second generalized Poisson bracket is given by

$$\begin{aligned}\left\{\tilde{R}_{(0)}(\lambda), \lambda + \tilde{A}\right\}_{(2)} &= \frac{\partial^2}{\partial x^\mu \partial x^\nu}\frac{1}{\left(\lambda + \tilde{A}\right)}\frac{\partial^2}{\partial p_\mu \partial p_\nu}\left(\lambda + \tilde{A}\right) \\ &\quad - \frac{\partial^2}{\partial x^\mu \partial p_\nu}\frac{1}{\left(\lambda + \tilde{A}\right)}\frac{\partial^2}{\partial x^\nu \partial p_\mu}\left(\lambda + \tilde{A}\right) \\ &\quad + \frac{\partial^2}{\partial p_\mu \partial p_\nu}\frac{1}{\left(\lambda + \tilde{A}\right)}\frac{\partial^2}{\partial x^\mu \partial x^\nu}\left(\lambda + \tilde{A}\right) \\ &= \frac{4}{\left(\lambda + \tilde{A}\right)^3}\frac{\partial \tilde{V}}{\partial x^\mu}\frac{\partial \tilde{V}}{\partial x_\mu} + \frac{8p^\mu p^\nu}{\left(\lambda + \tilde{A}\right)^3}\frac{\partial^2 \tilde{V}}{\partial x^\mu \partial x^\nu} - \frac{4}{\left(\lambda + \tilde{A}\right)^2}\frac{\partial^2 \tilde{V}}{\partial x^\mu \partial x_\mu}\,.\end{aligned} \tag{2.51}$$

The first nonzero contribution is

$$\tilde{R}_{(2)}(\lambda) = -\frac{4}{\left(\lambda + \tilde{A}\right)^3}\frac{\partial^2 \tilde{V}}{\partial x^\mu \partial x_\mu} + \frac{4}{\left(\lambda + \tilde{A}\right)^4}\frac{\partial \tilde{V}}{\partial x^\mu}\frac{\partial \tilde{V}}{\partial x_\mu} + \frac{8p^\mu p^\nu}{\left(\lambda + \tilde{A}\right)^4}\frac{\partial^2 \tilde{V}}{\partial x^\mu \partial x^\nu}\,. \tag{2.52}$$

Therefore, the semiclassical expansion for the resolvent symbol can be given by

$$\tilde{R}(\lambda) = \frac{1}{\left(\lambda + \tilde{A}\right)} - \frac{1}{2\left(\lambda + \tilde{A}\right)^3}\frac{\partial^2 \tilde{V}}{\partial x^\mu \partial x_\mu} + \frac{1}{2\left(\lambda + \tilde{A}\right)^4}\frac{\partial \tilde{V}}{\partial x^\mu}\frac{\partial \tilde{V}}{\partial x_\mu} + \frac{p^\mu p^\nu}{\left(\lambda + \tilde{A}\right)^4}\frac{\partial^2 \tilde{V}}{\partial x^\mu \partial x^\nu} + \cdots\,, \tag{2.53}$$

in which $\hbar$ is set to 1. The next step is to evaluate the semigroup integrals of this resolvent symbol in order to find the symbol of any complex power of the operator as an expansion,

$$\sigma[A^{-s}] = \frac{\sin \pi s}{\pi} \int_0^\infty d\lambda\, \lambda^{-s} \left[\frac{1}{\left(\lambda + \tilde{A}\right)} - \frac{1}{2\left(\lambda + \tilde{A}\right)^3} \frac{\partial^2 \tilde{V}}{\partial x^\mu \partial x_\mu} + \frac{1}{2\left(\lambda + \tilde{A}\right)^4} \frac{\partial \tilde{V}}{\partial x^\mu} \frac{\partial \tilde{V}}{\partial x_\mu} + \frac{p^\mu p^\nu}{\left(\lambda + \tilde{A}\right)^4} \frac{\partial^2 \tilde{V}}{\partial x^\mu \partial x^\nu} \right] + \cdots . \tag{2.54}$$

All the desired integrals are standard residue integrals and they can be evaluated by means of the following semigroup integral formula,

$$\int_0^\infty d\lambda \frac{\lambda^{-s}}{\left(\lambda + \tilde{A}\right)^n} = \frac{\tilde{A}^{1-n-s}\Gamma(1-s)\Gamma(-1+n+s)}{\Gamma(n)}, \tag{2.55}$$

if $\Re(\tilde{A} > 0)$, $\Re(n+s) > 1$, $\pi + (-1+s)\arg(1/\tilde{A}) \geq 0$ and $\Re(s) < 1$. After evaluating these integrals, one ends up with

$$\sigma[A^{-s}] = \frac{1}{\tilde{A}^s} - \frac{s(s+1)}{4\tilde{A}^{s+2}} \frac{\partial^2 \tilde{V}}{\partial x^\mu \partial x_\mu} + \frac{s(s+1)(s+2)}{6\tilde{A}^{s+3}} \left[p^\mu p^\nu \frac{\partial^2 \tilde{V}}{\partial x^\mu \partial x^\nu} + \frac{1}{2} \frac{\partial \tilde{V}}{\partial x^\mu} \frac{\partial \tilde{V}}{\partial x_\mu} \right] + \cdots . \tag{2.56}$$

Since we found the symbol of the inverse complex power of the operator, taking the trace of the symbol in phase space is left. We take the momentum integral first,

$$\int \frac{d^4p}{(2\pi)^4} \sigma\left[A^{-s}\right] = \int \frac{d^4p}{(2\pi)^4} \frac{1}{\left(p^2 + \tilde{V}\right)^s} - \frac{s(s+1)}{4} \frac{\partial^2 \tilde{V}}{\partial x^\mu \partial x_\mu} \int \frac{d^4p}{(2\pi)^4} \frac{1}{\left(p^2 + \tilde{V}\right)^{s+2}} + \frac{s(s+1)(s+2)}{6} \frac{\partial^2 \tilde{V}}{\partial x^\mu \partial x^\nu} \int \frac{d^4p}{(2\pi)^4} \frac{p^\mu p^\nu}{\left(p^2 + \tilde{V}\right)^{s+3}} + \frac{s(s+1)(s+2)}{6} \frac{1}{2} \frac{\partial \tilde{V}}{\partial x^\mu} \frac{\partial \tilde{V}}{\partial x_\mu} \int \frac{d^4p}{(2\pi)^4} \frac{1}{\left(p^2 + \tilde{V}\right)^{s+3}} + \cdots . \tag{2.57}$$

The momentum integrals in Eq. 2.57 can be calculated by the following d dimensional Euclidean integration formulae,

$$\int \frac{d^dp}{(2\pi)^d}\frac{p^{2m}}{(p^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(n-m-d/2)\Gamma(m+d/2)}{\Gamma(d/2)\Gamma(n)}\Delta^{d/2+m-n}\,, \tag{2.58}$$

$$\int \frac{d^dp}{(2\pi)^d}\frac{p_{\mu_1}\cdots p_{\mu_{2m}}}{(p^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(n-m-d/2)}{\Gamma(n)2^m}\delta_{\mu_1\cdots\mu_{2m}}\Delta^{d/2+m-n}\,, \tag{2.59}$$

in which the generalised Krocneker delta is defined as

$$\delta_{\mu_1\mu_2\cdots\mu_{n-1}\mu_n} = \delta_{\mu_1\mu_2}\cdots\delta_{\mu_{n-1}\mu_n} + (\text{permutations})\,. \tag{2.60}$$

Taking the momentum integral results in

$$\int \frac{d^4p}{(2\pi)^4}\sigma\left[A^{-s}\right] = \frac{1}{16\pi^2}\left[\frac{\tilde{V}^{2-s}}{(s-1)(s-2)} - \frac{\tilde{V}^{-s}}{6}\frac{\partial^2\tilde{V}}{\partial x^\mu\partial x_\mu} + \frac{s\tilde{V}^{-s-1}}{12}\frac{\partial\tilde{V}}{\partial x^\mu}\frac{\partial\tilde{V}}{\partial x_\mu}\right] + \cdots\,. \tag{2.61}$$

Thus, the zeta function of the operator A in ordinary space-time is just the x-integral of the equation above,

$$\zeta(s|A) = \frac{1}{16\pi^2}\int d^4x\left[\frac{V^{2-s}}{(s-1)(s-2)} - \frac{V^{-s}}{6}\frac{\partial^2 V}{\partial x^\mu\partial x_\mu} + \frac{sV^{-s-1}}{12}\frac{\partial V}{\partial x^\mu}\frac{\partial V}{\partial x_\mu}\right] + \cdots\,. \tag{2.62}$$

The determinant of the operator is given by

$$\ln\det\left[-\partial^2+V\right] = -\zeta'\left(0|-\partial^2+V\right)\,, \tag{2.63}$$

and the derivative of the zeta function with respect to s is

$$\begin{aligned}\frac{\partial}{\partial s}\zeta(s) = \frac{1}{16\pi^2}\int d^4x\Bigg[&-\frac{V^{2-s}}{(s-1)^2(s-2)} - \frac{V^{2-s}}{(s-1)(s-2)^2} - \frac{V^{2-s}\ln V}{(s-1)(s-2)}\\ &+\frac{V^{-s}\ln V}{6}\frac{\partial^2 V}{\partial x^\mu\partial x_\mu} + \frac{V^{-s-1}}{12}\frac{\partial V}{\partial x^\mu}\frac{\partial V}{\partial x_\mu} - \frac{sV^{-s-1}\ln V}{12}\frac{\partial V}{\partial x^\mu}\frac{\partial V}{\partial x_\mu}\Bigg] + \cdots\,.\end{aligned} \tag{2.64}$$

The zeta-regularized determinant of the operator $-\partial^2 + V$ is, thus,

$$\ln\det\left[-\partial^2 + V\right] = \frac{1}{32\pi^2}\int d^4x \left[V^2 \ln\left(e^{-3/2}\frac{V}{\mu^{[V]}}\right) + \frac{1}{6V}\frac{\partial V}{\partial x^\mu}\frac{\partial V}{\partial x_\mu}\right] + \cdots\,, \tag{2.65}$$

where the scale μ is introduced for dimensional bookkeeping of the logarithm and $[V]$ is the mass dimension of the potential. This result agrees with the ones in the literature [11, 58, 59]. Replacing $V(x)$ with $\lambda\phi^2(x)/2$, which is the potential for the massless ϕ^4 theory as it is said at the beginning of this section, the potential part of Eq. (2.65) yields

$$\frac{1}{32\pi^2}\int d^4x \frac{\lambda^2\phi^4}{4}\ln\left(e^{-3/2}\frac{\lambda\phi^2}{2\mu^2}\right)\,. \tag{2.66}$$

This is in agreement with the well known unrenormalized results which can be found in [4, 60].

2.6. The Determinant of the Dirac Operator

In this section, we will consider a Dirac operator which is massless and contains a scalar field, $D = \gamma\cdot\partial + \phi$, where $\cdot$ stands for the $4d$ Euclidean inner product and the gamma matrices are chosen to be hermitian. As we have done for the scalar determinant, our first task is to evaluate the resolvent symbol. For the Dirac operator under consideration, the symbol of the operator is

$$\tilde{A} = i\gamma\cdot p + \tilde{\phi}\,. \tag{2.67}$$

After using the same expansion in Eq. (2.42) with the symbol (2.67), Eqs. (2.43) and (2.44) give, respectively, one can obtain the zeroth and the first order resolvent symbols easily. whereas the zeroth order resolvent symbol is equal to

$$\tilde{R}_{(0)}(\lambda) = \frac{1}{\left(\lambda + \tilde{A}\right)}\,, \tag{2.68}$$

the first order resolvent is given by

$$\tilde{R}_{(1)}(\lambda) = -\frac{i}{2}\frac{\partial\tilde{\phi}}{\partial x^{\mu}}\left[-\frac{1}{\left(\lambda+\tilde{A}\right)^2}i\gamma^{\mu} + \frac{1}{\left(\lambda+\tilde{A}\right)}i\gamma^{\mu}\frac{1}{\left(\lambda+\tilde{A}\right)}\right]\frac{1}{\left(\lambda+\tilde{A}\right)}\,. \tag{2.69}$$

However, when the Dirac operator is considered, there is another trace which is over the spinor indices and if one takes it into account, it can be easily seen that the term in $\hbar$ is zero due to the cyclicity of the trace,

$$\mathrm{Tr}\tilde{R}_{(1)}(\lambda) = 0\,. \tag{2.70}$$

As in the scalar case, there is not any term which contains just one derivative of the field for the Dirac operator. The next term, which is an $\hbar^2$-order term, contains the second derivative of the scalar field and is just given by the second generalized Poisson bracket due to the fact that $\tilde{R}_{(1)}(\lambda)$ is zero,

$$\begin{aligned}\tilde{R}_{(2)}(\lambda) &= \frac{1}{8}\left\{\tilde{R}_{(0)}(\lambda), \lambda+\tilde{A}\right\}_{(2)}\frac{1}{\left(\lambda+\tilde{A}\right)} \\ &= \frac{1}{8}\left[\frac{\partial^2}{\partial p_{\mu}\partial p_{\nu}}\frac{1}{\left(\lambda+\tilde{A}\right)}\frac{\partial^2}{\partial x^{\mu}\partial x^{\nu}}\left(\lambda+\tilde{A}\right)\right]\frac{1}{\left(\lambda+\tilde{A}\right)}\,.\end{aligned} \tag{2.71}$$

This is the only nonvanishing term in the Poisson bracket, and it is equal to

$$\begin{aligned}&\frac{1}{8}\partial_{\mu\nu}\tilde{\phi}\left[\frac{1}{\left(\lambda+\tilde{A}\right)}i\gamma^{\mu}\frac{1}{\left(\lambda+\tilde{A}\right)}i\gamma^{\nu}\frac{1}{\left(\lambda+\tilde{A}\right)}\right. \\ &\qquad\left.+\frac{1}{\left(\lambda+\tilde{A}\right)}i\gamma^{\nu}\frac{1}{\left(\lambda+\tilde{A}\right)}i\gamma^{\mu}\frac{1}{\left(\lambda+\tilde{A}\right)}\right]\frac{1}{\left(\lambda+\tilde{A}\right)}\,.\end{aligned} \tag{2.72}$$

For the whole expression is multiplied by the second derivative of the field $\tilde{\phi}$, and symmetric in μ and ν, these terms simplify more. By using the cyclicity of the trace, one can rearrange the fractions. Being symmetric in μ and ν allows us to relabel these indices such that the first and second line in Eq. (2.72) becomes equal to each other.

The first nonzero contribution coming from the semiclassical expansion is, therefore, given by

$$\tilde{R}_{(2)}(\lambda) = -\frac{1}{4}\frac{1}{\left(\lambda+\tilde{A}\right)^3}\gamma^{\mu}\frac{1}{\left(\lambda+\tilde{A}\right)}\gamma^{\nu}\partial_{\mu\nu}\tilde{\phi}\,. \tag{2.73}$$

Thus, the resolvent symbol up to $\hbar^2$-order is given by

$$\tilde{R}(\lambda) = \frac{1}{\left(\lambda+\tilde{A}\right)} - \frac{1}{4}\partial_{\mu\nu}\tilde{\phi}\frac{1}{\left(\lambda+\tilde{A}\right)^3}\gamma^{\mu}\frac{1}{\left(\lambda+\tilde{A}\right)}\gamma^{\nu} + \cdots\,. \tag{2.74}$$

What should be done afterward is to take the residue integrals in order to find the symbol of the inverse complex power of our operator,

$$\begin{aligned}\sigma[A^{-s}] &= \frac{\sin\pi s}{\pi}\int_0^{\infty} d\lambda\,\lambda^{-s}\left[\frac{1}{\left(\lambda+\tilde{A}\right)} - \frac{1}{4}\partial_{\mu\nu}\tilde{\phi}\frac{1}{\left(\lambda+\tilde{A}\right)^3}\gamma^{\mu}\frac{1}{\left(\lambda+\tilde{A}\right)}\gamma^{\nu}\right] + \cdots \\ &= \frac{1}{\tilde{A}^s} - \frac{\sin\pi s}{\pi}\int_0^{\infty} d\lambda\,\lambda^{-s}\frac{1}{4}\partial_{\mu\nu}\tilde{\phi}\frac{1}{\left(\lambda+\tilde{A}\right)^3}\gamma^{\mu}\frac{1}{\left(\lambda+\tilde{A}\right)}\gamma^{\nu} + \cdots\,. \end{aligned} \tag{2.75}$$

So as to succeed in taking the second integral, we should reorganize the order of the gamma matrices and the fractions such that the gamma matrices and the fractions should be separated. This can be achieved by means of passing one of the gamma matrices over the fraction between them. By expanding the fraction as,

$$\frac{1}{\lambda+\tilde{\phi}+i\gamma\cdot p} = \frac{1}{\lambda+\tilde{\phi}}\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(1+n)}{n!\Gamma(1)}\left(\frac{i\gamma\cdot p}{\lambda+\tilde{\phi}}\right)^n\,, \tag{2.76}$$

and using an analytical continuation argument, one can easily show that

$$\gamma^{\mu}\frac{1}{\lambda+\tilde{A}}\gamma^{\nu} = \frac{1}{\lambda+\tilde{A}^*}\gamma^{\mu}\gamma^{\nu} + \frac{1}{\lambda+\tilde{A}}\frac{p^{\mu}}{p^2}p\cdot\gamma\gamma^{\nu} - \frac{1}{\lambda+\tilde{A}^*}\frac{p^{\mu}}{p^2}p\cdot\gamma\gamma^{\nu}\,, \tag{2.77}$$

in which $\tilde{A}^* = \tilde{\phi} - i\gamma\cdot p$. After placing Eq. (2.77) into Eq. (2.75) and doing little algebra, we get

$$\sigma[A^{-s}] = \frac{1}{\tilde{A}^s} - \frac{\sin \pi s}{\pi} \int_0^\infty d\lambda\, \lambda^{-s} \frac{1}{4} \frac{1}{\left(\lambda + \tilde{A}\right)^3} \frac{1}{\left(\lambda + \tilde{A}^*\right)} \gamma^\mu \gamma^\nu \partial_{\mu\nu} \tilde{\phi}$$
$$- \frac{\sin \pi s}{\pi} \int_0^\infty d\lambda\, \lambda^{-s} \frac{1}{4} \frac{1}{\left(\lambda + \tilde{A}\right)^4} \frac{p^\mu}{p^2} p \cdot \gamma \gamma^\nu \partial_{\mu\nu} \tilde{\phi}$$
$$+ \frac{\sin \pi s}{\pi} \int_0^\infty d\lambda\, \lambda^{-s} \frac{1}{4} \frac{1}{\left(\lambda + \tilde{A}\right)^3} \frac{1}{\left(\lambda + \tilde{A}^*\right)} \frac{p^\mu}{p^2} p \cdot \gamma \gamma^\nu \partial_{\mu\nu} \tilde{\phi} + \cdots . \quad (2.78)$$

We start with the first term: the trace, both over continuous indices, x and p, and over spinor indices, is going to be taken. In order to do this, it is a good idea to take the momentum integral in d space-time first and then take the limit as $d \to 4$,

$$\mathrm{tr} \int \frac{d^d p}{(2\pi)^d} \frac{1}{\left(\tilde{\phi} + ip \cdot \gamma\right)^s} = \mathrm{tr} \int \frac{d^d p}{(2\pi)^d} \phi^{-s} \sum_{n=0}^\infty (-1)^n \frac{\Gamma(s+n)}{n! \Gamma(s)} \left(\frac{ip \cdot \gamma}{\tilde{\phi}}\right)^n$$
$$= \frac{2\tilde{\phi}^{-s}}{(2\pi)^{d/2} \Gamma(d/2)} \int_0^\infty dp p^{d-1} {}_2F_1\left(\frac{s}{2}, \frac{s+1}{2}; \frac{1}{2}; -\frac{p^2}{\tilde{\phi}^2}\right)$$
$$= \frac{2^{d/2} \pi^{1/2 - d/2} \tilde{\phi}^{-s} \left(\tilde{\phi}^2\right)^{d/2} \Gamma(s-d)}{\Gamma(1/2 - d/2) \Gamma(s)}, \quad \left|\arg(1/\tilde{\phi}^2)\right| < \pi,$$
$$0 < \Re(d/2) < \Re(s/2) < \Re(s/2 + 1/2), \quad (2.79)$$

where the little trace stands for the one over the spinor indices and the fact that only even number of gamma matrices is nonvanishing is used. The result of integral in the third line is given by Marichev *et al.* [61]. As $d \to 4$, the expression above becomes

$$\mathrm{tr} \int \frac{d^4 p}{(2\pi)^4} \frac{1}{\left(\tilde{\phi} + ip \cdot \gamma\right)^s} = \frac{3\tilde{\phi}^{4-s} \Gamma(s-4)}{\pi^2 \Gamma(s)}. \quad (2.80)$$

If one evaluates minus the derivative of this momentum integral with respect to s at $s = 0$, then the zeroth order term will be obtained and this is equal to

$$\frac{1}{16\pi^2} \int d^4 x \phi^4 \ln\left(\frac{\phi^2}{\mu^2} e^{-25/6}\right), \quad (2.81)$$

where the scale μ is again introduced for dimensional bookkeeping of the logarithm. This agrees with a previous result [33].

The third term in Eq. (2.78) can also easily be calculated in the same manner

$$
\begin{aligned}
-\frac{1}{4}\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\frac{\sin\pi s}{\pi}\int_0^\infty d\lambda\,\lambda^{-s}\frac{1}{\left(\lambda+\tilde{A}\right)^4}\frac{p^\mu}{p^2}p\cdot\gamma\gamma^\nu\partial_{\mu\nu}\tilde{\phi}
&= -\frac{1}{4!}\partial_{\mu\nu}\frac{\Gamma(s+3)}{\Gamma(s)}\int\frac{d^4p}{(2\pi)^4}\frac{p^\mu}{p^2}\mathrm{tr}\frac{1}{\tilde{A}^{s+3}}p\cdot\gamma\gamma^\mu \\
&= -\frac{1}{4!}\frac{\Gamma(s+3)}{\Gamma(s)}\lim_{d\to 4}\partial_{\mu\nu}\int\frac{d^dp}{(2\pi)^d}\frac{p^\mu}{p^2}\mathrm{tr}\frac{1}{\tilde{A}^{s+3}}p\cdot\gamma\gamma^\mu \\
= -\frac{1}{4!}\frac{\Gamma(s+3)}{\Gamma(s)}\lim_{d\to 4}\partial_{\mu\nu}\frac{2g^{\mu\nu}}{\Gamma(d/2)d}\frac{\tilde{\phi}^{-s-3}}{(2\pi)^{d/2}}&\int_0^\infty dp\,p^{d-1}{}_2F_1\left(\frac{s+3}{2},\frac{s+4}{2};\frac{1}{2};-\frac{p^2}{\tilde{\phi}^2}\right) \\
&= -\frac{1}{4!}\frac{\Gamma(1/2)}{\Gamma(s)}\lim_{d\to 4}\partial_{\mu\nu}\frac{\tilde{\phi}^{-s-3}}{d(2\pi)^{d/2}}\frac{\Gamma(s+3-d)}{\Gamma(1/2-d/2)}\left(\tilde{\phi}^2\right)^{d/2}2^dg^{\mu\nu}\,, \quad (2.82)
\end{aligned}
$$

if $0<\Re(d/2)<\Re(s/2+3/2)\,,\Re(s/2+2)$ and $|\arg(1/\tilde{\phi}^2)|<\pi$. So one can get

$$
-\frac{1}{4}\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\frac{\sin\pi s}{\pi}\int_0^\infty d\lambda\,\lambda^{-s}\frac{1}{\left(\lambda+\tilde{A}\right)^4}\frac{p^\mu}{p^2}p\cdot\gamma\gamma^\nu\partial_{\mu\nu}\tilde{\phi} = -\frac{1}{32\pi^2}\frac{\tilde{\phi}^{1-s}}{s-1}\partial^2\tilde{\phi}\,. \quad (2.83)
$$

However, one should be cautious for the second and the fourth terms due to the fact that these terms contain the product of two operators noncommuting with each other. Therefore, a method must be suggested to take the residue integrals properly.

The first one is to use the Feynman parametrization so as to convert this multiplication of inverse powers into a single inverse power,

$$
\begin{aligned}
\frac{1}{\left(\lambda+\tilde{A}\right)^3}\frac{1}{\left(\lambda+\tilde{A}^*\right)} &= \mathcal{FP}\int_0^1 dt\frac{3t^2}{\left[t(\lambda+\tilde{A})+(1-t)(\lambda+\tilde{A}^*)\right]^4} \\
&= \mathcal{FP}\int_0^1 dt\frac{3t^2}{\left[\lambda+\tilde{\phi}+(2t-1)ip\cdot\gamma\right]^4}\,, \quad (2.84)
\end{aligned}
$$

in which the above integral is given as a Hadamard finite part integral [62, 63, 64] since an extraneous singularity is introduced while this parametrization is being utilized.

Thus, the second term in Eq. (2.78) is

$$
\begin{aligned}
&-\frac{1}{4}\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\frac{\sin\pi s}{\pi}\int_0^\infty d\lambda\,\lambda^{-s}\frac{1}{(\lambda+\tilde{A})^3(\lambda+\tilde{A}^*)}\gamma^\mu\gamma^\nu\partial_{\mu\nu}\tilde{\phi} \\
&= -\frac{3}{4}\partial^2\tilde{\phi}\mathcal{FP}\int_0^1 dt t^2\,\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\frac{\sin\pi s}{\pi}\int_0^\infty d\lambda\,\lambda^{-s}\frac{1}{(\lambda+\tilde{\phi}+(2t-1)ip\cdot\gamma)^4}\,, \qquad (2.85)
\end{aligned}
$$

where there is just one inverse power in the residue integral. After the residue integral, the expression above becomes

$$
-\frac{1}{8}\partial^2\tilde{\phi}\mathcal{FP}\int_0^1 dt t^2\frac{\Gamma(s+3)}{\Gamma(s)}\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\frac{1}{\left(\tilde{\phi}+(2t-1)ip\cdot\gamma\right)^{s+3}}\,. \qquad (2.86)
$$

The momentum integral in d dimensions is given by

$$
\begin{aligned}
\mathrm{tr}\int\frac{d^dp}{(2\pi)^d}\frac{1}{\left(\tilde{\phi}+it'p\cdot\gamma\right)^{s+3}} &= \frac{2\tilde{\phi}^{-s-3}}{(2\pi)^{d/2}\Gamma(d/2)}\int_0^\infty dp\,p^{d-1}{}_2F_1\left(\frac{s+3}{2},\frac{s+4}{2};\frac{1}{2};-\frac{t'^2}{\tilde{\phi}^2}p^2\right) \\
&= \frac{2^{d/2}\pi^{1/2-d/2}\tilde{\phi}^{-s-3}\left(\tilde{\phi}^2\right)^{d/2}\Gamma(s+3-d)}{(t'^2)^{d/2}\Gamma(1/2-d/2)\Gamma(s+3)}\,, \qquad (2.87)
\end{aligned}
$$

where $t' = 2t-1$, $0 < \Re(d/2) < \Re(s/2+3/2)\,,\Re(s/2+2)$ and $|\arg(1/\tilde{\phi}^2)| < \pi$. Equation (2.86) is, then, equal to

$$
-\frac{3}{8\pi^2}\partial^2\tilde{\phi}\frac{\tilde{\phi}}{s-1}\mathcal{FP}\int_0^1 dt\frac{t^2}{(2t-1)^4}\,. \qquad (2.88)
$$

Since the Hadamard finite part of the integral is $-\frac{1}{3}$, the second term in the expansion is

$$
-\frac{1}{4}\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\frac{\sin\pi s}{\pi}\int_0^\infty d\lambda\,\lambda^{-s}\frac{1}{(\lambda+\tilde{A})^3(\lambda+\tilde{A}^*)}\gamma^\mu\gamma^\nu\partial_{\mu\nu}\tilde{\phi} = \frac{1}{8\pi^2}\frac{\tilde{\phi}^{1-s}}{s-1}\partial^2\tilde{\phi}\,. \qquad (2.89)
$$

The fourth term in Eq. (2.78) can be dealt with in the same manner. This term contains also a multiplication of inverse powers, and they can be united by means of a Feynman parametrization. After using a Feynman parametrization and taking the resultant residue integral, this term becomes

$$\frac{\partial_{\mu\nu}\tilde{\phi}\Gamma(s+3)}{8\Gamma(s)}\mathcal{FP}\int_0^1 dt\, t^2 \int \frac{d^4}{(2\pi)^4}\frac{p^\mu}{p^2}\mathrm{tr}\frac{1}{\left(\tilde{\phi}+(2t-1)ip\cdot\gamma\right)^{s+3}}p\cdot\gamma\gamma^\nu\,. \tag{2.90}$$

The momentum integral can be calculated as a d dimensional integral, and it is equal to

$$\begin{aligned}
&\int \frac{d^dp}{(2\pi)^d}\frac{p^\mu}{p^2}\mathrm{tr}\frac{1}{\left(\tilde{\phi}+(2t-1)ip\cdot\gamma\right)^{s+3}}p\cdot\gamma\gamma^\nu \\
&= \frac{2g^{\mu\nu}\tilde{\phi}^{-s-3}}{d(2\pi)^{d/2}\Gamma(d/2)}\int_0^\infty dp\, p^{d-1}{}_2F_1\left(\frac{s+3}{2},\frac{s+4}{2};\frac{1}{2};-\frac{t'^2}{\tilde{\phi}^2}p^2\right) \\
&= \frac{g^{\mu\nu}2^{d/2}\pi^{1/2-d/2}\tilde{\phi}^{-s-3}\left(\tilde{\phi}^2\right)^{d/2}\Gamma(s+3-d)}{d\Gamma(1/2-d/2)\Gamma(s+3)\left((2t-1)^2\right)^{d/2}}\,,
\end{aligned} \tag{2.91}$$

if $0<\Re(d/2)<\Re(s/2+3/2)\,,\Re(s/2+2)$ and $|\arg(1/\tilde{\phi}^2)|<\pi$. Taking the limit $d\to 4$ and evaluating the Feynman parametrized integral give rise to the following result,

$$\frac{1}{4}\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\frac{\sin\pi s}{\pi}\int_0^\infty d\lambda\,\lambda^{-s}\frac{1}{\left(\lambda+\tilde{A}\right)^3}\frac{1}{\left(\lambda+\tilde{A}^*\right)}\frac{p^\mu}{p^2}p\cdot\gamma\gamma^\nu\partial_{\mu\nu}\tilde{\phi} = -\frac{1}{32\pi^2}\frac{\tilde{\phi}^{1-s}}{s-1}\partial^2\tilde{\phi}\,. \tag{2.92}$$

The zeta function of the operator up to $\hbar^2$-order is, then,

$$\zeta\left(s|\gamma\cdot\partial+\phi\right) = \frac{1}{\pi^2}\int d^4x\left[\frac{3\tilde{\phi}^{4-s}}{(s-1)(s-2)(s-3)(s-4)}+\frac{1}{16}\frac{\tilde{\phi}^{1-s}}{s-1}\partial^2\tilde{\phi}\right]+\cdots\,. \tag{2.93}$$

The regularized determinant is equal to minus times the first derivative of the above zeta function with respect to its parameter s, evaluated at zero. The determinant of the Dirac operator with a scalar field ϕ up to order $\hbar^2$ can, therefore, be given by

$$\ln\det(\gamma\cdot\partial+\phi) = \frac{1}{16\pi^2}\int d^4x\left[\phi^4\ln\left(\frac{\phi^2}{\mu^2}e^{-25/6}\right)+\ln\left(\frac{\phi^2}{\mu^2}\right)\frac{1}{2}\partial_\mu\phi\partial^\mu\phi\right]+\cdots\,. \tag{2.94}$$

The alternative way of attacking the second and the fourth terms in Eq. (2.78) is to introduce an operator identity in order to get rid off the multiplication of the resolvents in the residue integrals such that all residue integrals contain just operators commuting with each other with arbitrary powers of them. It turns out that the same result which has been found with the assistance of the Feynman parametrization will be obtained as it should be. This is possible with the following operator identity:

$$\frac{1}{(\lambda+\tilde{A})^3}\frac{1}{(\lambda+\tilde{A}^*)} = \frac{1}{8}\frac{1}{(\lambda+\tilde{\phi})^3}\frac{1}{(\lambda+\tilde{A})} + \frac{1}{8}\frac{1}{(\lambda+\tilde{\phi})^3}\frac{1}{(\lambda+\tilde{A}^*)} + \frac{1}{4}\frac{1}{(\lambda+\tilde{\phi})^2}\frac{1}{(\lambda+\tilde{A})^2} + \frac{1}{2}\frac{1}{(\lambda+\tilde{\phi})}\frac{1}{(\lambda+\tilde{A})^3}\,. \tag{2.95}$$

However, if one pays more attention to the expression above, it is easy to notice that all the terms have a factor which can be located on the cut. On account of that, a complex parameter is supposed to be introduced to the symbol of the scalar field $\tilde{\phi}$, so it becomes $\tilde{\phi}+i\varepsilon$, where ε is positive but small and will eventually be made to approach zero, that is,

$$\begin{aligned}
\tilde{C} &= \tilde{\phi}+ip\cdot\gamma+i\varepsilon \quad , \quad \tilde{A} = \lim_{\varepsilon\to 0^+}\tilde{C}\,,\\
\tilde{C}' &= \tilde{\phi}-ip\cdot\gamma+i\varepsilon \quad , \quad \tilde{A}^* = \lim_{\varepsilon\to 0^+}\,,\\
\tilde{B} &= \tilde{\phi}+i\varepsilon \quad , \quad \tilde{\phi} = \lim_{\varepsilon\to 0^+}\,.
\end{aligned} \tag{2.96}$$

By means of these new operators with the operator identity already given, the following integral can be given as

$$\begin{aligned}
&\frac{\sin\pi s}{\pi}\int_0^\infty d\lambda\lambda^{-s}\frac{1}{(\lambda+\tilde{A})^3}\frac{1}{(\lambda+\tilde{A}^*)}\\
&= \lim_{\varepsilon\to 0^+}\frac{\sin\pi s}{\pi}\int_0^\infty d\lambda\lambda^{-s}\frac{1}{(\lambda+\tilde{C})^3}\frac{1}{(\lambda+\tilde{C}')}\\
&= \lim_{\varepsilon\to 0^+}\frac{\sin\pi s}{\pi}\left[\int_0^\infty d\lambda\lambda^{-s}\frac{1}{8}\frac{1}{(\lambda+\tilde{B})^3}\frac{1}{(\lambda+\tilde{C})} + \int_0^\infty d\lambda\lambda^{-s}\frac{1}{8}\frac{1}{(\lambda+\tilde{B})^3}\frac{1}{(\lambda+\tilde{C}')}\right.\\
&\quad\left. + \int_0^\infty d\lambda\lambda^{-s}\frac{1}{4}\frac{1}{(\lambda+\tilde{B})^2}\frac{1}{(\lambda+\tilde{C})^2} + \int_0^\infty d\lambda\lambda^{-s}\frac{1}{2}\frac{1}{(\lambda+\tilde{B})}\frac{1}{(\lambda+\tilde{C})^3}\right]\,.
\end{aligned} \tag{2.97}$$

After taking these residue integrals with the ε small limit afterward, the above integral is given by a sum of four integrals. We list them below,

$$\frac{\sin \pi s}{\pi} \int_0^\infty d\lambda\, \lambda^{-s} \frac{1}{(\lambda + \tilde{\phi})^3} \frac{1}{(\lambda + \tilde{A})} = -\frac{1}{(ip\cdot\gamma)^3} \tilde{A}^{-s} + \frac{1}{(ip\cdot\gamma)^3} \tilde{\phi}^{-s} - \frac{s}{(ip\cdot\gamma)^2} \tilde{\phi}^{-s-1} + \frac{s(s+1)}{2(ip\cdot\gamma)} \tilde{\phi}^{-s-2}, \tag{2.98}$$

$$\frac{\sin \pi s}{\pi} \int_0^\infty d\lambda\, \lambda^{-s} \frac{1}{(\lambda + \tilde{\phi})^3} \frac{1}{(\lambda + \tilde{A}^*)} = \frac{1}{(ip\cdot\gamma)^3} \tilde{A}^{*-s} - \frac{1}{(ip\cdot\gamma)^3} \tilde{\phi}^{-s} - \frac{s}{(ip\cdot\gamma)^2} \tilde{\phi}^{-s-1} - \frac{s(s+1)}{2(ip\cdot\gamma)} \tilde{\phi}^{-s-2}, \tag{2.99}$$

$$\frac{\sin \pi s}{\pi} \int_0^\infty d\lambda\, \lambda^{-s} \frac{1}{(\lambda + \tilde{\phi})^2} \frac{1}{(\lambda + \tilde{A})^2} = \frac{2}{(ip\cdot\gamma)^3} \tilde{A}^{-s} - \frac{2}{(ip\cdot\gamma)^3} \tilde{\phi}^{-s} + \frac{s}{(ip\cdot\gamma)^2} \tilde{A}^{-s-1} + \frac{s}{(ip\cdot\gamma)^2} \tilde{\phi}^{-s-1}, \tag{2.100}$$

$$\frac{\sin \pi s}{\pi} \int_0^\infty d\lambda\, \lambda^{-s} \frac{1}{(\lambda + \tilde{\phi})} \frac{1}{(\lambda + \tilde{A})^3} = -\frac{1}{(ip\cdot\gamma)^3} \tilde{A}^{-s} + \frac{1}{(ip\cdot\gamma)^3} \tilde{\phi}^{-s} - \frac{s}{(ip\cdot\gamma)^2} \tilde{A}^{-s-1} - \frac{s(s+1)}{2(ip\cdot\gamma)} \tilde{A}^{-s-2}. \tag{2.101}$$

After some cancelations, the residue integral in Eq. (2.97) is given by

$$\frac{\sin \pi s}{\pi} \int_0^\infty d\lambda \lambda^{-s} \frac{1}{(\lambda + \tilde{A})^3} \frac{1}{(\lambda + \tilde{A}^*)} = -\frac{1}{8(ip\cdot\gamma)^3} \left[\tilde{A}^{-s} - \tilde{A^*}^{-s} \right] - \frac{s}{4(ip\cdot\gamma)} \left[\frac{1}{(ip\cdot\gamma)} \tilde{A}^{-s-1} + (s+1)\tilde{A}^{-s-2} \right]. \tag{2.102}$$

The second term is, then,

$$-\frac{\partial_{\mu\nu}\tilde{\phi}}{4} \mathrm{tr} \int \frac{d^4p}{(2\pi)^4} \frac{\sin \pi s}{\pi} \int_0^\infty d\lambda \lambda^{-s} \frac{1}{(\lambda + \tilde{A})^3} \frac{1}{(\lambda + \tilde{A}^*)} \gamma^\mu \gamma^\nu = \frac{\partial^2 \phi}{32} \mathrm{tr} \int \frac{d^4p}{(2\pi)^4} \frac{ip\cdot\gamma}{p^4} \left(\tilde{A}^{-s} - \tilde{A^*}^{-s} \right) - \frac{s\partial^2\phi}{16} \mathrm{tr} \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2} \tilde{A}^{-s-1} - \frac{s(s+1)\partial^2\phi}{16} \mathrm{tr} \int \frac{d^4p}{(2\pi)^4} \frac{ip\cdot\gamma}{p^2} \tilde{A}^{-s-2}. \tag{2.103}$$

Now this term contains separate inverse powers. To be explicit, the first momentum integral can be calculated by treating it as a d dimensional, and then one lets $d \to 4$.

This is done as follows:

$$\begin{aligned}
&\mathrm{tr}\int \frac{d^dp}{(2\pi)^d}\frac{ip\cdot\gamma}{p^4}\left[\frac{1}{(\tilde{\phi}+ip\cdot\gamma)^s}-\frac{1}{(\tilde{\phi}-ip\cdot\gamma)^s}\right]\\
&\quad=\frac{4s}{\Gamma(d/2)}\frac{\tilde{\phi}^{-s-1}}{(2\pi)^{d/2}}\int_0^\infty dp\,p^{d-3}{}_2F_1\left(\frac{s+1}{2},\frac{s+2}{2};\frac{3}{2};-\frac{p^2}{\tilde{\phi}^2}\right)\\
&\quad=\frac{2s}{\Gamma(d/2)}\frac{\tilde{\phi}^{-s-1}}{(2\pi)^{d/2}}\frac{\Gamma(3/2)\Gamma(d/2-1)}{\Gamma(5/2-d/2)}\frac{\Gamma(s+3-d)}{\Gamma(s+1)}2^{d-2}\left(\tilde{\phi}^2\right)^{d/2-1}\ ,
\end{aligned}\tag{2.104}$$

if $0<\Re(d/2-1)<\Re(s/2+1/2),\Re(s/2+1)$ and $\left|\arg(1/\tilde{\phi}^2)\right|<\pi$. The first term in Eq. (2.103) becomes the following after letting $d\to 4$ in the above d dimensional expression:

$$\frac{\partial^2\phi}{32}\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\frac{ip\cdot\gamma}{p^4}\left(\tilde{A}^{-s}-\tilde{A}^{*^{-s}}\right)=\frac{\partial^2\tilde{\phi}}{32\pi^2}\frac{\phi^{1-s}}{(s-1)}\,. \tag{2.105}$$

The other terms in Eq. (2.103) can be calculated in the same manner, and one gets

$$-\frac{s\partial^2\phi}{16}\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\frac{1}{p^2}\tilde{A}^{-s-1}=\frac{\partial^2\tilde{\phi}}{32\pi^2}\frac{\phi^{1-s}}{(s-1)} \tag{2.106}$$

$$-\frac{s(s+1)\partial^2\phi}{16}\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\frac{ip\cdot\gamma}{p^2}\tilde{A}^{-s-2}=\frac{\partial^2\tilde{\phi}}{16\pi^2}\frac{\phi^{1-s}}{(s-1)}\,. \tag{2.107}$$

After placing Eqs. (2.105), (2.106) and (2.107) into Eq. (2.103), the second term in the expansion is, then,

$$-\frac{\partial_{\mu\nu}\tilde{\phi}}{4}\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\frac{\sin\pi s}{\pi}\int_0^\infty d\lambda\lambda^{-s}\frac{1}{(\lambda+\tilde{A})^3}\frac{1}{(\lambda+\tilde{A}^*)}\gamma^\mu\gamma^\nu=\frac{\partial^2\tilde{\phi}}{8\pi^2}\frac{\phi^{1-s}}{(s-1)}\,, \tag{2.108}$$

which is the same result already found with the assistance of the Feynman parametrization.

The next thing should be done is to do the same tedious calculations for the fourth term in the expansion (2.78). Using the operator identity in the semigroup integral representation, one can again convert the terms containing the multiplication of the inverse powers into separate ones. This term can, then, be given by

$$\begin{aligned}
&\frac{\partial_{\mu\nu}\tilde{\phi}}{4}\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\frac{\sin\pi s}{\pi}\int_0^\infty d\lambda\lambda^{-s}\frac{1}{(\lambda+\tilde{A})^3}\frac{1}{(\lambda+\tilde{A}^*)}\frac{p^\mu}{p^2}p\cdot\gamma\gamma^\nu \\
&\quad=\frac{\partial_{\mu\nu}\tilde{\phi}}{32}\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\frac{p^\mu}{p^6}ip\cdot\gamma\left(\tilde{A^*}^{-s}-\tilde{A}^{-s}\right)p\cdot\gamma\gamma^\nu+\frac{s\partial_{\mu\nu}\tilde{\phi}}{16}\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\frac{p^\mu}{p^4}\tilde{A}^{-s-1}p\cdot\gamma\gamma^\nu \\
&\qquad+\frac{s(s+1)\partial_{\mu\nu}\tilde{\phi}}{16}\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\frac{p^\mu}{p^4}ip\cdot\gamma\tilde{A}^{-s-2}p\cdot\gamma\gamma^\nu\,. \qquad (2.109)
\end{aligned}$$

We will just give the results for the momentum integrals since the calculations are the same as the ones which have been done for the second term in the expansion (2.78). The desired integrals are given by

$$\frac{\partial_{\mu\nu}\tilde{\phi}}{32}\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\frac{p^\mu}{p^6}ip\cdot\gamma\left(\tilde{A^*}^{-s}-\tilde{A}^{-s}\right)p\cdot\gamma\gamma^\nu=-\frac{\partial^2\tilde{\phi}}{128\pi^2}\frac{\tilde{\phi}^{1-s}}{s-1}\,, \qquad (2.110)$$

$$\frac{s\partial_{\mu\nu}\tilde{\phi}}{16}\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\frac{p^\mu}{p^4}\tilde{A}^{-s-1}p\cdot\gamma\gamma^\nu=-\frac{\partial^2\tilde{\phi}}{128\pi^2}\frac{\tilde{\phi}^{1-s}}{s-1}\,, \qquad (2.111)$$

$$\frac{s(s+1)\partial_{\mu\nu}\tilde{\phi}}{16}\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\frac{p^\mu}{p^4}ip\cdot\gamma\tilde{A}^{-s-2}p\cdot\gamma\gamma^\nu=-\frac{\partial^2\tilde{\phi}}{64\pi^2}\frac{\tilde{\phi}^{1-s}}{s-1}\,. \qquad (2.112)$$

The fourth term in the expansion (2.78) is, therefore,

$$\frac{\partial_{\mu\nu}\tilde{\phi}}{4}\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\frac{\sin\pi s}{\pi}\int_0^\infty d\lambda\lambda^{-s}\frac{1}{(\lambda+\tilde{A})^3}\frac{1}{(\lambda+\tilde{A}^*)}\frac{p^\mu}{p^2}p\cdot\gamma\gamma^\nu=-\frac{\partial^2\tilde{\phi}}{32\pi^2}\frac{\tilde{\phi}^{1-s}}{s-1}\,. \qquad (2.113)$$

If these terms are collected together, the $\hbar^2$-order correction in the semiclassical expansion is found to be equal to

$$\begin{aligned}
&\frac{\partial_{\mu\nu}\tilde{\phi}}{4}\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\frac{\sin\pi s}{\pi}\int_0^\infty d\lambda\lambda^{-s}\frac{1}{(\lambda+\tilde{A})^3}\frac{1}{(\lambda+\tilde{A}^*)}\frac{p^\mu}{p^2}p\cdot\gamma\gamma^\nu \\
&\quad-\frac{\partial_{\mu\nu}\tilde{\phi}}{4}\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\frac{\sin\pi s}{\pi}\int_0^\infty d\lambda\lambda^{-s}\frac{1}{(\lambda+\tilde{A})^3}\frac{1}{(\lambda+\tilde{A}^*)}\gamma^\mu\gamma^\nu \\
&\qquad-\frac{\partial_{\mu\nu}\tilde{\phi}}{4}\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\frac{\sin\pi s}{\pi}\int_0^\infty d\lambda\lambda^{-s}\frac{1}{(\lambda+\tilde{A})^4}\frac{p^\mu}{p^2}p\cdot\gamma\gamma^\nu=\frac{\partial^2\tilde{\phi}}{16\pi^2}\frac{\tilde{\phi}^{1-s}}{s-1}\,, \qquad (2.114)
\end{aligned}$$

which is exactly the same result that we have found by means of the Feynman parametrization.

Thus, the semiclassical expansion of the zeta function and the determinant of the Dirac operator with a scalar field ϕ can, respectively, be given by

$$\zeta\left(s|\gamma\cdot\partial+\phi\right)=\frac{1}{\pi^2}\int d^4x\left[3\tilde{\phi}^{4-s}\frac{\Gamma(s-4)}{\Gamma(s)}+\frac{1}{16}\tilde{\phi}^{1-s}\frac{\Gamma(s-1)}{\Gamma(s)}\partial^2\tilde{\phi}\right]+\cdots, \tag{2.115}$$

$$\ln\det(\gamma\cdot\partial+\phi)=\frac{1}{16\pi^2}\int d^4x\left[\phi^4\ln\left(\frac{\phi^2}{\mu^2}e^{-25/6}\right)+\ln\left(\frac{\phi^2}{\mu^2}\right)\frac{1}{2}\partial_\mu\phi\partial^\mu\phi\right]+\cdots, \tag{2.116}$$

which agrees with the result given by Li *et al.* [65] up to a numerical factor.

2.7. The Large–N Yukawa Theory

In this section, a set of $N+1$ massless Dirac fermions that are both $U(N+1)$ symmetric and couples to a scalar field via the well known Yukawa coupling is studied at large–N regime. The self-coupling of the scalar field is left unspecified. The Euclidean action of this theory, similar to the ones [14, 66], is given by

$$S\left[\bar{\Psi},\Psi,\phi\right]=\int d^4x\left\{-\bar{\Psi}\left(\gamma\cdot\partial+g\phi\right)\Psi+\frac{1}{2}\partial_\mu\phi\partial^\mu\phi+NV\left[\frac{\phi^2}{N}\right]\right\}. \tag{2.117}$$

For the large-N analysis, N fermion fields are integrated out and the fields ψ_{N+1}, $\bar{\psi}_{N+1}$, and ϕ are rescaled as $\sqrt{N}\psi_{N+1}$, $\sqrt{N}\bar{\psi}_{N+1}$, and ϕ/g, respectively, with a redefined coupling constant $\tilde{g}^2=g^2N$ which is kept fixed as $N\to\infty$. In this way, one gets

$$\begin{aligned}S_{eff}\left[\bar{\psi},\psi,\phi\right]=N\int d^4x&\left\{-\bar{\psi}\left(\gamma\cdot\partial+\phi\right)\psi+\frac{1}{2\tilde{g}^2}\partial_\mu\phi\partial^\mu\phi+V\left[\frac{\phi^2}{\tilde{g}^2}\right]\right\}\\&-N\ln\det(\gamma\cdot\partial+\phi)\,.\end{aligned} \tag{2.118}$$

In the limit $N\to\infty$, the contributions coming from the extremals of this action dominates the functional integral. The functional to zeroth order in $1/N$ is just the $N=\infty$ quantum effective action without any corrections coming from the next $1/N$ orders. If one plugs the semiclassical expansion for the determinant of the same Dirac operator, which has been calculated in the previous section, into Eq. (2.118), the $N = \infty$ quantum effective action is, thus, given by

$$\Gamma_0\left[\bar{\psi}, \psi, \phi\right] = \int d^4x \left\{ -\bar{\psi}\left(\gamma \cdot \partial + \phi\right)\psi + \left[\frac{1}{\tilde{g}^2} - \frac{1}{16\pi^2}\ln\left(\frac{\phi^2}{\mu^2}\right)\right]\frac{1}{2}\partial_\mu\phi\partial^\mu\phi \right.$$
$$\left. + V\left[\frac{\phi^2}{\tilde{g}^2}\right] - \frac{\phi^4}{16\pi^2}\ln\left(\frac{\phi^2}{\mu^2}e^{-25/6}\right)\right\} . \tag{2.119}$$

If one pays more attention to this effective action, it is easy to notice that there is a term involving gradients which is multiplied by a logarithm of the field. This combination, of course, arises from the fact that the scalar field is assumed to be inhomogeneous or in other words nonconstant while the determinant coming from integrating the N fermions out is being calculated. Here, it is appropriate to make a comparison between the ϕ^4–theory and the Yukawa theory as far as the perturbation theory in terms of ε expansion is concerned. What is known from the perturbation theory is that there is no one-loop wave function renormalization for the ϕ^4 theory, whereas one has to deal with that for the Yukawa theory even at one loop. The renormalization constants for the fields as well as for the other parameters in the theory should be defined in terms of the poles of ε and appropriate counterterms should also be introduced in order to end up with a finite set of parameters in the theory. Although one does not have to worry about wave function renormalization to first order in the ϕ^4 theory, the kinetic term is multiplied by a nontrivial function of the field [67]. It is easy to obtain this factor by replacing $V(x)$ with $m^2 + \lambda\phi^2(x)/2$ in Eq. (2.65) and this yields

$$\frac{\lambda^2}{6(4\pi)^2}\frac{\phi^2}{(2m^2 + \lambda\phi^2)} , \tag{2.120}$$

which is actually the first nontrivial term of the $Z_{eff}(\phi)$, which is the function multiplier of the kinetic term in the expansion of the effective action in terms of the field itself. However, to second order in the loop expansion there is a wave function renormalization in the ϕ^4 theory and the terms involving gradients multiplied by logarithms of the field actually occur as in the Yukawa theory. Therefore, it can presumably be thought that the occurrence of that type of kinetic terms with logarithmic multipliers in the effective action expansion is a manifestation of the divergences encountered in parameter renormalization for the fields themselves. Thus, the reason that the kinetic term multiplied by a logarithm of the field is encountered in the Yukawa theory even in the first order in the loop expansion as in Eq. (2.119) arises from the fact that there is an inevitable wave function renormalization in the Yukawa theory at one loop. As it is written, this theory looks unstable for the different regimes for the scalar fields; however, one has to find the physical fields with the correct classical configuration and then define the renormalized action afterward.

3. THE RELATIVISTIC LEE MODEL ON RIEMANNIAN MANIFOLDS

3.1. The Relativistic Lee Model in $\mathbb{R}^{3+1}$

We would like to give a brief introduction to the Lee model at the first place before constructing the model with our tools. The Lee model is a simple field theory model, which requires a mass, coupling constant and wave function renormalization [35, 68]. What is so special about the model is that the renormalizations can be carried out nonperturbatively; that is, the renormalization of the parameters can be performed exactly and be found in a closed form. The model consists of three fictitious particles V, N and θ, and describes the interaction between them. The interaction can occur via the fundamental virtual processes represented the following reaction,

$$V \rightleftharpoons N + \theta \,, \tag{3.1}$$

that is, the V particle transforms into the N particle by emitting the θ particle. However, the N particle is not allowed to transform into the V particle via emitting the θ particle. Rather, it absorbs a θ particle and transforms into a V particle. Therefore, the crossed interaction $N \rightleftharpoons V + \theta$ is forbidden, which actually makes the model solvable. This restriction on the model can be thought of a consequence of a priori superselection rules. Furthermore, no antiparticles such as $\overline{\theta}$ are included. The V and N particles can be considered as the internal states of a heavy particle. One of the interpretation of this model which provides a good physical insight regards the V particle as proton, the N particle as neutron and θ particle as π^+ meson. In this interpretation the restriction can be explained as the consequence of the charge conservation. While the charge conservation permits the interaction $p \to n + \pi^+$, it forbids the interaction $n \to p + \pi^+$. The Hamiltonian describing this field theory is given by

$$H = H_0 + H_I \,, \tag{3.2}$$

in which

$$H_0 = m_V \int \frac{d^3p}{(2\pi)^3} V^\dagger(\mathbf{p})V(\mathbf{p}) + m_N \int \frac{d^3p}{(2\pi)^3} N^\dagger(\mathbf{p})N(\mathbf{p}) + \int \frac{d^3p}{(2\pi)^3} \omega(\mathbf{p})a^\dagger(\mathbf{p})a(\mathbf{p}) , \tag{3.3}$$

and

$$H_I = \lambda \int \frac{d^3p}{(2\pi)^3} \left[V^\dagger(\mathbf{p})N(\mathbf{p}-\mathbf{q})a(\mathbf{p}) + N^\dagger(\mathbf{p}-\mathbf{q})a^\dagger(\mathbf{q})V(\mathbf{p})\right] , \tag{3.4}$$

$V^\dagger, V$ and $N^\dagger, N$ being the creation and annihilation operators for the particles V and N respectively and obey the usual anticommutation rules whereas $a^\dagger(\mathbf{p}), a(\mathbf{p})$ obeying the usual commutation rule play the same role for the θ particle. m_V and m_N represent the masses of the V particle and the N particle, respectively. Recoil effects of the particles V and N are not included. $\omega(\mathbf{p}) = \sqrt{m^2 + \mathbf{p}^2}$ is the energy of a θ particle of mass m and momentum $\mathbf{p}$. The Lee model admits two rather restricting conserved quantities. One of which is the conservation of the total number of the fermion species, that is,

$$\mathcal{N}_V + \mathcal{N}_N = \mathcal{Q}_1 , \tag{3.5}$$

where $\mathcal{N}_V$ and $\mathcal{N}_N$ are the total number of V and N, respectively. Furthermore, the sum of bosons and V type fermions is conserved. In other words,

$$\mathcal{N}_V + \mathcal{N}_\theta = \mathcal{Q}_2 , \tag{3.6}$$

$\mathcal{N}_\theta$ being the total number of the θ particle. The number operators for V, N and θ particles can be given by, respectively,

$$\mathcal{N}_V = \int \frac{d^3p}{(2\pi)^3} V^\dagger(\mathbf{p})V(\mathbf{p}) , \tag{3.7}$$

$$\mathcal{N}_N = \int \frac{d^3p}{(2\pi)^3} N^\dagger(\mathbf{p})N(\mathbf{p}) , \tag{3.8}$$

$$\mathcal{N}_\theta = \int \frac{d^3p}{(2\pi)^3} a^\dagger(\mathbf{p})a(\mathbf{p}) . \tag{3.9}$$

It is obvious that $\mathcal{Q}_1$ and $\mathcal{Q}_2$ commute with the total Hamiltonian H. Due to Eqs. (3.5) and (3.6) the theory is highly constrained, which allows only a finite number of particles interacting at any given time, and consequently makes the theory exactly solvable. To put it another way, the model is realized throughout being divided into independent sectors which are defined by the integral values of Eqs. (3.5) and (3.6). Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ represent the constants in Eqs. (3.5) and (3.6), respectively. The physical vacuum, which coincides with their corresponding bare states, is the state in which $\mathcal{Q}_1 = 0$ and $\mathcal{Q}_2 = 0$, and can be defined by

$$N(\mathbf{p})|0\rangle = V(\mathbf{p})|0\rangle = a(\mathbf{p})|0\rangle \quad \text{for all } \mathbf{p}\,. \tag{3.10}$$

The one-particle θ state is the one with $\mathcal{Q}_1 = 0$ and $\mathcal{Q}_2 = 1$ such that

$$|\theta_{\mathbf{p}}\rangle = a^{\dagger}(\mathbf{p})|0\rangle\,, \tag{3.11}$$

whereas the one-particle N state is the one with $\mathcal{Q}_1 = 1$ and $\mathcal{Q}_2 = 0$ such that

$$|N_{\mathbf{p}}\rangle = N^{\dagger}(\mathbf{p})|0\rangle\,. \tag{3.12}$$

The V particle corresponds to the first nontrivial subspace and whose quantum numbers are $\mathcal{Q}_1 = 1$ and $\mathcal{Q}_2 = 1$. Now comes an important point: The solvable part of the one-boson Lee model with a heavy fermion extends to the case $V + (n-1)\theta$ in which $\mathcal{Q}_1 = 1$ and $\mathcal{Q}_2 = n$ [69]. The following diagram briefly summarizes the allowed interactions, which only occur in each independent sector and are denoted by the diagonal arrows, that is, subspaces do not shuffle:

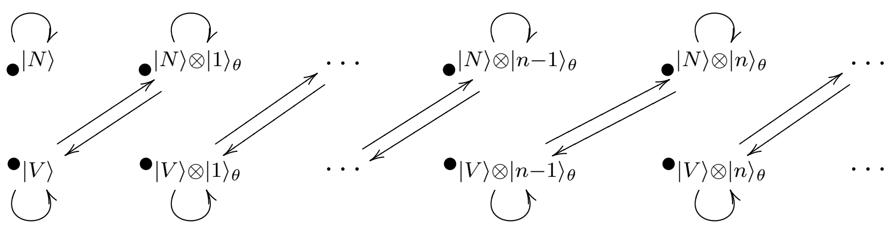

Figure 3.1. Allowed interactions

Here $|n\rangle_\theta$ being the n-particle θ state, and the notation for V and N particles is self-explanatory. It is easy to check that $|\theta_{\mathbf{p}}\rangle$ and $|N_{\mathbf{p}}\rangle$ are eigenfunctions of the total Hamiltonian H. This results in the fact that the θ and N particles are actually the same as physical particles. Therefore, the masses corresponding to these particles are assumed to be observed masses. However, the same is not true for the V particle. The bare and the physical states do not coincide for this particle, and at that point renormalization is brought into play. A string of $N\theta$ loops connected by V lines contributes to the V propagator. While computing the complete amplitude for $V \to N + \theta$, which is the sum of the following geometric series,

$$V \to N + \theta \to V \to N + \theta \to V \to \cdots \to N + \theta \to \cdots , \tag{3.13}$$

one encounters a divergence, which makes applying a renormalization prescription necessary. The pole in the V propagator gives the physical mass after applying some renormalization conditions, and whose residue is nothing but the wave-function renormalization constant. For the renormalized coupling constant one should evaluate the exact $N\theta$ scattering amplitude at the physical V particle pole, (i.e., at zero momentum or at threshold $m_N + m$) and then require that this is equal to the Born approximation to the scattering amplitude in which the bare coupling constant and mass are replaced by their renormalized counterparts at the lowest order. At that point, we stop discussing the calculations to be done for the renormalization of the Lee model with conventional methods (e.g. regularization with an ultraviolet cut-off or dimensional regularization, which are extensively studied [35, 38, 39, 68]), and we will rather construct the following simplified Lee model with our tool.

The model which we will construct in the rest of this section describes the interaction between a field of relativistic bosons and a heavy fermionic source with an internal degree of freedom which actually corresponds to *two distinct states* of the source. Since the source is heavy, we can effectively consider it as sitting at some fixed point in space-time. This results in the neglect of recoil for the source, which means that the energies of the states do not depend on their momentum. The cut-off

Hamiltonian of the model in matrix form is given by

$$H_\epsilon = H_0 \left[\chi_+ \otimes \chi_+^\dagger + Z(\epsilon)\chi_- \otimes \chi_-^\dagger\right] + H_{I,\epsilon}\,, \tag{3.14}$$

where H_0 and $H_{I,\epsilon}$ are the free and the interaction parts of the cut-off Hamiltonian respectively and are given by

$$H_0 = \int \frac{d^3p}{(2\pi)^3}\,\omega(\mathbf{p})a^\dagger(\mathbf{p})a(\mathbf{p})\,, \tag{3.15}$$

$$H_{I,\epsilon} = Z(\epsilon)\mu(\epsilon)\frac{1-\sigma_3}{2} + \sqrt{Z(\epsilon)}\lambda(\epsilon)\left[\sigma_+\phi_\epsilon^{(-)}(0) + \sigma_-\phi_\epsilon^{(+)}(0)\right]\,. \tag{3.16}$$

The form of the interaction Hamiltonian above seems to be different from the ones widely exposed in the literature at first glance. The θ particle in the Lee model is represented by the field ϕ, and we are restricting the model to the total fermion number equal to 1-sector. Our choice for the form of the coupling between N, V and θ particles can be easily obtained by the well-known isomorphism between bilinear products of fermion operators and the Pauli spin matrices [70], as well as the kinetic part of the fermionic sector. At this moment ϵ is an unspecified cut-off prescription, the meaning of which will become clear when we will renormalize the parameters of the theory. Here, $\phi_\epsilon^{(\pm)}(0)$ are the positive and negative frequency parts of the real bosonic field, respectively, defined through this cut-off prescription. A more precise definition of these field operators will be given in the manifold case. $\chi_\pm$ in Eq. (3.14) are the standard spin states, which describe the two states of the system. Due to the fact that there can be various divergences hidden inside the theory, we allow two states of the system to have different normalizations. With hindsight we choose this to be χ_- state.

The theory has a conserved charge, which can be written as,

$$\mathcal{Q} = \frac{1-\sigma_3}{2} + \int \frac{d^3p}{(2\pi)^3}\,a^\dagger(\mathbf{p})a(\mathbf{p})\,. \tag{3.17}$$

It means that the theory decouples into independent sectors as $\mathcal{F}_\mathcal{B}^{(n)} \otimes \chi_+ \oplus \mathcal{F}_\mathcal{B}^{(n-1)} \otimes \chi_-$.

The construction of the model is merely based on finding the resolvent of the cut-off Hamiltonian, which describes the system completely. While computing the resolvent, one introduces the principal operator $\Phi(E)$, that can be regarded as an effective Hamiltonian of the theory. The reason that the Krein formula can be applied lies in the observation that there is a constraining conserved quantity, namely, $\mathcal{Q}$. The idea of using this operator comes from the fact that the zero eigenvalues of it determine implicitly the bound state energies of the theory. The ultra-violet divergence takes place in the theory when the size of the source goes to zero, which causes the difference of the energy levels to become infinite. In the following, by renormalization it is solely meant to search for a well-defined limit of the cut-off Hamiltonian in matrix form as $\epsilon \to 0^+$. This is accomplished by curing the principal operator in the same limit.

Following Rajeev [48] the cut-off Hamiltonian minus energy is given in a 2×2 block form,

$$H_\epsilon - E = \begin{bmatrix} H_0 - E & \sqrt{Z(\epsilon)}\lambda(\epsilon)\phi_\epsilon^{(-)}(0) \\ \sqrt{Z(\epsilon)}\lambda(\epsilon)\phi_\epsilon^{(+)}(0) & Z(\epsilon)\left[H_0 - E + \mu(\epsilon)\right] \end{bmatrix} . \tag{3.18}$$

The resolvent is simply the formal inverse of Eq. (3.18) and this inverse can be calculated algebraically. If the Hamiltonian is parametrized as,

$$H_\epsilon - E = \begin{pmatrix} a & b^\dagger \\ b & d \end{pmatrix} , \tag{3.19}$$

and if the resolvent is parametrized as,

$$R_\epsilon(E) = \begin{pmatrix} \alpha & \beta^\dagger \\ \beta & \delta \end{pmatrix} , \tag{3.20}$$

then one ends up with the following algebraic equalities, which allow one to calculate the resolvent,

$$\alpha = a^{-1} + a^{-1}b^{\dagger}\left(d - ba^{-1}b^{\dagger}\right)^{-1} ba^{-1}\,, \tag{3.21}$$

$$\beta = -\left(d - ba^{-1}b^{\dagger}\right)^{-1} ba^{-1}\,, \tag{3.22}$$

$$\delta = \left(d - ba^{-1}b^{\dagger}\right)^{-1} = \delta^{\dagger}\,, \tag{3.23}$$

$$\Phi = d - ba^{-1}b^{\dagger}\,. \tag{3.24}$$

Equation (3.24) is just the cut-off principal operator and is given by

$$\Phi_{\epsilon}(E) = Z(\epsilon)\Bigg\{ H_0 - E + \mu(\epsilon) - \lambda^2(\epsilon)\int \frac{d^3p}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3}\,\frac{a(\mathbf{p})}{\sqrt{2\omega(\mathbf{p})}}\,\frac{1}{(H_0 - E)}\,\frac{a^{\dagger}(\mathbf{q})}{\sqrt{2\omega(\mathbf{q})}}\Bigg\}\,, \tag{3.25}$$

where E is considered as a complex parameter and the formulae below should be analytically continued to their largest domains of analyticity. As one can easily notice the annihilation and the creation operators are in the wrong order with respect to normal ordering prescription, so we should normal-order them. After being normal-ordered, the principal operator becomes

$$\Phi_{\epsilon}(E) = Z(\epsilon)\Bigg\{ H_0 - E + \mu(\epsilon) - \lambda^2(\epsilon)\int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2\omega(\mathbf{p})}\,\frac{1}{H_0 - E + \omega(\mathbf{p})} - \lambda^2(\epsilon)\int \frac{d^3p}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3}\,\frac{a^{\dagger}(\mathbf{q})}{\sqrt{2\omega(\mathbf{q})}}\,\frac{1}{H_0 - E + \omega(\mathbf{q}) + \omega(\mathbf{p})}\,\frac{a(\mathbf{p})}{\sqrt{2\omega(\mathbf{p})}}\Bigg\}\,. \tag{3.26}$$

The two fractions in the fourth term above can be united by a Feynman parametrization,

$$\frac{1}{\omega(\mathbf{p})}\,\frac{1}{H_0 - E + \omega(\mathbf{p})} = \int_0^1 d\xi\,\frac{1}{\left[(H_0 - E)\xi + \omega(\mathbf{p})\right]^2}\,. \tag{3.27}$$

Afterwards, it can be exponentiated as

$$\frac{1}{\omega(\mathbf{p})}\frac{1}{H_0 - E + \omega(\mathbf{p})} = \int_0^1 d\xi \int_0^\infty ds\, e^{-s\omega(\mathbf{p})} e^{-s(H_0-E)\xi}$$
$$= \int_0^\infty ds\, e^{-s\omega(\mathbf{p})} \frac{1}{H_0 - E}\left[1 - e^{-s(H_0-E)}\right] \tag{3.28}$$

In order to evaluate the momentum integral one more identity is needed, and this is the so-called subordination identity:

$$e^{-s\omega(\mathbf{p})} = \frac{s}{2\sqrt{\pi}} \int_0^\infty du\, \frac{1}{u^{3/2}} e^{-s^2/4u} e^{-u\omega^2(\mathbf{p})}\,. \tag{3.29}$$

With the help of that identity, we can convert $\omega(\mathbf{p})$ in the exponential into $\omega^2(\mathbf{p})$ such that after calculating the momentum integral the second term can be given by

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega(\mathbf{p})} \frac{1}{H_0 - E + \omega(\mathbf{p})}$$
$$= \frac{1}{4\sqrt{\pi}} \int_\epsilon^\infty du\, \frac{e^{-um^2}}{u^{3/2}} \int \frac{d^3p}{(2\pi)^3} e^{-up^2} \int_0^\infty ds\, s e^{-s^2/4u} \frac{1}{H_0 - E}\left[1 - e^{-s(H_0-E)}\right]$$
$$= \frac{1}{32\pi^2} \int_\epsilon^\infty du\, \frac{e^{-um^2}}{u^{3/2}} \int_0^\infty ds\, s e^{-s^2/4} \frac{\left[1 - e^{-s\sqrt{u}(H_0-E)}\right]}{\sqrt{u}(H_0 - E)}\,. \tag{3.30}$$

The use of that identity is to convert the divergence of the momentum integral into a divergence emerging from the lower limit of the u-integral, which is now mollified by ϵ, explicitly. The momentum integral is no longer divergent, and thus, can safely be computed. If the same calculations are done step by step for the three fractions in the fifth term in Eq. (3.26) without calculating the momentum integral, one obtains

$$\frac{1}{\sqrt{2\omega(\mathbf{q})}} \frac{1}{H_0 - E + \omega(\mathbf{q}) + \omega(\mathbf{p})} \frac{1}{\sqrt{2\omega(\mathbf{p})}}$$
$$= \frac{2}{\pi} \int_0^\infty d\alpha \int_0^\infty d\beta \int_0^\infty ds\, e^{-\omega(\mathbf{q})(s+\alpha^2)} e^{-\omega(\mathbf{p})(s+\beta^2)} e^{-s(H_0-E)}$$
$$= \frac{1}{2\pi^2} \int_0^\infty ds \int_0^\infty d\alpha\, \left(s + \alpha^2\right) \int_0^\infty d\beta\, \left(s + \beta^2\right) \int_0^\infty du_1\, \frac{e^{-(s+\alpha^2)^2/4u_1}}{u_1^{3/2}}$$
$$\times \int_0^\infty du_2\, \frac{e^{-(s+\beta^2)^2/4u_2}}{u_2^{3/2}} e^{-u_1\omega^2(\mathbf{q})} e^{-u_2\omega^2(\mathbf{p})} e^{-s(H_0-E)}\,. \tag{3.31}$$

After placing Eqs. (3.30) and (3.31) into Eq. (3.26), the cut-off principal operator becomes

$$\begin{aligned}
\Phi_\epsilon(E) = Z(\epsilon)\Bigg\{ & H_0 - E + \mu(\epsilon) - \frac{\lambda^2(\epsilon)}{32\pi^2} \int_\epsilon^\infty du \, \frac{e^{-um^2}}{u^{3/2}} \int_0^\infty ds \, s e^{-s^2/4} \frac{\left[1 - e^{-s\sqrt{u}(H_0 - E)}\right]}{\sqrt{u}(H_0 - E)} \\
& - \frac{\lambda^2(\epsilon)}{2\pi^2} \int_0^\infty ds \int_0^\infty d\alpha \, \left(s + \alpha^2\right) \int_0^\infty d\beta \, \left(s + \beta^2\right) \\
& \times \int_0^\infty du_1 \, \frac{e^{-(s+\alpha^2)^2/4u_1}}{u_1^{3/2}} \int_0^\infty du_2 \, \frac{e^{-(s+\beta^2)^2/4u_2}}{u_2^{3/2}} \\
& \times \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \, e^{-u_1 \omega^2(\mathbf{q})} e^{-u_2 \omega^2(\mathbf{p})} a^\dagger(\mathbf{q}) e^{-s(H_0 - E)} a(\mathbf{p}) \Bigg\} . \qquad (3.32)
\end{aligned}$$

It is now time to renormalize the cut-off principal operator. If the exponential in the second integral term is expanded in power series in s, one can notice that the only terms which produce divergence are just the ones which are up to order s^2. On the basis of this expansion, one can redefine the coupling constant and the mass, whereby the cut-off principal operator can, easily, be regularized. Thus, we are able to achieve the renormalized counterparts of both those parameters and the principal operator. In order to accomplish this, it is appropriate to divide the principal operator by the square of the coupling constant. The main difference of the relativistic Lee model from the nonrelativistic one resides in not only that there is a coupling constant renormalization besides the mass renormalization but also there is a wave function renormalization. Therefore, the ratio of the principal operator to the square of the coupling constant, $\Phi(E)/\lambda^2$, should be renormalized instead of just the principal operator, $\Phi(E)$, in the relativistic Lee model. In the light of the above discussion, one can renormalize whole parameters of the model. The following choices regularize the principal operator by canceling the divergences,

$$\frac{\mu(\epsilon)}{\lambda^2(\epsilon)} = \frac{\mu_R}{\lambda_R^2} + \frac{1}{32\pi^2} \int_\epsilon^\infty du \, \frac{e^{-um^2}}{u^{3/2}} \int_0^\infty ds \, s^2 e^{-s^2/4} , \qquad (3.33)$$

$$\frac{1}{\lambda^2(\epsilon)} = \frac{1}{\lambda_R^2} - \frac{1}{64\pi^2} \int_\epsilon^\infty du \, \frac{e^{-um^2}}{u} \int_0^\infty ds \, s^3 e^{-s^2/4} . \qquad (3.34)$$

The principal operator is, thereby, given by

$$\frac{\Phi_\epsilon(E)}{\lambda^2(\epsilon)} = Z(\epsilon)\Bigg\{\frac{(H_0-E)}{\lambda_R^2} + \frac{\mu_R}{\lambda_R^2} - \frac{1}{32\pi^2}\int_\epsilon^\infty du\, \frac{e^{-um^2}}{u^{3/2}} \int_0^\infty ds\, s e^{-s^2/4} \frac{1}{\sqrt{u}(H_0-E)} \\ \times \left[1 - s\sqrt{u}(H_0-E) + \frac{1}{2}s^2 u (H_0-E)^2 - e^{-s\sqrt{u}(H_0-E)}\right] \\ - \frac{1}{2\pi^2}\int_0^\infty ds \int_0^\infty d\alpha\, \left(s+\alpha^2\right) \int_0^\infty d\beta\, \left(s+\beta^2\right) \\ \times \int_0^\infty du_1\, \frac{e^{-(s+\alpha^2)^2/4u_1}}{u_1^{3/2}} \int_0^\infty du_2\, \frac{e^{-(s+\beta^2)^2/4u_2}}{u_2^{3/2}} \\ \times \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3}\, e^{-u_1\omega^2(\mathbf{q})} e^{-u_2\omega^2(\mathbf{p})} a^\dagger(\mathbf{q}) e^{-s(H_0-E)} a(\mathbf{p}) \Bigg\}. \tag{3.35}$$

We notice that the subtractions in the second line resemble the regularization of the infinite Fredholm determinants. This is analogous to the quantum effective action calculations via regularized determinants in the path integral formalism. It is obvious that this operator has a well-defined limit as $\epsilon \to 0^+$ when both sides are divided by $Z(\epsilon)$:

$$\lim_{\epsilon\to 0^+} \frac{\Phi_\epsilon(E)}{\lambda^2(\epsilon) Z(\epsilon)} = \frac{\Phi_R(E)}{\lambda_R^2}, \tag{3.36}$$

and the renormalized principal operator in terms of the renormalized mass and the renormalized coupling constant can be given by

$$\frac{\Phi_R(E)}{\lambda_R^2} = \frac{(H_0-E)}{\lambda_R^2} + \frac{\mu_R}{\lambda_R^2} - \frac{1}{32\pi^2}\int_0^\infty du\, \frac{e^{-um^2}}{u^{3/2}} \int_0^\infty ds\, s e^{-s^2/4} \frac{1}{\sqrt{u}(H_0-E)} \\ \times \left[1 - s\sqrt{u}(H_0-E) + \frac{1}{2}s^2 u (H_0-E)^2 - e^{-s\sqrt{u}(H_0-E)}\right] \\ - \frac{1}{2\pi^2}\int_0^\infty ds \int_0^\infty d\alpha\, \left(s+\alpha^2\right) \int_0^\infty d\beta\, \left(s+\beta^2\right) \\ \times \int_0^\infty du_1\, \frac{e^{-(s+\alpha^2)^2/4u_1}}{u_1^{3/2}} \int_0^\infty du_2\, \frac{e^{-(s+\beta^2)^2/4u_2}}{u_2^{3/2}} \\ \times \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3}\, e^{-u_1\omega^2(\mathbf{q})} e^{-u_2\omega^2(\mathbf{p})} a^\dagger(\mathbf{q}) e^{-s(H_0-E)} a(\mathbf{p}). \tag{3.37}$$

It is easily seen that this limit also fixes the wave function renormalization constant $Z(\epsilon)$ to be equal to $\lambda_R^2/\lambda^2(\epsilon)$. Now the elements of the resolvent can also be given in terms of the renormalized principal operator. To see this, we look at the expression for α,

$$\alpha = \frac{1}{H_0 - E} + \frac{1}{H_0 - E}\phi_\epsilon^{(-)}(0)\frac{Z(\epsilon)\lambda^2(\epsilon)}{\Phi_\epsilon(E)}\phi_\epsilon^{(+)}(0)\frac{1}{H_0 - E}\,. \tag{3.38}$$

If we now use the wave function renormalization constant in this expression, we may take the limit $\epsilon \to 0^+$, giving us

$$\alpha = \frac{1}{H_0 - E} + \frac{1}{H_0 - E}\phi^{(-)}(0)\frac{\lambda_R^2}{\Phi_R(E)}\phi^{(+)}(0)\frac{1}{H_0 - E}\,. \tag{3.39}$$

Similarly, for the others, we find

$$\beta = -\frac{\lambda_R}{\Phi_R(E)}\phi^{(-)}(0)\frac{1}{H_0 - E}\,, \tag{3.40}$$

$$\delta = \frac{1}{\Phi_R(E)}\,. \tag{3.41}$$

These equations tell us that zero eigenvalues of the renormalized principal operator determine the bound states and the corresponding energies as nonlinear eigenvalue equations. Notice that the renormalized operator $\Phi_R(E)$ converts a divergent linear problem in the Schrödinger picture into a highly nonlinear but a well-defined problem.

It is also important to know how the divergences are controlled by the cut-off parameter ϵ in the redefinition of mass and coupling constant. In order to find it out, the integrals in Eqs. (3.33) and (3.34) should be calculated. The cut-off dependent parameters can be given as an asymptotic series in ϵ,

$$\frac{\mu(\epsilon)}{\lambda^2(\epsilon)} \simeq \frac{\mu_R}{\lambda_R^2} + \frac{1}{8\pi^{3/2}}\frac{1}{\sqrt{\epsilon}} \quad \text{as} \quad \epsilon \to 0^+\,, \tag{3.42}$$

$$\frac{1}{\lambda^2(\epsilon)} \simeq \frac{1}{\lambda_R^2} + \frac{1}{8\pi^2}\ln\epsilon \quad \text{as} \quad \epsilon \to 0^+\,. \tag{3.43}$$

In the above formulation, we see that as $\epsilon \to 0^+$ the bare coupling constant squared can become negative, especially, if the renormalized coupling is too strong. In the usual treatment, this is an indication of the appearance of a ghost state. In our approach, this means that the cut-off Hamiltonian becomes unbounded from below as the cut-off is removed.

To make contact with the usual perturbative renormalization, we will recast the Hamiltonian into a renormalized part and a counterterm Hamiltonian. We will see that there are no other counterterms needed other than the ones existing already in the original Hamiltonian. So as to establish that, we should go back in the calculations and replace $Z(\epsilon)$ by $\lambda_R^2/\lambda^2(\epsilon)$. The mass term of the source becomes

$$\begin{aligned} Z(\epsilon)\mu(\epsilon)\frac{1-\sigma_3}{2} &= \frac{\mu(\epsilon)}{\lambda^2(\epsilon)}\lambda_R^2\frac{1-\sigma_3}{2} \\ &= \left(\mu_R + \lambda_R^2\Delta\mu\right)\frac{1-\sigma_3}{2}\,, \end{aligned} \tag{3.44}$$

in which the term $\Delta\mu$ is nothing but the divergent part in the redefinition of the mass. The same replacement should also be done in the interaction terms and one can get

$$\sqrt{Z(\epsilon)}\lambda(\epsilon)\left[\sigma_+\phi_\epsilon^{(-)}(0) + \sigma_-\phi_\epsilon^{(+)}(0)\right] = \lambda_R\left[\sigma_+\phi^{(-)}(0) + \sigma_-\phi^{(+)}(0)\right]\,. \tag{3.45}$$

The next step to perform is plug those into the Hamiltonian such that the renormalized Hamiltonian can be determined. After placing them, the Hamiltonian becomes,

$$\begin{aligned} H_\epsilon = H_0 &\left[\chi_+\otimes\chi_+^\dagger + Z(\epsilon)\chi_-\otimes\chi_-^\dagger\right] + Z(\epsilon)\mu(\epsilon)\frac{1-\sigma_3}{2} \\ &+ \sqrt{Z(\epsilon)}\lambda(\epsilon)\left[\sigma_+\phi_\epsilon^{(-)}(0) + \sigma_-\phi_\epsilon^{(+)}(0)\right] \\ = H_0 &\left[\chi_+\otimes\chi_+^\dagger + Z(\epsilon)\chi_-\otimes\chi_-^\dagger\right] + \left(\mu_R + \lambda_R^2\Delta\mu\right)\frac{1-\sigma_3}{2} \\ &+ \lambda_R\left[\sigma_+\phi^{(-)}(0) + \sigma_-\phi^{(+)}(0)\right]\,. \end{aligned} \tag{3.46}$$

We know from the theory of renormalization that if one would like to give the Hamiltonian of the theory in terms of renormalized parameters instead of bare or cut-off parameters, then the bare Hamiltonian is given by the renormalized Hamiltonian containing only the renormalized parameters plus the appropriate counterterms. Therefore if we choose the cut-off Hamiltonian as

$$H_\epsilon = H_R + H_0[Z(\epsilon) - 1]\chi_- \otimes \chi_-^\dagger + \lambda_R^2 \Delta\mu \frac{1-\sigma_3}{2}\,, \tag{3.47}$$

then the renormalized Hamiltonian of the theory can be given by

$$H_R = H_0 \left[\chi_+ \otimes \chi_+^\dagger + \chi_- \otimes \chi_-^\dagger\right] + \mu_R \frac{1-\sigma_3}{2} + \lambda_R \left[\sigma_+ \phi^{(-)}(0) + \sigma_- \phi^{(+)}(0)\right]\,. \tag{3.48}$$

The renormalized Hamiltonian H_R should not be confused with what we call the quantum Hamiltonian H_Q, which determines the time evolution of the quantum system. The resolvent that we have found in the Fock space should correspond to the resolvent of the Hamiltonian H_Q defined in this Fock space. The existence of this Hamiltonian cannot be proved by a straightforward application of the resolvent convergence as is done for a different model [71]. This question is delicate in our case. This Hamiltonian may not be written as an explicit formula. However its resolvent can be explicitly derived.

Although the renormalized parameters had been found, we did not complete the renormalization. Since after regularizing the parameters by removing the divergences, there remains a finite arbitrariness [67]. In order to fix these finite parts, which results in determining the physical parameters of the theory, one should impose the renormalization conditions. In perturbative field theories, these conditions should be imposed on the superficially divergent Green's functions to determine the coefficients of the counterterms and one demands that Green's functions satisfy them order by order if these conditions are satisfied to lowest order. In our formulation we should also specify similar conditions. In this approach the Schrödinger equation is replaced by the equation $\Phi(E)\Psi = 0$. So a natural choice is the one related to the simple composite which consists of a single boson and χ_+ state giving us a dressed χ_- state. We can fix the mass difference of χ_- and χ_+. Therefore, we impose the following,

$$\Phi_R(E = \mu_p)|0\rangle \equiv 0\,, \tag{3.49}$$

where μ_p is the physical mass difference. If the calculations of the principal operator are followed backwards, one can obtain a much more compact version of the principal operator. After a little algebra, we get

$$\Phi_R(E) = H_0 - E + \mu_R$$
$$- \frac{\lambda_R^2}{2} \int \frac{d^3p}{(2\pi)^3} \left[\frac{1}{\omega(\mathbf{p})} \frac{1}{H_0 - E + \omega(\mathbf{p})} - \frac{1}{\omega^2(\mathbf{p})} + \frac{H_0 - E}{\omega^3(\mathbf{p})} \right] + \cdots , \quad (3.50)$$

where the dots stand for the normal-ordered interaction term. Incidentally, if we could expand the first term in the integral into a power series in $H_0 - E$, the second and the third terms are canceled, leading to a series of finite terms. If we add and subtract the second line above with $E = \mu_p$ and $H_0 = 0$, the resultant operator will satisfy the desired condition and fix the finite part of the renormalization.

$$\Phi_R(E) = H_0 - E + \mu_p + \frac{\lambda_R^2}{2} \int \frac{d^3p}{(2\pi)^3} \left[\frac{1}{\omega(\mathbf{p})} \frac{1}{-\mu_p + \omega(\mathbf{p})} - \frac{1}{\omega^2(\mathbf{p})} + \frac{-\mu_p}{\omega^3(\mathbf{p})} \right]$$
$$- \frac{\lambda_R^2}{2} \int \frac{d^3p}{(2\pi)^3} \left[\frac{1}{\omega(\mathbf{p})} \frac{1}{H_0 - E + \omega(\mathbf{p})} - \frac{1}{\omega^2(\mathbf{p})} + \frac{H_0 - E}{\omega^3(\mathbf{p})} \right] \cdots$$
$$\Phi_R(E) = (H_0 - E + \mu_p)$$
$$\times \left\{ 1 + \frac{\lambda_R^2}{2} \int \frac{d^3p}{(2\pi)^3} \left[\frac{1}{\omega(\mathbf{p}) \left[H_0 - E + \omega(\mathbf{p})\right] \left[-\mu_p + \omega(\mathbf{p})\right]} - \frac{1}{\omega^3(\mathbf{p})} \right] \right\}$$
$$- \lambda_R^2 \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{a^\dagger(\mathbf{q})}{\sqrt{2\omega(\mathbf{q})}} \frac{1}{H_0 - E + \omega(\mathbf{q}) + \omega(\mathbf{p})} \frac{a(\mathbf{p})}{\sqrt{2\omega(\mathbf{p})}} . \quad (3.51)$$

In Section 3.6, we will use Eq. (3.51) to analyze the asymptotic limit of the theory with the assistance of the asymptotic limit of the principal operator.

3.2. Quantum Field Theory on General Static Riemannian Manifolds

We will summarize the necessary tools before going into the details of the construction of the model on Riemannian manifolds [73]. We consider $4d$ Riemannian manifold equipped with a metric structure which is static. That is to say, there is a timelike Killing vector field and there is a family of spacelike hypersurfaces orthogonal to the Killing vector everywhere. Alternatively, there is a coordinate system in which not only are the metric components $g_{\mu\nu}$ independent of the time coordinate, but also $g_{0j} = 0$ for $j \neq 0$. Therefore, the spacetime under consideration is in the class of globally hyperbolic spacetimes with the topology $\mathcal{M} = \mathcal{R} \times \Sigma$, which means that $\mathcal{M}$ admits a foliation of $\mathcal{M}$ into 3-dimensional submanifolds Σ, all of whose topologies are the same. Global hyperbolicity is crucial in order to obtain the solution of the wave equation. Since we need to have a well-posed Cauchy problem and being globally hyperbolic is equivalent to possesssing a Cauchy surface. If one has a causal curve, which is defined as the curve whose tangent vector is everywhere timelike or null, then one can define a Cauchy surface. A spacelike hypersurface is called a Cauchy surface for a pseudo Riemannian manifold $\mathcal{M}$ only if it is intersected exactly once by every causal curve in $\mathcal{M}$. In this respect, each leaf of the foliation of $\mathcal{M}$ is a Cauchy surface. Therefore it is possible to have a complete and nonredundant set of normal modes provided that one can solve the wave equation by separation of variables and this is the case for static Riemannian manifolds.

It is assumed that the action of a bosonic field can be given by

$$S = \int d^4x \sqrt{\det g} \frac{1}{2} \left(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi^2 - \xi R \phi^2 \right) , \tag{3.52}$$

where ξ is a dimensionless constant and R is the curvature scalar of the manifold. The indefinite analogue of the Laplace-Beltrami operator, the so-called wave operator obtained through the covariant derivative, and the resulting field equations are given by, respectively,

$$\Box \phi = \frac{1}{\sqrt{\det g}} \partial_\mu \left[g^{\mu\nu} \sqrt{\det g} \partial_\nu \phi \right] , \tag{3.53}$$

$$\Box \phi + (m^2 + \xi R) \phi = 0 . \tag{3.54}$$

Since the metric is static, the field equations can be solved by separation of variables. The problem is, then, reduced to solving the eigenvalue equation for the operator L.

If the bosonic field is decomposed as

$$\phi(t,x) = \phi_j(x)e^{\mp i\omega(j)t}\,, \tag{3.55}$$

then the eigenvalue equation can be given by

$$\begin{aligned} L\phi_j &= g_{00}\left[\frac{1}{\sqrt{\det g}}\partial_k(\sqrt{\det g}g^{kl}\partial_l\phi_j) + (m^2+\xi R)\phi_j\right] \\ &= \omega^2(j)\phi_j\,. \end{aligned} \tag{3.56}$$

The operator L is formally self-adjoint with respect to $\mathcal{L}^2$ inner product defined through

$$(\phi_1,\phi_2) = \int d^3x\,\sqrt{\det g}g^{00}\phi_1^*(x)\phi_2(x)\,. \tag{3.57}$$

Any function in the Hilbert space defined by that inner product can be expanded in terms of the solutions of Eq. (3.56), namely the eigenfunctions of that operator, as

$$\phi(x) = \int d\mu(j)\,\phi(j)\phi_j(x)\,, \tag{3.58}$$

where $\int d\mu(j)$ is the measure and it contains point spectrum or discrete spectrum or both. By means of that expansion, the scalar product defined by Eq. (3.57) can be given by

$$(\phi_1,\phi_2) = \int d\mu(j)\,\phi_1^*(j)\phi_2(j)\,. \tag{3.59}$$

We have also the orthonormality and the completeness relations. They are,

$$\delta(j,k) = \int d^3x\,\sqrt{\det g}g^{00}\phi_j^*(x)\phi_k(x)\,, \tag{3.60}$$

$$\delta_g^{(3)}(x,x') = \int d\mu(j)\,\phi_j^*(x)\phi_j(x')\,. \tag{3.61}$$

The general solution of the field equation can be decomposed into positive and negative parts and they are given by, respectively,

$$\phi(t,x) = \int \frac{d\mu(j)}{\sqrt{2\omega(j)}} \left[a(j)\phi_j(x)e^{-i\omega(j)t} + a^\dagger(j)\phi_j^*(x)e^{i\omega(j)t}\right] , \tag{3.62}$$

$$\phi^{(+)}(x) = \int \frac{d\mu(j)}{\sqrt{2\omega(j)}} \phi_j(x)a(j) , \tag{3.63}$$

$$\phi^{(-)}(x) = \int \frac{d\mu(j)}{\sqrt{2\omega(j)}} \phi_j^*(x)a^\dagger(j) , \tag{3.64}$$

where $a(j)$ and $a^\dagger(j)$ are the annihilation and the creation operators. A conjugate momentum and a Hamiltonian should be defined in order to quantize the field canonically. The conjugate momentum is

$$\pi(t,x) = g^{00}\sqrt{\det g}\partial_0\phi , \tag{3.65}$$

and the Hamiltonian is just the Legendre transform of the Lagrangian. With the help of them, one can calculate the equal-time canonical commutation relations both between the field and the conjugate momentum, and then between the creation and annihilation operators.

$$[\phi(t,x), \pi(t,x')] = i\sqrt{\det g}g^{00}\delta_g^{(3)}(x,x') , \quad \left[a(j), a^\dagger(k)\right] = \delta(j,k) . \tag{3.66}$$

The free Hamiltonian in terms of creation and annihilation operators is given by

$$H_0 = \int d\mu(j)\, \omega(j)a^\dagger(j)a(j) . \tag{3.67}$$

3.3. Construction of the Heat Kernel on Manifolds

Before carrying on to the construction of the heat kernel on a manifold, we would like to give a brief introduction to how the Laplacian operator can be generalized to manifold cases [72, 73, 74, 75]. In Section 2.1 we used the heat kernel for the Euclidean space $\mathbb{R}^4$ in the zeta-function regularization of the corresponding functional determinant. For the d-dimensional Euclidean space $\mathbb{R}^d$ the heat kernel can be defined by the unique positive solution of the following the Cauchy problem in $(0, +\infty) \times \mathbb{R}^d$ with the specified initial condition,

$$\left[\frac{\partial}{\partial u} + \Delta\right] K_u(x,y) = 0\,,$$
$$K_{u\to 0^+}(x,y) = \delta(x,y)\,, \tag{3.68}$$

such that the heat kernel is given by the formula,

$$K_u(x,y) = \frac{1}{(4\pi u)^{n/2}} e^{-|x-y|^2/4u}\,. \tag{3.69}$$

From the probabilistic point of view, the heat kernel can also be thought as the transition density of the Brownian motion on the manifold, and in the current situation it is the one in $\mathbb{R}^d$.

What we know from the differential geometry is that the natural way to generalize the Laplace operator on $\mathbb{R}$ resides on introducing additional geometric data in the form of a Riemannian metric. A metric structure can be thought as a geometric datum in the sense that in order to do geometry we need to know how to measure lengths of curves on the manifold or basically how to measure the length of the tangent vector of the corresponding curve. Moreover, if one treats the Laplacian as an operator acting on the Hilbert space $\mathcal{L}^2(\mathcal{M})$ of real valued functions, it is crucial to introduce a volume form so as to define a positive definite inner product in this space, which is a must to have a Hilbert space, and we also need a metric structure to define a volume form. Therefore, it can be thought that we have a hermitian vector bundle whose base manifold is the Riemannian manifold $\mathcal{M}$ with the metric g and each fiber on it is equipped with the inner product defined on the sections with respect to the top volume form. We can also call this structure a $U(r)$ bundle, where r stands for the dimensions of the fiber, since the natural symmetry group of the complex inner product in each fiber is the unitary group $U(r)$.

Let $\mathcal{M}$ be an arbitrary smooth connected d-dimensional Riemannian manifold with the Riemannian metric g on it. Although there are no preferred coordinate systems on $\mathcal{M}$, one can still define the gradient of a smooth function f as a vector field on $\mathcal{M}$ in a local chart $x^1, x^2, \ldots, x^d$ and it is

$$(\nabla f)^\mu = g^{\mu\nu} \frac{\partial f}{\partial x^\nu} \tag{3.70}$$

where summation is assumed over the repeated indices which appear as both superscripts and subscripts. Let X be a smooth vector field on $\mathcal{M}$, the divergence $\mathrm{div}X$ defines a scalar function on $\mathcal{M}$ and it is given in the same local chart by

$$\mathrm{div}X = \frac{1}{\sqrt{\mathrm{det}g}} \frac{\partial}{\partial x^\mu} \left(\sqrt{\mathrm{det}g} X^\mu \right) . \tag{3.71}$$

The Riemannian volume form of a Riemannian metric on $\mathcal{M}$ is defined to be the top dimensional form dvol which in that local chart is given by

$$\mathrm{dvol} = \sqrt{\mathrm{det}g}\, dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n . \tag{3.72}$$

For any smooth functions f and h, the Stokes' theorem implies that

$$-\int_\Omega \mathrm{dvol}\, \langle \nabla f, \nabla h \rangle = \int_\Omega \mathrm{dvol}\, f \mathrm{div} \nabla h \,, \tag{3.73}$$

Ω being a pre-compact open subset of $\mathcal{M}$, and provided that one of the functions above has compact support which means that one of them vanishes in a neighbourhood of the boundary $\partial\Omega$. $\langle \cdot, \cdot \rangle$ stands for the global inner products on functions and the vector fields induced by the standard dot product. The operator

$$\Delta = -\mathrm{div} \circ \nabla \tag{3.74}$$

is called the Laplace or Laplace-Beltrami operator of the Riemannian manifold $\mathcal{M}$ and it can be given in any local chart by

$$\Delta = -\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^{\mu}}\left(\sqrt{\det g}g^{\mu\nu}\frac{\partial}{\partial x^{\nu}}\right) , \tag{3.75}$$

which is covariant since in any other chart this operator will have the same form.

We can now construct the heat kernel on a manifold by using the Laplace operator defined on it. Let $\mathcal{M}$ be an arbitrary smooth connected Riemannian manifold and Δ be the Laplace operator associated with the Riemannian metric g. By definition, the heat kernel is the fundamental solution of the heat equation on a d-dimensional manifold $\mathcal{M}$,

$$\left[\frac{\partial}{\partial u} + L\right] f = 0 \tag{3.76}$$

where L is a formally self-adjoint, second-order operator on the functions on $\mathcal{M}$ and it will be taken as the Laplace operator of the manifold for the rest. That is to say, one has the following Cauchy problem

$$\begin{aligned} \left[\frac{\partial}{\partial u} + \Delta\right] f &= 0 \\ f|_{u=0} &= f_0(x) \end{aligned} \tag{3.77}$$

which has a solution,

$$f(u,x) = \int_{\mathcal{M}} d_g^d y\, K(u|x,y) f_0(y) \,, \tag{3.78}$$

$K(u|x,y)$ being the heat kernel on $\mathcal{M}$ and $d_g^d y$ stands for the top volume form $\mathrm{dvol}(y)$. The heat kernel is a function on $(0,+\infty)\times\mathcal{M}\times\mathcal{M}$ and it is, also, the solution of the heat equation.

$$\left[\frac{\partial}{\partial u} + \Delta\right] K(u|x,y) = 0 \,. \tag{3.79}$$

We will abuse the notation such that $K(u|x,y)$ and $K_u(x,y)$ will be used interchangeably for the rest. Moreover, if $f_0 \geq 0$, then this solution defines the minimal positive solution of the Cauchy problem. The initial condition implies that

$$\lim_{u\to 0^+} K(u|x,y) \to \delta_g(x,y)\,, \tag{3.80}$$

$\delta_g(x,y)$ being the covariant Dirac distribution and it is represented by the ordinary Dirac delta density divided by the square root of the metric determinant in local coordinates. It is easy to see that at least formally, the heat equation has the following operator solution,

$$f = e^{-\Delta t} f_0\,. \tag{3.81}$$

One can write down this equation in the orthonormal basis $\{\phi_k\}$ of $\mathcal{L}^2(\mathcal{M})$ satisfying $\Delta\phi_k = \lambda_k\phi_k$ such that the expansion of the function $f_0 \in \mathcal{L}^2(\mathcal{M})$ in this basis can be given by

$$f = \sum_k a_k\phi_k\,, \tag{3.82}$$

which implies

$$a_k = \int_{\mathcal{M}} d_g^d y\, f_0(y)\phi_k(y)\,. \tag{3.83}$$

After rewriting Eq. (3.81) in the orthonormal basis, one has

$$\begin{aligned} f(u,x) &= \sum_k a_k e^{-\lambda_k(\mathcal{M})t}\phi_k(x) \\ &= \int_{\mathcal{M}} d_g^d y \left\{ \sum_k e^{-\lambda_k(\mathcal{M})t}\phi_k(x)\phi_k(y) \right\} f_0(y)\,, \end{aligned} \tag{3.84}$$

in which the expression in the curly brackets is the formal expansion of the heat kernel in the orthonormal basis ϕ_k of $\mathcal{L}^2(\mathcal{M})$ so that the heat kernel $K(u|x,y)$ is just the integral kernel of the operator $e^{-\Delta t}$ realized in this orthonormal basis.

Besides being the fundamental solution, the heat kernel possesses general properties. One of which is the symmetry property and it is the reflection of the self-adjointness of the Laplace operator, which leads to the self-adjointness of the heat operator. The usage of the self-adjointness of the heat operator,

$$\begin{aligned}\langle e^{-\Delta t} f, h\rangle &= \int_{\mathcal{M}} d_g^d x \int_{\mathcal{M}} d_g^d y\, K(u|x,y) f(y) h(x) \\ &= \langle f, e^{-\Delta t} h\rangle \\ &= \int_{\mathcal{M}} d_g^d y \int_{\mathcal{M}} d_g^d x\, K(u|y,x) h(x) f(y)\,, \end{aligned} \tag{3.85}$$

then, brings about the following symmetry property,

$$K(u|x,y) = K(u|y,x)\,. \tag{3.86}$$

Other property of the heat kernel is the reproducing or the semigroup property which is

$$K(u|,x,y) = \int_{\mathcal{M}} d_g^d z\, K(v|x,z) K(u-v|z,y)\,,\ \forall x,y \in \mathcal{M}\,,\ 0 < v < u\,. \tag{3.87}$$

This property results in the semigroup theory of the heat operator and we can write down this property as the following identity,

$$e^{-\Delta(u-v)} e^{-\Delta v} = e^{-\Delta u}\,. \tag{3.88}$$

It is not hard to verify this property. By means of the identity above, one has

$$\begin{aligned} e^{-\Delta(u-v)} e^{-\Delta v} f_0(x) &= e^{-\Delta(u-v)} \left(\int_{\mathcal{M}} d_g^d y\, K(v|z,y) f_0(y) \right)(x) \\ &= \int_{\mathcal{M}} d_g^d y \left(\int_{\mathcal{M}} d_g^d z\, K(u-v|x,z) K(v|z,y) \right) f_0(y)\,, \end{aligned} \tag{3.89}$$

which gives rise to that the integral kernel of the operator $e^{-\Delta(u-v)}e^{-\Delta v}$ is

$$\int_{\mathcal{M}} d_g^d z\, K(u-v|x,z)K(v|z,y)\,. \tag{3.90}$$

What we know is that this kernel should satisfy the heat kernel, as well. Therefore, applying the initial condition converts this kernel into a covariant Dirac delta distribution.

$$\begin{aligned}
&\lim_{u\to 0^+}\int_{\mathcal{M}} d_g^d y \left(\int_{\mathcal{M}} d_g^d z\, K(u-v|x,z)K(v|z,y)\right) f_0(y) \\
&\quad= \lim_{u\to 0^+}\int_{\mathcal{M}} d_g^d z\, K(u-v|,x,z) \lim_{v\to 0^+}\int_{\mathcal{M}} d_g^d y\, K(v|,z,y) f_0(y) \\
&\quad= \lim_{u\to 0^+}\int_{\mathcal{M}} d_g^d z\, K(u|x,z) f_0(z) \\
&\quad= f_0(x)\,,
\end{aligned} \tag{3.91}$$

which assures that the followings are equivalent,

$$\begin{aligned}
K(u|,x,y) &= \int_{\mathcal{M}} d_g^d z\, K(v|x,z)K(u-v|z,y)\,, \\
e^{-\Delta u} &= e^{-\Delta v}e^{-\Delta(u-v)}\,.
\end{aligned} \tag{3.92}$$

This property is intuitively obvious since the heat flow on $\mathcal{M}$ with the given initial temperature distribution is described by the solution of the heat equation. Thus, two different heat flows up to time v and $u-v$, modeled by the corresponding heat operators $e^{-\Delta v}$ and $e^{-\Delta(u-v)}$, are supposed to constitute a heat flow at time u, generated by the heat operator $e^{-\Delta u}$ with the given initial temperature distribution.

Apart from the properties mentioned above, the heat kernel has also two crucial properties which are the nontrivial consequences of the maximum/minimum principle, [75, 76]. First one is the positivity of the heat kernel:

$$K(u|x,y) \geq 0\,,\ \forall x,y \in \mathcal{M} \text{ and } u>0\,. \tag{3.93}$$

Although it seems that the positivity of the heat kernel may be verified with the assistance of the eigenfunction expansion at first glance, this is not at all obvious, on the grounds that the eigenfunctions ϕ_k are signed except for ϕ_1. Second one is that the total probability does not exceed 1:

$$\int_{\mathcal{M}} d_g^d y \, K(u|x,y) \leq 1 \,, \, \forall x \in \mathcal{M} \text{ and } u > 0 \,, \tag{3.94}$$

in which the equal sign holds only for the Riemannian manifolds which are stochastically complete.

Furthermore, the heat kernel admits an asymptotic expansion of the kind,

$$K(u|x,y) \simeq \frac{e^{-d(x,y)^2/4u}}{(4\pi u)^{d/n}} \sum_{k=0}^{\infty} a_k(x,y) u^{(k-d)/n} \,, \tag{3.95}$$

where n is the order of the operator, d is the dimension of the manifold and $d(x,y) \equiv \text{dist}(x,y)$ is the geodesic distance between the points x and y. This expansion is truly local, which means that it exists when x is near y and u is near 0. There is also similar expansion when the manifold has a boundary. When the manifold does not have a boundary, the coefficients with odd k are vanishing, that allows us to obtain the expansion in a much more simpler form. For the Laplace operators on manifolds without boundary, we have the following expansion for the heat kernel, whose values are on the diagonal,

$$K(u|x,x) \simeq \frac{1}{(4\pi u)^{d/2}} \sum_{k=0}^{\infty} a_k(x) u^k \,, \tag{3.96}$$

in which the expansion coefficients, widely known as the Seeley-DeWitt coefficients, $a_k(x) \equiv a_k(x,x)$ are independent of the coordinate systems. In other words, they can be thought as an operator in the fiber of the gauge on the point x on the manifold. Thus, they are local invariant polynomials depending on curvature, torsion, the potential part of the Laplace operator and their covariant derivatives. The literature on the calculation of these coefficients is vast, and some of the elaborate methods using different algorithms to calculate them exist [11, 77, 78, 79, 80, 81].

3.4. The Relativistic Lee Model on Riemannian Manifolds

Since the source is heavy and essentially sits at a point in space, one has to find a way to describe this situation. We use the same trick which was used by Erman *et al.* [51]. The interaction is introduced by a convolution of the bosonic field with a heat kernel whose index is just a short-time cut-off. In the limit as the cut-off goes to zero, the heat kernel becomes a Dirac delta function, and hence the convolution in this limit allows us to find the interaction occurring at some fixed point in space. Utilizing the short-time behavior of the heat kernel is a nice way to analyze and control the high energy behavior of the expressions, so this allows us to deal with the ultra-violet divergence in the theory. The cut-off Hamiltonian of the theory on a Riemannian manifold, specified previously, is

$$\begin{aligned} H_\epsilon = H_0 \left[\chi_+ \otimes \chi_+^\dagger + Z(\epsilon)\chi_- \otimes \chi_-^\dagger\right] + Z(\epsilon)\mu(\epsilon)\frac{1-\sigma_3}{2} \\ + \sqrt{Z(\epsilon)}\lambda(\epsilon)\left[\sigma_+\phi_\epsilon^{(-)}(\overline{x}) + \sigma_-\phi_\epsilon^{(+)}(\overline{x})\right] , \end{aligned} \tag{3.97}$$

in which the smeared out positive and negative frequency parts of the field are given by

$$\phi_\epsilon^{(+)}(\overline{x}) = \int d_g^3x\, K_{\epsilon/2}(\overline{x}, x)\phi^{(+)}(x)\,, \tag{3.98}$$

$$\phi_\epsilon^{(-)}(\overline{x}) = \int d_g^3x\, K_{\epsilon/2}(\overline{x}, x)\phi^{(-)}(x)\,, \tag{3.99}$$

where $\int d_g^3x \equiv \int d^3x\,\sqrt{\det g}g^{00}$, $\overline{x}$ is a fixed point on the manifold, and $\epsilon/2$ is chosen for convenience.

The resolvent is again the formal inverse of the operator,

$$H_\epsilon - E = \begin{bmatrix} H_0 - E & \lambda(\epsilon)\sqrt{Z(\epsilon)}\phi_\epsilon^{(-)}(\overline{x}) \\ \sqrt{Z(\epsilon)}\lambda(\epsilon)\phi_\epsilon^{(+)}(\overline{x}) & Z(\epsilon)\left[H_0 - E + \mu(\epsilon)\right] \end{bmatrix} . \tag{3.100}$$

The cut-off principal operator can be calculated algebraically by the resolvent as in the flat case and is given by

$$\begin{aligned}
\Phi_\epsilon(E) &= Z(\epsilon)\left\{H_0 - E + \mu(\epsilon) - \lambda^2(\epsilon)\phi_\epsilon^{(+)}(\overline{x})\frac{1}{H_0 - E}\phi_\epsilon^{(-)}(\overline{x})\right\} \\
&= Z(\epsilon)\Bigg\{H_0 - E + \mu(\epsilon) - \lambda^2(\epsilon)\int d_g^3x\, d_g^3y\, K_{\epsilon/2}(\overline{x},x)K_{\epsilon/2}(\overline{x},y) \\
&\quad \times \int \frac{d\mu(j)}{\sqrt{2\omega(j)}}\frac{d\mu(k)}{\sqrt{2\omega(k)}}\,\phi_j(x)\phi_k^*(y)a(j)\frac{1}{H_0 - E}a^\dagger(k)\Bigg\}\,. \qquad (3.101)
\end{aligned}$$

Henceforth, the same game is played in order to renormalize the theory. First of all, one should normal-order this object by letting the creation operator stand on the right and the annihilation operator stand on the left in the fourth term in Eq. (3.101). If the following operator equalities are used,

$$\frac{1}{H_0 - E}a^\dagger(k) = a^\dagger(k)\frac{1}{H_0 - E + \omega(k)}\,, \qquad (3.102)$$

$$a(j)\frac{1}{H_0 - E + \omega(k)} = \frac{1}{H_0 - E + \omega(k) + \omega(j)}a(j)\,, \qquad (3.103)$$

then the principal operator becomes

$$\begin{aligned}
\Phi_\epsilon(E) = Z(\epsilon)\Bigg\{& H_0 - E + \mu(\epsilon) - \lambda^2(\epsilon)\int d_g^3x\, d_g^3y\, K_{\epsilon/2}(\overline{x},x)K_{\epsilon/2}(\overline{x},y) \\
&\times\Bigg[\int d\mu(j)\,\phi_j(x)\phi_j^*(y)\frac{1}{2\omega(j)}\frac{1}{H_0 - E - \omega(j)} \\
&+ \int \frac{d\mu(j)}{\sqrt{2\omega(j)}}\frac{d\mu(k)}{\sqrt{2\omega(k)}}\,\phi_j(x)\phi_k^*(y)a^\dagger(k)\frac{1}{H_0 - E + \omega(k) + \omega(j)}a(j)\Bigg]\Bigg\}\,. \qquad (3.104)
\end{aligned}$$

We will again use a Feynman parametrization in order to combine the inverse powers in the second line of the above equation,

$$\frac{1}{\omega(j)}\frac{1}{H_0 - E + \omega(j)} = \int_0^1 d\zeta\,\frac{1}{[(H_0 - E)\zeta + \omega(j)]^2}\,, \qquad (3.105)$$

and do an exponentiation,

$$\frac{1}{[(H_0 - E)\zeta + \omega(j)]^2} = \int_0^\infty ds\, s e^{-s\omega(j)} e^{-s(H_0-E)\zeta}\,. \tag{3.106}$$

After these tricks, the multiplication of the inverse powers can be given by

$$\frac{1}{\omega(j)} \frac{1}{H_0 - E + \omega(j)} = \int_0^\infty ds\, e^{-s\omega(j)} \frac{1}{H_0 - E} \left[1 - e^{-s(H_0-E)}\right]\,. \tag{3.107}$$

By means of the subordination identity, $\omega(j)$ can be turned into $\omega^2(j)$ which allows us to convert $e^{-s\omega^2(j)}$ into a heat kernel via sandwiching it with the eigenfunctions of the operator L:

$$K_u(y, x) = \int d\mu(j)\, \phi_j(x)\phi_j^*(y) e^{-u\omega^2(j)}\,. \tag{3.108}$$

Note that in some cases, especially when the scalar curvature coupling is ignored, it is more natural to take the mass away and define a heat kernel via the Laplacian part only. We hope that the context makes this clear in the following discussions. The semigroup property of the heat kernel also allows us to combine the convoluted heat kernels as

$$K_{u+\epsilon}(\overline{x}, \overline{x}) = \int d_g^3x\, d_g^3y\, K_{\epsilon/2}(\overline{x}, x) K_u(x, y) K_{\epsilon/2}(y, \overline{x})\,. \tag{3.109}$$

If all of them are taken into account, we reach the following form of the principal operator,

$$\begin{aligned}\Phi_\epsilon(E) = Z(\epsilon)\Bigg\{ & H_0 - E + \mu(\epsilon) - \lambda^2(\epsilon) \int d_g^3x\, d_g^3y\, K_{\epsilon/2}(\overline{x}, x) K_{\epsilon/2}(\overline{x}, y) \\ & \times \int \frac{d\mu(j)}{\sqrt{2\omega(j)}} \frac{d\mu(k)}{\sqrt{2\omega(k)}}\, \phi_j(x)\phi_k^*(y) a^\dagger(k) \frac{1}{H_0 - E + \omega(k) + \omega(j)} a(j) \\ & - \frac{\lambda^2(\epsilon)}{4\sqrt{\pi}} \int_0^\infty du \int_0^\infty ds\, s e^{-s^2/4} K_{u+\epsilon}(\overline{x}, \overline{x}) \frac{\left[1 - e^{-s\sqrt{u}(H_0-E)}\right]}{\sqrt{u}(H_0 - E)} \Bigg\}\,. \end{aligned} \tag{3.110}$$

We can also exponentiate the fraction in the fourth term:

$$\frac{1}{H_0 - E + \omega(k) + \omega(j)} = \int_0^\infty ds\, e^{-s\omega(k)} e^{-s\omega(j)} e^{-s(H_0-E)}\,, \tag{3.111}$$

Moreover both $a(j)$ and $a^\dagger(j)$ can be given by the field itself by an inverse transform:

$$a(j) = \sqrt{2\omega(j)} \int d_g^3 z\, \phi_j^*(z) \phi^{(+)}(z)\,, \tag{3.112}$$

$$a^\dagger(k) = \sqrt{2\omega(k)} \int d_g^3 z\, \phi_k(z) \phi^{(-)}(z)\,. \tag{3.113}$$

Equations (3.111), (3.112) and applying reproducing property one more time brings the principal operator to the following form

$$\begin{aligned}
\Phi_\epsilon(E) = Z(\epsilon) \Bigg\{ & H_0 - E + \mu(\epsilon) \\
& - \frac{\lambda^2(\epsilon)}{4\sqrt{\pi}} \int_0^\infty du \int_0^\infty ds\, s e^{-s^2/4} K_{u+\epsilon}(\overline{x}, \overline{x}) \frac{\left[1 - e^{-s\sqrt{u}(H_0-E)}\right]}{\sqrt{u}(H_0-E)} \\
& - \frac{\lambda^2(\epsilon)}{4\pi} \int d_g^3 x\, d_g^3 y \int_0^\infty ds\, s^2 \int_0^\infty du_1 \frac{e^{-s^2/4u_1}}{u_1^{3/2}} \int_0^\infty du_2 \frac{e^{-s^2/4u_2}}{u_2^{3/2}} \\
& \times K_{\epsilon/2+u_1}(\overline{x}, y) K_{\epsilon/2+u_2}(\overline{x}, x) \phi^{(-)}(y) e^{-s(H_0-E)} \phi^{(+)}(x) \Bigg\}\,.
\end{aligned} \tag{3.114}$$

We are now ready to renormalize the cut-off principal operator via redefining the cut-off dependent parameters in terms of renormalized ones and divergent parts so as to cancel the divergences emerging from the normal ordering. As in the flat case we, first, determine which powers of u in the u-integral produce divergence in the fourth term. In order to do that one should use the short-time expansion of the heat kernel which generates some inverse powers of u. Afterwards, we combine it with the powers of u coming from the expansion of the exponential in s. Well-known short-time expansion of the heat kernel (see [74], for example) is given by

$$K_u(x,x) \simeq \frac{1}{(4\pi u)^{d/2}} \sum_{n=0}^\infty a_n(x) u^n\,, \tag{3.115}$$

where a_n are universal polynomials in the curvature tensor, its covariant derivatives and various contractions thereof. We choose a normalization which makes a_0 is equal to 1. The mass term in the Laplacian does not affect the asymptotic expansion. It can immediately be seen that only the first term in this short-time expansion contributes to divergences when it is combined with the factors coming from the exponential. The following choices are sufficient to kill all the divergences,

$$\begin{aligned} \frac{\mu(\epsilon)}{\lambda^2(\epsilon)} &= \frac{\mu_R}{\lambda_R^2} + \frac{1}{4\sqrt{\pi}} \int_0^\infty du \int_0^\infty ds\, s^2 e^{-s^2/4} K_{u+\epsilon}(\overline{x}, \overline{x}) \\ &= \frac{\mu_R}{\lambda_R^2} + \frac{1}{2} \int_0^\infty du\, K_{u+\epsilon}(\overline{x}, \overline{x})\,, \end{aligned} \tag{3.116}$$

$$\begin{aligned} \frac{1}{\lambda^2(\epsilon)} &= \frac{1}{\lambda_R^2} - \frac{1}{8\sqrt{\pi}} \int_0^\infty du\, \sqrt{u} \int_0^\infty ds\, s^3 e^{-s^2/4} K_{u+\epsilon}(\overline{x}, \overline{x}) \\ &= \frac{1}{\lambda_R^2} - \frac{1}{\sqrt{\pi}} \int_0^\infty du\, \sqrt{u} K_{u+\epsilon}(\overline{x}, \overline{x})\,. \end{aligned} \tag{3.117}$$

By placing the short-time expansion of the heat kernel into above redefinitions and then taking the asymptotic limits of the integrals above successively, one can find out how the cut-off parameter ϵ controls the divergences and one gets

$$\frac{\mu(\epsilon)}{\lambda^2(\epsilon)} \simeq \frac{\mu_R}{\lambda_R^2} + \frac{1}{8\pi^{3/2}} \frac{1}{\sqrt{\epsilon}} \quad \text{as} \quad \epsilon \to 0^+\,, \tag{3.118}$$

$$\frac{1}{\lambda^2(\epsilon)} \simeq \frac{1}{\lambda_R^2} + \frac{1}{8\pi^2} \ln(\epsilon) \quad \text{as} \quad \epsilon \to 0^+\,. \tag{3.119}$$

It is striking that these are exactly the same results which we have found in the flat case. This equality arises from the fact that in this language short-time behavior captures the high energy behavior. Therefore, we would not expect any contribution from the geometry itself, only extreme local structure which is Euclidean determines the divergence. It is also easy to conclude that point from the short-time expansion of the heat kernel since only the first term contributes to divergences. Moreover, the first expansion coefficient a_0 does not contain the curvature scalar, it is just equal to 1. Yet, the geometry is very important for the spectrum of the theory. The principal operator is given in terms of the heat kernel at arbitrary times as well as its values at separate points.

After replacing the parameters by their renormalized counterparts and successively taking the limit $\epsilon \to 0^+$, we can obtain the renormalized principal operator, which is given by

$$\begin{aligned}
\frac{\Phi_R(E)}{\lambda_R^2} = \frac{(H_0 - E)}{\lambda_R^2} + \frac{\mu_R}{\lambda_R^2} - \frac{1}{4\sqrt{\pi}} \int_0^\infty du \int_0^\infty ds\, s e^{-s^2/4} K_u(\overline{x}, \overline{x}) \frac{1}{\sqrt{u}(H_0 - E)} \\
\times \left[1 - s\sqrt{u}(H_0 - E) + \frac{1}{2} s^2 u (H_0 - E)^2 - e^{-s\sqrt{u}(H_0 - E)}\right] \\
- \frac{1}{4\pi} \int d_g^3 x\, d_g^3 y \int_0^\infty ds\, s^2 \int_0^\infty du_1 \frac{e^{-s^2/4u_1}}{u_1^{3/2}} \int_0^\infty du_2 \frac{e^{-s^2/4u_2}}{u_2^{3/2}} \\
\times K_{u_1}(\overline{x}, y) K_{u_2}(\overline{x}, x) \phi^{(-)}(y) e^{-s(H_0 - E)} \phi^{(+)}(x) \,. \qquad (3.120)
\end{aligned}$$

After imposing $\Phi_R(E = \mu_p)|0\rangle \equiv 0$ and doing little algebra, one can also obtain the principal operator in terms of physical mass difference as in the flat case,

$$\begin{aligned}
\Phi_R(E) = (H_0 - E + \mu_p) \Bigg\{ 1 + \frac{\lambda_R^2}{2} \int d\mu(j)\, \phi_j^*(\overline{x}) \phi_j(\overline{x}) \\
\times \left[-\frac{1}{\omega(j)^3} + \frac{1}{\omega(j) \left[H_0 - E + \omega(j)\right] \left[-\mu_p + \omega(j)\right]} \right] \Bigg\} \\
- \lambda_R^2 \int \frac{d\mu(j)}{\sqrt{2\omega(j)}} \frac{d\mu(k)}{\sqrt{2\omega(k)}} \phi_j(\overline{x}) \phi_k^*(\overline{x}) a^\dagger(k) \frac{1}{H_0 - E + \omega(k) + \omega(j)} a(j) \,. \\
(3.121)
\end{aligned}$$

We see that the renormalized Hamiltonian, after the same calculations done in the flat case, is given by

$$H_R = H_0 \left[\chi_+ \otimes \chi_+^\dagger + \chi_- \otimes \chi_-^\dagger\right] + \mu_R \frac{1 - \sigma_3}{2} + \lambda_R \left[\sigma_+ \phi^{(-)}(\overline{x}) + \sigma_- \phi^{(+)}(\overline{x})\right] \,. \quad (3.122)$$

3.5. Cartan–Hadamard Manifolds

The renormalization of the Lee model both on a $\mathbb{R}^{3+1}$ space-time and on a general static Riemannian manifold without reference to the type of the manifold has been studied so far since it has been found that only the short-time asymptotics of the heat kernel determine the divergence structure of the model, and only a_0 term in the short-time expansion of the heat kernel is needed so as to cure the divergence encountered in the calculations. In the following sections we will study the aymptotic limits of the Lee model in the large boson limit, specifically $n \to \infty$, on different space-times with different dimensions. We will mainly concentrate on the following space-times which can be identified as $\mathbb{R}^{d-1+1}$ and $\mathbb{H}^{d-1} \times \mathbb{R}$. Here $\mathbb{H}^{d-1}$ is nothing but the hyperbolic space with the dimensions of $d-1$. $\mathbb{R}^d$ and $\mathbb{H}^d$ belong to the same class of manifolds, the so-called Cartan-Hadamard manifolds.

A manifold $\mathcal{M}$ is called a Cartan-Hadamard manifold if $\mathcal{M}$ is geodesically complete simply connected noncompact Riemannian manifold with nonpositive sectional curvature [72, 75]. It is diffeomorphic to $\mathbb{R}^d$ but not isometric. We mention here two important properties of this type of manifolds. One of which is that they are stochastically complete provided their sectional curvatures are bounded below. Another one is that these manifolds admit isoperimetric functions, which will be discussed later.

In terms of the heat kernel on the manifold, the former property is given by

$$\int_{\mathcal{M}} \mathrm{dvol}(y) K(u|x,y) = 1 \,, \quad \forall x \in \mathcal{M} \text{ and } t > 0 \,, \tag{3.123}$$

which basically means the conservation of heat property. In terms of the Brownian motion, this property implies that the total probability of the particle to be found on $\mathcal{M}$ is equal to 1. One of the important consequences of this property is that if the manifold is stochastically complete, then the minimal positive heat kernel of $\mathcal{M}$ is the unique heat kernel among all positive heat kernels satisfying, [75, 82],

$$\int_{\mathcal{M}} \mathrm{dvol}(y) K(u|x,y) \leq 1 \,, \quad \forall x \in \mathcal{M} \text{ and } t > 0 \,. \tag{3.124}$$

Moreover, the stochastic completeness can be tested by the criterion given in terms of the volume function of the geodesic ball. Let $B(x,r)$ be the geodesic ball on $\mathcal{M}$ of radius r centered at x and $V(x,r)$ be the volume function of $(\mathcal{M},\mu)$, to put it another way

$$B(x,r) \equiv \{y \in \mathcal{M} : d(x,y) < r\} \ ,$$
$$V(x,r) \equiv \mu\left(B(x,r)\right) \ , \tag{3.125}$$

$d(x,y)$ being the geodesic distance on $\mathcal{M}$ and μ being the Riemannian d-volume or measure. A geodesically complete manifold is called stochastically complete if, for some point $x \in \mathcal{M}$,

$$\int^{\infty} \frac{rdr}{\log V(x,r)} = \infty \, . \tag{3.126}$$

A convenient way to meet this criterion is mostly using the volume comparison theorems. For example, [72, 83], if the manifold in question is a geodesically complete manifold with bounded below Ricci curvature, then it follows from the Bishop-Gromov volume comparison theorem that

$$V(x,r) \leq e^{Cr} \, , \tag{3.127}$$

C being a positive constant and thus the manifold is stochastically complete.

The latter property to be worth mentioning is related to the isoperimetric inequalities. Here, we would like to emphasize that there are two classes of isoperimetric inequalities, namely geometric and physical isoperimetric inequalties, [75, 76, 82]. Given a d-dimensional noncompact Riemannian manifold $\mathcal{M}$, a geometric isoperimetric inequality relates the $(d-1)$-dimensional volume of $\partial\Omega$, where Ω is the union of a finite number of regular domains with compact closure in $\mathcal{M}$, to the d-dimensional volume of Ω. On the other hand, a physical isoperimetric inequality is an inequality relating the eigenvalues of an arbitrary domain Ω, with compact closure in $\mathcal{M}$, to the d-dimensional volume of Ω for a given metric. Let's make the notion of a geometric isoperimetric inequality more precise. A manifold $\mathcal{M}$ is called to admit the isoperimetric function I

if, for any precompact open set $\Omega \subset \mathcal{M}$ with smooth boundary,

$$\sigma(\partial\Omega) \geq I(|\Omega|)\,, \tag{3.128}$$

in which

$$|\Omega| \equiv \mu(\Omega)\,. \tag{3.129}$$

This isoperimetric function should be monotonically decreasing. It is a well-known fact that any Cartan-Hadamard manifold $\mathcal{M}$ of the dimension d admits the following isoperimetric function [75, 84, 85],

$$I(v) = cv^{(d-1)/d}\,, \quad \forall c > 0\,. \tag{3.130}$$

The derivation of physical isoperimetric inequalities from geometric ones is an important research area on its own, both in Riemannian geometry and in spectral theory, [75, 76, 82, 84, 86].

Apart from being an independent research area, isoperimetric inequalities come in useful when one needs to study large time decay of the heat kernel or more specifically to obtain upper bounds to the heat kernel under consideration as sharp as possible. Obtaining an upper bound to a heat kernel is, generally, accomplished in two stages. In the first place one tries to get an on-diagonal estimate with the assistance of appropriate inequalities, e.g. Sobolev's, Nash's, log-Sobolev, the Faber-Krahn's inequality and so on, generally deduced from geometric inequalities [75, 82]. Especially, it can be thought that obtaining a Gaussian upper bound of a heat kernel is nothing but obtaining an on-diagonal estimates since off-diagonal estimates are implied by on-diagonal estimates suitably modified with the Riemannian distance factor. Therefore, in the second stage this factor is introduced as an off-diagonal correction to the upper bounds in a Gausian exponential form containing the Riemannian distance. By Davies' theorem [87], this factor is shown to be relevant to the heat kernel upper bounds on arbitrary manifolds.

Let's go back to Cartan-Hadamard manifolds after emphasising the derivation of the physical isoperimetric inequalities form the geometric inequalities. As it is said before, $\mathbb{R}^d$ and $\mathbb{H}^d$ are Cartan-Hadamard manifolds. Whereas the heat kernel in the 3-dimensonal Euclidean space $\mathbb{R}^3$ [75, 86] takes the form

$$K(u|x,y) = \frac{1}{(4\pi u)^{3/2}} e^{-d^2(x,y)/4u}\,, \tag{3.131}$$

the heat kernel in the 3-dimensional hyperbolic space $\mathbb{H}^3$ [75, 86, 88] is given by

$$K(u|x,y) = \frac{1}{(4\pi u)^{3/2}} \frac{d(x,y)}{\sinh d(x,y)} e^{-d^2(x,y)/4u-u}\,. \tag{3.132}$$

$d(x,y)$ being the geodesic distance, whose definition is given in Appendix C. The distinctions between the geometries of the Euclidean and hyperbolic spaces are encoded in the extra terms to be found when their heat kernels are compared. It turns out that the Gaussian exponential term is not accounted for possessing specific information of the geometry of the manifold for it is little sensitive to the underlying geometry(apart from the metric). On the other hand, it rather reflects characteristic of the heat equation. This is also true for the off-diagonal correction to the upper bounds of the heat kernel estimates mentioned in the previous paragraph. It is obvious from the heat kernel of $\mathbb{R}^3$ that its on-diagonal behaviour in the limit $t \to 0^+$ is the same as the one in the limit $t \to \infty$ due to the fact that there is a scale transformation in $\mathbb{R}^d$. However, this is not the case for $\mathbb{H}^3$. Whereas the on-diagonal behaviour of the heat kernel of $\mathbb{H}^3$ can be characterized by the function $ct^{-d/2}$ with $c > 0$ as $t \to 0^+$, its heat kernel decreases as e^{-ct} with $c > 0$ as $t \to \infty$. Therefore, sharp estimates for small and large time should be considered separately. The sharp uniform estimates of the heat kernel on the generalized hyperbolic space $\mathbb{H}^n_k$ of the constant negative curvature $-k^2$ can be found in [84, 89, 90].

On the other hand, one can intuitively attain an upper bound for the heat kernel estimates on Cartan-Hadamard manifolds in the light of both those asymptotics of the heat kernels and the well-known relation between the on-diagonal behaviour of the heat kernels and the first Dirichlet eigenvalue, which is a pure geometric object [75, 84]. Let

$K_\Omega(u|x,y)$ be the heat kernel in a precompact region $\Omega \subset \mathcal{M}$ subject to the Dirichlet boundary condition. It is possible to show that the heat kernel $K(u|x,y)$ on the entire manifold $\mathcal{M}$ can be constructed by taking the limit as $\Omega \to \mathcal{M}$. This stems form the monotonicity of the heat kernel $K_\Omega(u|x,y)$ with respect to Ω. We can exhaust $\mathcal{M}$ by a sequence of precompact open sets $\{\Omega_k\}$ such that $\partial\Omega_k$ is smooth and $\overline{\Omega_k} \subset \Omega_{k+1}$ and

$$\bigcup_{k=1}^{\infty} \Omega_k = \mathcal{M}\,. \tag{3.133}$$

One creates an increasing sequence of Dirichlet heat kernels $K_{\Omega_{k+1}}(u|x,y) \geq K_{\Omega_k}(u|x,y)$ and thereby define the minimal positive heat kernel of $\mathcal{M}$ as the limit of the former ones,

$$K(u|x,y) = \lim_{k\to\infty} K_{\Omega_k}(u|x,y)\,. \tag{3.134}$$

Under this construction, whole properties of $K_\Omega(u|x,y)$ are inherited by $K(u|x,y)$ except for the eigenfunction expansion. This results from the fact that the Laplace operator $\Delta_\mathcal{M}$ on $\mathcal{M}$ does not necessarily have a discrete spectrum as for a precompact region Ω. This is the reason why the eigenfunction expansion of the heat kernel and the way to define it via its eigenfunction expansion are called formal. Since the limit $K_\Omega(u|x,y) \to K(u|x,y)$ is independent of the choice of $\{\Omega_k\}$, and the eigenvalue $\lambda_1(\Omega)$ (note that it corresponds to the eigenfunction ϕ_1 such that ϕ_k forms an orthonormal basis in $\mathcal{L}^2(\Omega)$) is a simple eigenvalue, it is plausible to expect that the following limit of $K_\Omega(u|x,y)$,

$$K_\Omega(u|,x,y) \sim e^{-\lambda_1(\Omega)u} \text{ as } u \to \infty\,, \tag{3.135}$$

also holds for $K(u|x,y)$, that is,

$$K(u|,x,y) \sim e^{-\lambda_1(\mathcal{M})u} \text{ as } u \to \infty\,, \tag{3.136}$$

provided $\lambda_1(\Omega) \to \lambda_1(\mathcal{M})$ as $\Omega \to \mathcal{M}$. Here, $\lambda_1(\mathcal{M})$ is called the spectral gap/radius of the Laplacian $\Delta_{\mathcal{M}}$ as an operator in $\mathcal{L}^2(\mathcal{M})$, and it corresponds to the bottom of the spectrum of $\Delta_{\mathcal{M}}$ [76]. Therefore, it is natural to expect that the on-diagonal upper bound estimate should possess a polynomial term like $t^{-n/2}$ and an exponential decay term like e^{-Cu}. Recall that there is an off-diagonal correction term to the upper bound estimate in a Gaussian exponential form containing the Riemannian distance. For example, the following off-diagonal upper bound estimate for Cartan-Hadamard manifolds takes into account all of the previous discussions,

$$K(u|x,y) \leq \frac{C}{\min(1, t^{n/2})} e^{-\lambda_1(\mathcal{M})u - \frac{d^2(x,y)}{2Du}} , \tag{3.137}$$

C and D being positive constants. If the manifold in question is a Cartan-Hadamard manifold with the sectional curvature bounded from above by $-k^2$ then McKean's theorem [91] gives a lower bound for the spectral gap and it is

$$\lambda_1(\mathcal{M}) \geq \frac{(n-1)^2}{4} k^2 . \tag{3.138}$$

Thus, the heat kernel estimate takes the following form

$$K(u|x,y) \leq \frac{C}{\min(1, t^{n/2})} e^{-\frac{(n-1)^2}{4} k^2 u - \frac{d^2(x,y)}{2Du}} . \tag{3.139}$$

It is possible to attain this upper bound by using the Faber-Krahn inequality with appropriate modifications. For rigorous but pedagogical derivations of these kinds of upper bounds and much more, Grigor'yan [75] can be consulted.

3.6. Asymptotic Limits

In this section, we will study the asymptotic behavior of the operator $\Phi_R(E)$ in the limit of large number of bosons, $n \to \infty$, and the flat case is our starting point. Combining the fractions in Eq. (3.51) by Feynman parametrization, exponentiating the resultant fractions, applying the subordination identity to $e^{-s\omega(\mathbf{p})}$ and taking the momentum integral, successively, brings the principal operator to the following form

$$\Phi_R(E) = (H_0 - E + \mu_p)\left\{1 + \frac{\lambda_R^2}{32\pi^2}\int_0^1 d\xi \int_0^{1-\xi} d\zeta \int_0^\infty ds\, \frac{1}{s} \right.$$
$$\left. \times \int_0^\infty du\, \frac{e^{-1/4u - us^2m^2}}{u^3}\left[e^{-s[(H_0-E)\xi - \mu_p\zeta]} - 1\right]\right\} - \cdots . \quad (3.140)$$

The u-integral is just the integral representation of the modified Bessel function of the second kind $K_2(ms)$ multiplied with $8m^2s^2$. After letting $u \to us^2$, calculating the u-integral and scaling s as $s \to s/m$, one gets

$$\Phi_R(E) = (H_0 - E + \mu_p)\left\{1 + \frac{\lambda_R^2}{4\pi^2}\int_0^1 d\xi \int_0^{1-\xi} d\zeta \right.$$
$$\left. \times \int_0^\infty ds\, sK_2(s)\left[e^{-s[(H_0-E)\xi - \mu_p\zeta]/m} - 1\right]\right\} - \cdots . \quad (3.141)$$

We can add and subtract e^{-s} to the right hand side of this expression, which regularizes the s-integral and also generates a constant $-C$, which is numerically computable and approximately equal to -2.67. Thus, the principal operator is

$$\Phi_R(E) = (H_0 - E + \mu_p)\left\{1 - C\frac{\lambda_R^2}{4\pi^2} + \frac{\lambda_R^2}{4\pi^2}\int_0^1 d\xi \int_0^{1-\xi} d\zeta \right.$$
$$\left. \times \int_0^\infty ds\, sK_2(s)\left[e^{-s[(H_0-E)\xi - \mu_p\zeta]/m} - e^{-s}\right]\right\} - \cdots . \quad (3.142)$$

Being calculated by some mathematical software, the s-integral is equal to

$$\frac{5am\left[8a^2 + \left(-12 + \sqrt{2 - 2a/m}\right)m^2 - \sqrt{2}m^{3/2}\sqrt{-a+m}\right]\pi}{20\sqrt{1 - a/m}(m-a)\left[m(a+m)\right]^{3/2}}$$
$$- \frac{32(a-m)(a+m)^2\sqrt{-a^2+m^2}\,{}_3F_2\left(1,1,5;2,\frac{7}{2};\frac{a+m}{2m}\right)}{20\sqrt{1-a/m}(m-a)\left[m(a+m)\right]^{3/2}}, \quad (3.143)$$

where $a = (H_0 - E)\xi - \mu_p\zeta$. After converting the Hypergeometric function into an elementary function and doing some simplifications, the principal operator can be given by

$$\Phi_R(E) = (H_0 - E + \mu_p)\left\{1 - C\frac{\lambda_R^2}{4\pi^2} + \frac{\lambda_R^2}{4\pi^2}\int_0^1 d\xi \int_0^{1-\xi} d\zeta \left[\frac{5}{3} - \frac{\tilde{m}^2}{\left(\zeta - \tilde{\zeta}\right)\left(\zeta - \tilde{\zeta} - 2\tilde{m}\right)} + \frac{\left(\zeta - \tilde{\zeta} - \tilde{m}\right)\left[\tilde{m}^2 + 2\left(\zeta - \tilde{\zeta}\right)\left(2\tilde{m} - \zeta + \tilde{\zeta}\right)\right]}{\left(\zeta - \tilde{\zeta}\right)^{3/2}\left(2\tilde{m} - \zeta + \tilde{\zeta}\right)^{3/2}} \times \left(\pi - 2\arcsin\sqrt{1 - \frac{\zeta}{2\tilde{m}} + \frac{\tilde{\zeta}}{2\tilde{m}}}\right)\right]\right\} - \cdots, \tag{3.144}$$

where $\tilde{m} \equiv \frac{m}{\mu_p}$ and $\tilde{\zeta} \equiv \frac{(H_0 - E)\xi - m}{\mu_p}$.

Whether $\tilde{\zeta}$ is between the limits of the ζ-integral or not is important for calculating this integral. Taking the integration interval of the ξ-integral and $H \geq m$ into account tells us that $\tilde{\zeta}$ is in the integration interval. This could cause poles since the denominators has some powers of $\zeta - \tilde{\zeta}$. If this is the case, then the integral should be defined either by a principal value prescription or by a Hadamard finite part prescription. In order to answer this question, one should expand the integrand around $\zeta = \tilde{\zeta}$ in series. For this expansion, logarithmic form of the inverse trigonometric function is more suitable:

$$-2\arcsin\sqrt{1 - \frac{\zeta}{2\tilde{m}} + \frac{\tilde{\zeta}}{2\tilde{m}}} = 2i\ln\left(\sqrt{\frac{\zeta}{2\tilde{m}} - \frac{\tilde{\zeta}}{2\tilde{m}}} + i\sqrt{1 - \frac{\zeta}{2\tilde{m}} + \frac{\tilde{\zeta}}{2\tilde{m}}}\right). \tag{3.145}$$

The series expansion of the combination of the first and the second term is

$$\lim_{\zeta\to\tilde{\zeta}}\left[\frac{5}{3} - \frac{\tilde{m}^2}{\left(\zeta - \tilde{\zeta}\right)\left(\zeta - \tilde{\zeta} - 2\tilde{m}\right)}\right] = \frac{\tilde{m}}{2\left(\zeta - \tilde{\zeta}\right)} + \frac{23}{12} + \frac{\zeta - \tilde{\zeta}}{8\tilde{m}} + \mathrm{O}\left(\zeta - \tilde{\zeta}\right)^2, \tag{3.146}$$

and the expansion of the third term is

$$\lim_{\zeta\to\tilde{\zeta}} \left\{ \frac{\left(\zeta-\tilde{\zeta}-\tilde{m}\right)\left[\tilde{m}^2+2\left(\zeta-\tilde{\zeta}\right)\left(2\tilde{m}-\zeta+\tilde{\zeta}\right)\right]}{\left(\zeta-\tilde{\zeta}\right)^{3/2}\left(2\tilde{m}-\zeta+\tilde{\zeta}\right)^{3/2}} \right.$$
$$\left. \times \left[\pi + 2i\ln\left(\sqrt{\frac{\zeta}{2\tilde{m}}-\frac{\tilde{\zeta}}{2\tilde{m}}}+i\sqrt{1-\frac{\zeta}{2\tilde{m}}+\frac{\tilde{\zeta}}{2\tilde{m}}}\right)\right]\right\}$$
$$= -\frac{\tilde{m}}{2\left(\zeta-\tilde{\zeta}\right)} - \frac{23}{12} + \frac{59\left(\zeta-\tilde{\zeta}\right)}{40\tilde{m}} + \mathrm{O}\left(\zeta-\tilde{\zeta}\right)^{3/2}. \tag{3.147}$$

It is astonishing that not only the singular parts but also the constant parts of the integrand in the expansion cancel each other and that limit of the integrand is just given by

$$\lim_{\zeta\to\tilde{\zeta}}(\text{integrand}) = \frac{8}{5\tilde{m}}(\zeta-\tilde{\zeta}) + \mathrm{O}(\zeta-\tilde{\zeta})^{3/2}. \tag{3.148}$$

Although $\tilde{\zeta}$ is between the integration limits, the series expansions tell us the fact that the ζ integral is just an ordinary integral due to the fact that the integrand does not really have any poles at $\zeta = \tilde{\zeta}$. Thus, we do not need to introduce any prescription in order to make this integral well-defined and compute it. After doing very long and tedious calculations, the renormalized principal operator can be obtained as

$$\Phi_R(E) = (H_0 - E + \mu_p)\left[1 - \left(C - \frac{7}{3}\right)\frac{\lambda_R^2}{4\pi^2}\right]$$
$$- 2\frac{\lambda_R^2}{4\pi^2}\sqrt{(H_0-E-m)(H_0-E+m)}$$
$$\times \ln\left(\sqrt{\frac{H_0-E-m}{2m}} + \sqrt{\frac{H_0-E+m}{2m}}\right)$$
$$- 2\frac{\lambda_R^2}{4\pi^2}\sqrt{(m-\mu_p)(m+\mu_p)}\arccos\sqrt{\frac{m-\mu_p}{2m}}$$
$$- \lambda_R^2\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}\frac{a^\dagger(\mathbf{q})}{\sqrt{2\omega(\mathbf{q})}}\frac{1}{H_0-E+\omega(\mathbf{q})+\omega(\mathbf{p})}\frac{a(\mathbf{p})}{\sqrt{2\omega(\mathbf{p})}}. \tag{3.149}$$

Here, we would like to emphasize that this pirincipal operator is exact.

In the flat case, the asymptotic behavior of the principal operator in the limit of large number of bosons, that is $H_0 \geq nm \gg m > \mu_p$, is, then, given by

$$\begin{aligned}\Phi_R(E) \simeq H_0 \left[1 - \left(C - \frac{7}{3} + \ln 2\right) \frac{\lambda_R^2}{4\pi^2}\right] - \frac{\lambda_R^2}{4\pi^2} H_0 \ln\left(\frac{H_0}{m}\right) \\ - \text{(the normal-ordered interaction term)} \\ + \text{(the lower order terms in } H_0) \,. \end{aligned} \tag{3.150}$$

This asymptotic behavior has a striking feature, the interaction term is positive, multiplied by a minus sign gives a negative contribution, and the leading term of the renormalized principal operator is also negative. Whatever the leading behavior of this interaction term is, these two terms are enhancing the negative value of $\Phi_R(E)$. We can show the positivity of the interaction term in general by studying the same term in the manifold case. The interaction term in Eq. (3.120) can be written as

$$\begin{aligned}&\frac{\lambda_R^2}{4\pi^2} \int_0^\infty ds \, s^2 \left[\int d_g^3 y \int_0^\infty du_1 \, \frac{e^{-s^2/4u_1}}{u_1^{3/2}} K_{u_1}(\overline{x}, y) \phi^{(+)}(y)\right]^\dagger \\ &\quad \times e^{-s(H_0 - E)} \left[\int d_g^3 x \int_0^\infty du_2 \, \frac{e^{-s^2/4u_2}}{u_2^{3/2}} K_{u_2}(\overline{x}, x) \phi^{(+)}(x)\right] \\ &= \frac{\lambda_R^2}{4\pi^2} \int_0^\infty ds \, s^2 \underbrace{A^\dagger(s) e^{-s(H_0 - E)} A(s)}_{>0} \,. \end{aligned} \tag{3.151}$$

Since the integrand is positive, the interaction term is positive-definite apart from the minus sign in front. This, in turn, implies the operator to have a negative-definite sign. Henceforth, the operator $\Phi_R(E)$ can not have zero eigenvalues for E positive but smaller than nm, so that as $n \to \infty$ $H_0 - E$ still grows by n (this is the case for example $E < nm\tau$ for $\tau < 1$). For large number of particles this proves the positivity of the energy, which is extremely important for the stability of the theory.

Secondly, we would like to analyze the leading behavior of the renormalized principal operator on a general ultra-static Riemannian manifold in the same limit. Having done similar calculations, Eq. (3.121) becomes ready to be studied in the limit $n \to \infty$.

$$\Phi_R(E) = (H_0 - E + \mu_p)\left\{1 - C(\overline{x}, m)\frac{\lambda_R^2}{4\sqrt{\pi}} + \frac{\lambda_R^2}{4\sqrt{\pi}}\int_0^1 d\xi \int_0^{1-\xi} d\zeta \int_0^\infty ds\, s^3 \right.$$
$$\left. \times \int_0^\infty du\, \frac{e^{-um^2 - s^2/4u}}{u^{3/2}} K_u(\overline{x}, \overline{x}) \left[e^{-s[(H_0-E)\xi - \mu_p\zeta]} - e^{-sm}\right]\right\} - \cdots . \quad (3.152)$$

Appropriate scalings of the variables in the above equation allow us to take that limit and the operator is given by

$$\Phi_R(E) = (H_0 - E + \mu_p)\left\{1 - C(\overline{x}, m)\frac{\lambda_R^2}{4\sqrt{\pi}} + \frac{\lambda_R^2}{4\sqrt{\pi}}\int_0^1 d\xi \int_0^{1-\xi} d\zeta \right.$$
$$\times \int_0^\infty ds\, \frac{s^3}{(nm)^3} \int_0^\infty du\, \frac{e^{-um^2/(nm)^2 - s^2/4u}}{u^{3/2}} K_{u/(nm)^2}(\overline{x}, \overline{x})$$
$$\left. \times \left[e^{-s[(H_0-E)\xi - \mu_p\zeta]/nm} - e^{-sm/nm}\right]\right\} - \cdots . \quad (3.153)$$

The asymptotic behavior of the heat kernel is given by

$$\lim_{n\to\infty} K_{u/(nm)^2}(\overline{x}, \overline{x}) \simeq \frac{(nm)^3}{(4\pi u)^{3/2}} . \quad (3.154)$$

Placing the equation above into Eq. (3.153) allows us to take the u-integral and we get

$$\Phi_R(E) \simeq (H_0 - E + \mu_p)\left\{1 - C(\overline{x}, m)\frac{\lambda_R^2}{4\sqrt{\pi}} + \frac{\lambda_R^2}{4\pi^2}\int_0^1 d\xi \int_0^{1-\xi} d\zeta \right.$$
$$\left. \times \int_0^\infty ds\, \frac{s}{n^2} K_2\left(\frac{s}{n}\right) \left[e^{-s[(H_0-E)\xi - \mu_p\zeta]/nm} - e^{-s/n}\right]\right\} + \cdots . \quad (3.155)$$

We should, now, be careful about the asymptotic expansion of the integral. Although asymptotic behavior of the function $K_2(s/n)$ can be used for s small enough, we are not allowed to use it when s becomes comparable with n because the other multiplying factors do not decay sufficiently fast with s. Since the upper limit of the s-integral is at infinity, this is the case. However, if we rescale s with n, this integral takes a form which is independent of n. Therefore, this expression becomes the same expression which we have found already in the previous case whose constant term C is, basically, replaced by $\pi^{3/2}C(\overline{x}, m)$. If one takes the next term in the short-time expansion of the heat kernel into account, then it can be seen that the contribution coming from that term is of the order of $1/n^2$, which is much smaller and hence neglected. Yet, there comes a contribution from the expansion of the first exponential, which results in a new constant C', multiplying H_0. Thus, the leading behavior of the operator $\Phi_R(E)$ in the asymptotic limit $H_0 \gg m$ on a general ultra-static Riemannian manifold can be given by

$$\begin{aligned}\Phi_R(E) \simeq H_0 \left[1 - \frac{\lambda_R^2}{4\pi^2}\left(\pi^{3/2}C(\overline{x}, m) + C' - \frac{7}{3} + \ln 2\right)\right] - \frac{\lambda_R^2}{4\pi^2} H_0 \ln\left(\frac{H_0}{m}\right) \\ - (\text{the normal-ordered interaction term}) \\ + (\text{the lower order terms in } H_0)\,. \qquad (3.156)\end{aligned}$$

At this stage, we are unable to give precise asymptotic analysis of the normal-ordered interaction term, which requires a delicate study. We would like to call readers' attention to the fact that the same remarks, which have been done in the flat case, are also valid for the relativistic Lee model defined on a general ultra-static Riemannian manifold.

At last, the manifold defined as $\mathcal{M} = \mathbb{R} \times \mathbb{H}_a^3$ will be considered as an example. $\mathbb{H}_a^3$ is, here, a three dimensional hyperbolic space. The reason why we study this manifold is based on the fact that its heat kernel is one of the simplest and explicitly known heat kernels.

The heat kernel of the hyperbolic space $\mathbb{H}_a^n$, found in [84, 88], and the diagonal heat kernel of $\mathbb{H}_a^3$ takes the form

$$\begin{aligned}K_u(\bar{x}, \bar{x}) &= \frac{1}{(4\pi u)^{3/2}} \lim_{\bar{y} \to \bar{x}} \frac{d(\bar{x}, \bar{y})}{\sinh d(\bar{x}, \bar{y})} e^{-a^2 u - d(\bar{x}, \bar{y})^2/4u} \\ &= \frac{e^{-a^2 u}}{(4\pi u)^{3/2}}\,, \qquad (3.157)\end{aligned}$$

where $d(\overline{x}, \overline{y}) = \text{dist}(\overline{x}, \overline{y})$ is the geodesic distance on $\mathbb{H}_a^3$ and $-a^2$ is the constant sectional curvature. Having used the diagonal heat kernel in Eq. (3.152), the following operator could be obtained,

$$\Phi_R(E) = (H_0 - E + \mu_p) \left\{ 1 + \frac{\lambda_R^2}{32\pi^2} \int_0^1 d\xi \int_0^{1-\xi} d\zeta \right.$$
$$\left. \times \int_0^\infty ds\, s^3 \int_0^\infty du\, \frac{e^{-u(m^2+a^2)}}{u^3} e^{-s^2/4u} \left[e^{-s[(H_0-E)\xi - \mu_p \zeta]} - 1 \right] \right\} + \cdots . \tag{3.158}$$

This is the same result, which was found already in the flat case with m^2 replaced by $m^2 + a^2$. Thus, the space $\mathbb{H}_a^3$ modifies the mass term of the exact principal operator only. After this slight modification, the exact principal operator takes the form

$$\begin{aligned}\Phi_R(E) = (H_0 - E + \mu_p) &\left[1 - \left(C - \frac{7}{3} \right) \frac{\lambda_R^2}{4\pi^2} \right] \\ &- 2\frac{\lambda_R^2}{4\pi^2} \sqrt{\left(H_0 - E - \sqrt{m^2+a^2} \right) \left(H_0 - E + \sqrt{m^2+a^2} \right)} \\ &\times \ln \left(\sqrt{\frac{H_0 - E - \sqrt{m^2+a^2}}{2\sqrt{m^2+a^2}}} + \sqrt{\frac{H_0 - E + \sqrt{m^2+a^2}}{2\sqrt{m^2+a^2}}} \right) \\ &- 2\frac{\lambda_R^2}{4\pi^2} \sqrt{\left(\sqrt{m^2+a^2} - \mu_p \right) \left(\sqrt{m^2+a^2} + \mu_p \right)} \arccos \sqrt{\frac{\sqrt{m^2+a^2} - \mu_p}{2\sqrt{m^2+a^2}}} \\ &- (\text{the normal-ordered interaction term}) \,. \end{aligned} \tag{3.159}$$

It is easy to see the asymptotic behavior of the operator $\Phi_R(E)$ a simple modification is sufficient to calculate it. Hence, one get the following,

$$\begin{aligned}\Phi_R(E) \simeq H_0 \left[1 + \frac{\lambda_R^2}{4\pi^2} \left(\frac{7}{3} - \ln 2 - C \right) \right] &- \frac{\lambda_R^2}{4\pi^2} H_0 \ln \left(\frac{H_0}{\sqrt{m^2+a^2}} \right) \\ &- (\text{the normal-ordered interaction term}) \\ &+ (\text{the lower order terms in } H_0) \,, \end{aligned} \tag{3.160}$$

and the normal-ordered interaction term, of course, changes drastically (see Eq. (3.120)).

3.7. The Lee model on $2+1$ dimensional Riemannian manifolds

In this section, we make a digression to an analysis of the two dimensional version of the Lee model. Our purpose here is two fold, we first would like to show that the two dimensional model is much simpler, which only requires a mass renormalization and secondly we would like to illustrate the power of this approach by obtaining an explicit bound on the ground state energy in each sector.

We write the model on a Riemannian manifold in the matrix form by using a heat kernel cut-off function:

$$H_\epsilon - E = \begin{bmatrix} H_0 - E & \lambda\phi_\epsilon^{(-)}(\overline{x}) \\ \lambda\phi_\epsilon^{(+)}(\overline{x}) & [H_0 - E + \mu(\epsilon)] \end{bmatrix} . \tag{3.161}$$

The model now neither requires a coupling constant renormalization nor a wave function one. We take the resolvent in the same way as before and find the principal operator as,

$$\begin{aligned} \Phi_\epsilon(E) = H_0 - E + \mu(\epsilon) - \lambda^2 \int d_g^3x\, d_g^3y\, K_{\epsilon/2}(\overline{x},x) K_{\epsilon/2}(\overline{x},y) \\ \times \left[\int \frac{d\mu(j)}{\sqrt{2\omega(j)}} \frac{d\mu(k)}{\sqrt{2\omega(k)}}\, \phi_j(x)\phi_k^*(y) a^\dagger(k) \frac{1}{H_0 - E + \omega(k) + \omega(j)} a(j) \right. \\ \left. + \int d\mu(j)\, \phi_j(x)\phi_j^*(y) \frac{1}{2\omega(j)} \frac{1}{H_0 - E - \omega(j)} \right] . \end{aligned} \tag{3.162}$$

Following the same steps in the $3+1$ dimensional case, we end up with,

$$\begin{aligned} \Phi_\epsilon(E) = H_0 - E + \mu(\epsilon) - \frac{\lambda^2}{4\sqrt{\pi}} \int_0^\infty du \int_0^\infty ds\, s e^{-s^2/4} K_{u+\epsilon}(\overline{x},\overline{x}) \frac{\left[1 - e^{-s\sqrt{u}(H_0-E)}\right]}{\sqrt{u}(H_0-E)} \\ - \frac{\lambda^2}{4\pi} \int d_g^3x\, d_g^3y \int_0^\infty ds\, s^2 \int_0^\infty du_1 \frac{e^{-s^2/4u_1}}{u_1^{3/2}} \int_0^\infty du_2 \frac{e^{-s^2/4u_2}}{u_2^{3/2}} \\ \times K_{\epsilon/2+u_1}(\overline{x},y) K_{\epsilon/2+u_2}(\overline{x},x) \phi^{(-)}(y) e^{-s(H_0-E)} \phi^{(+)}(x) . \end{aligned} \tag{3.163}$$

Using the behaviour of the heat kernel on a two dimensional Riemannian manifold,

$$K_u(x,x) \simeq \frac{1}{(4\pi u)} \sum_{n=0}^{\infty} a_n(x) u^n \,, \tag{3.164}$$

we see that the principal operator becomes finite if we define a mass renormalization given by

$$\begin{aligned}\mu(\epsilon) &= \mu_R + \frac{\lambda^2}{4\sqrt{\pi}} \int_0^\infty du \int_0^\infty ds\, s^2 e^{-s^2/4} K_{u+\epsilon}(\overline{x},\overline{x}) \\ &= \mu_R + \frac{\lambda^2}{2} \int_0^\infty du\, K_{u+\epsilon}(\overline{x},\overline{x}) \,. \end{aligned} \tag{3.165}$$

As a result we find the renormalized principal operator as,

$$\begin{aligned}\Phi_R(E) = (H_0 - E + \mu_R) &- \frac{\lambda^2}{4\sqrt{\pi}} \int_0^\infty du \int_0^\infty ds\, s e^{-s^2/4} \frac{K_u(\overline{x},\overline{x})}{\sqrt{u}(H_0-E)} \\ &\times \left[1 - s\sqrt{u}(H_0 - E) - e^{-s\sqrt{u}(H_0-E)}\right] \\ &- \frac{\lambda^2}{4\pi} \int d_g^3 x\, d_g^3 y \int_0^\infty ds\, s^2 \int_0^\infty du_1 \frac{e^{-s^2/4u_1}}{u_1^{3/2}} \int_0^\infty du_2 \frac{e^{-s^2/4u_2}}{u_2^{3/2}} \\ &\times K_{u_1}(\overline{x},y) K_{u_2}(\overline{x},x) \phi^{(-)}(y) e^{-s(H_0-E)} \phi^{(+)}(x) \,. \end{aligned} \tag{3.166}$$

If we now impose the physical mass condition $\Phi(E = \mu_p)|0 >= 0$, written in the eigenfunction expansion, we end up with,

$$\begin{aligned}\Phi_R(E) = (H_0 - E + \mu_p) &\left\{1 + \frac{\lambda^2}{2} \int d\mu(j) \frac{\phi_j^*(\overline{x})\phi_j(\overline{x})}{\omega(j)\left[H_0 - E + \omega(j)\right]\left[-\mu_p + \omega(j)\right]}\right\} \\ &- \lambda^2 \int \frac{d\mu(j)}{\sqrt{2\omega(j)}} \frac{d\mu(k)}{\sqrt{2\omega(k)}} a^\dagger(k) \frac{\phi_j(\overline{x})\phi_k^*(\overline{x})}{H_0 - E + \omega(k) + \omega(j)} a(j) \,. \end{aligned} \tag{3.167}$$

The change in the renormalized part is important, if we recall that $\mu_p < \omega(j)$ this part is actually always positive for $E < nm$ (the interesting case from the bound state spectrum point of view). Therefore, the interaction term now competes with these two terms. If we evaluate the answer for the flat case, then we see that it is given by

$$\Phi_R(E) = (H_0 - E + \mu_p) + \frac{\lambda^2}{4\pi} \ln\left[\frac{H_0 - E + m}{m - \mu_p}\right] \\ - \lambda^2 \int \frac{d^2p}{(2\pi)^2} \int \frac{d^2q}{(2\pi)^2} \frac{a^\dagger(\mathbf{q})}{\sqrt{2\omega(\mathbf{q})}} \frac{1}{H_0 - E + \omega(\mathbf{q}) + \omega(\mathbf{p})} \frac{a(\mathbf{p})}{\sqrt{2\omega(\mathbf{p})}} . \quad (3.168)$$

Since the flat case is sufficiently important we will give a bound on the ground state energy for all particle sectors first and discuss the general case of manifolds afterwards. Note that if we can show that the principal operator becomes positive for sufficiently small values of E, this means that it is invertible, hence, it cannot have a zero eigenvalue beyond that value. This give us a lower bound on the ground state energy. To accomplish this we rewrite the principal operator in the form,

$$\Phi_R(E) = \tilde{K}(E) - U(E) , \quad (3.169)$$

where

$$\tilde{K}(E) = (H_0 - E + \mu_p) + \frac{\lambda^2}{4\pi} \ln\left[\frac{H_0 - E + m}{m - \mu_p}\right] \\ U(E) = \lambda^2 \int \frac{d^2p}{(2\pi)^2} \int \frac{d^2q}{(2\pi)^2} \frac{a^\dagger(\mathbf{q})}{\sqrt{2\omega(\mathbf{q})}} \frac{1}{H_0 - E + \omega(\mathbf{q}) + \omega(\mathbf{p})} \frac{a(\mathbf{p})}{\sqrt{2\omega(\mathbf{p})}} . \quad (3.170)$$

Note that for real values of E, we can drop the logarithm and the resulting operator is smaller than $\tilde{K}(E)$. Thus following Rajeev [48], we write an inequality of the form,

$$\Phi_R(E) > K(E) - U(E) = K(E)^{1/2}(1 - K(E)^{-1/2}U(E)K(E)^{-1/2})K(E)^{1/2} , \quad (3.171)$$

where $K(E) = H_0 + \mu_p - E$. Hence to show that the operator $\Phi_R(E)$ to be invertible, it is sufficient to impose the condition, $||\tilde{U}(E)|| = ||K(E)^{-1/2}U(E)K(E)^{-1/2}|| < 1$. This will impose a condition on the ground state energy. If we write this out explicitly, after commuting the square root operators with the creation and annihilation operators of the interaction term,

$$\tilde{U}(E) = \lambda^2 \int \frac{d^2p}{(2\pi)^2} \int \frac{d^2q}{(2\pi)^2} \frac{a^\dagger(\mathbf{q})}{\sqrt{2\omega(\mathbf{q})}} \frac{1}{[H_0 - E + \mu_p + \omega(\mathbf{q})]^{1/2}}$$
$$\times \frac{1}{[H_0 - E + \omega(\mathbf{q}) + \omega(\mathbf{p})]} \frac{1}{[H_0 - E + \mu_p + \omega(\mathbf{p})]^{1/2}} \frac{a(\mathbf{p})}{\sqrt{2\omega(\mathbf{p})}} . \quad (3.172)$$

Now we can use the inequality $H_0 > (n-1)m$ in the n boson sector inside the operator and replacement of it results in a bigger operator function. Call this $\chi = (n-1)m + \mu_p - E$ and for $n > 1$ we find as a result,

$$\tilde{U}(E) \leq \lambda^2 \int \frac{d^2p}{(2\pi)^2} \int \frac{d^2q}{(2\pi)^2} \frac{a^\dagger(\mathbf{q})}{\sqrt{2\omega(\mathbf{q})}} \frac{1}{[\chi + \omega(\mathbf{q})]^{1/2}}$$
$$\times \frac{1}{[\chi - \mu_p + \omega(\mathbf{q}) + \omega(\mathbf{p})]} \frac{1}{[\chi + \omega(\mathbf{p})]^{1/2}} \frac{a(\mathbf{p})}{\sqrt{2\omega(\mathbf{p})}} . \quad (3.173)$$

If we now use an extension of the Cauchy-Schwartz inequality to the Fock-Space operators, we find

$$||\tilde{U}(E)|| \leq \frac{1}{2} n\lambda^2 \Bigg[\int \frac{d^2p}{(2\pi)^2} \int \frac{d^2q}{(2\pi)^2} \frac{1}{\omega(\mathbf{q})[\chi + \omega(\mathbf{q})]}$$
$$\times \frac{1}{[\chi - \mu_p + \omega(\mathbf{q}) + \omega(\mathbf{p})]^2} \frac{1}{[\chi + \omega(\mathbf{p})]\omega(\mathbf{p})} \Bigg]^{1/2} . \quad (3.174)$$

We now note that $\sqrt{\mathbf{p}^2 + m^2} \geq |\mathbf{p}| = p$ and $m > \mu_p$, and replace some of the terms by these lower ones and thus preserving direction of the inequalities,

$$||\tilde{U}(E)|| \leq \frac{1}{2} n\lambda^2 \left[\int \frac{pdpd\Omega_p}{(2\pi)^2} \int \frac{qdqd\Omega_q}{(2\pi)^2} \frac{1}{pq[\chi + q + p]^2 [\chi + q][\chi + p]} \right]^{1/2} . \quad (3.175)$$

Let us scale the momenta by $p = \chi\bar{p}, q = \chi\bar{q}$, we find

$$||\tilde{U}(E)|| \leq \frac{n\lambda^2}{8\pi^2} \frac{1}{\chi} \left[\int_0^\infty \int_0^\infty \frac{d\bar{p}d\bar{q}}{[1 + \bar{q} + \bar{p}]^2 [1 + \bar{q}][1 + \bar{p}]} \right]^{1/2} . \quad (3.176)$$

And the last integral is finite, let us call its value as C, we then impose the condition,

$$\frac{n\lambda^2 C}{8\pi^2\chi} < 1\,, \tag{3.177}$$

which guaranties that the $||\tilde{U}(E)|| < 1$ This implies the rigorous inequality on the ground state energy,

$$E_{gr}(n) \geq (n-1)m + \mu_p - \frac{\lambda^2 nC}{8\pi^2}. \tag{3.178}$$

If we want the energy to be positive in all sectors this in turn brings about a bound on the coupling constant. In fact, for global stability we should have the energy to be bounded by $(n-2)m$. But the present analysis is too crude to get a bound of this form, that requires a much more delicate analysis.

Next we will work out the same problem for the Riemannian manifolds, it is simpler to work on the eigenfunction expansions. We follow the same approach and estimate the leading behavior of the term resulting from the renormalization.

The denominator of the second term in Eq. (3.167) can be united by Feynman parametrization as,

$$\begin{aligned} &\frac{\lambda^2}{2}\int d\mu(j)\,\frac{|\phi_j(\overline{x})|^2}{\omega(j)\,[H_0 - E + \omega(j)]\,[-\mu_p + \omega(j)]} \\ &\quad = \frac{\lambda^2}{2}\int_0^1 d\xi \int_0^{1-\xi} d\zeta \int d\mu(j)\,\frac{|\phi_j(\overline{x})|^2}{[\omega(j)(H_0 - E)\xi - \mu_p\zeta]^3}\,. \end{aligned} \tag{3.179}$$

After converting the fraction into an exponential, utilizing subordination identity and the definition of the heat kernel, this term becomes,

$$\frac{\lambda^2}{4\sqrt{\pi}}\int_0^1 d\xi \int_0^{1-\xi} d\zeta \int_0^\infty ds\, s^3 \int_0^\infty du \frac{e^{-s^2/4u - um^2 - s(H_0-E)\xi + \mu_p s\zeta}}{u^{3/2}} K_u(\overline{x},\overline{x})\,. \tag{3.180}$$

Let $s \rightarrow s/(nm)$ and $u \rightarrow u/(nm)^2$, we obtain,

$$\frac{\lambda^2}{4\sqrt{\pi}(nm)^3} \int_0^1 d\xi \int_0^{1-\xi} d\zeta \int_0^\infty ds\, s^3 \\ \times \int_0^\infty du \frac{e^{-s^2/4u - u/n^2 - s(H_0-E)\xi/(nm) + \mu_p s\zeta/(nm)}}{u^{3/2}} K_{u/(nm)^2}(\overline{x}, \overline{x})\,. \tag{3.181}$$

The asymptotic behavior of the heat kernel for large n is given by,

$$\lim_{n\rightarrow\infty} K_{u/(nm)^2}(\overline{x}, \overline{x}) \simeq \frac{(nm)^2}{4\pi u}\,. \tag{3.182}$$

After pluging this asymptotic behavior, one gets,

$$\frac{\lambda^2}{16\pi^{3/2} nm} \int_0^1 d\xi \int_0^{1-\xi} d\zeta \int_0^\infty ds\, s^3 \\ \times \int_0^\infty du \frac{e^{-s^2/4u - u/n^2 - s(H_0-E)\xi/(nm) + \mu_p s\zeta/(nm)}}{u^{5/2}}\,. \tag{3.183}$$

Performing the integrals results in,

$$\frac{\lambda^2}{4\pi} \frac{1}{(H_0 - E + \mu_p)} \ln \left[\frac{H_0 - E + m}{m - \mu_p} \right]\,. \tag{3.184}$$

Taking the overall factor $(H_0 - E + \mu_p)$ into account, we find the same answer as the one in Eq. (3.168) and the leading contribution of the renormalization to the principal operator in the large number of particles limit results in

$$\Phi_R(E) \simeq (H_0 - E + \mu_p) + \frac{\lambda^2}{4\pi} \ln \left[\frac{H_0 - E + m}{m - \mu_p} \right] \\ - \lambda^2 \int \frac{d\mu(j)}{\sqrt{2\omega(j)}} \frac{d\mu(k)}{\sqrt{2\omega(k)}}\, a^\dagger(k) \frac{\phi_j(\overline{x})\phi_k^*(\overline{x})}{H_0 - E + \omega(k) + \omega(j)} a(j) \\ + (\text{smaller order terms})\,. \tag{3.185}$$

The term in Eq. (3.179) is always positive, and we see that its leading term is of smaller order. Hence this can be dropped out safely without affecting the inequalities,

$$\Phi_R(E) > (H_0 - E + \mu_p)^{1/2}[1 - \tilde{U}(E)](H_0 - E + \mu_p)^{1/2}. \tag{3.186}$$

We will now work on a noncompact manifold. We expand the $\tilde{U}(E)$ in the eigenfunction basis,

$$\tilde{U}(E) = \lambda^2 \int d\mu(j)d\mu(k)\frac{a^\dagger(j)}{\sqrt{2\omega(j)}}\frac{\phi_j^*(\bar{x})}{[H_0 - E + \mu_p + \omega(j)]^{1/2}} \\ \times \frac{1}{[H_0 - E + \omega(j) + \omega(k)]}\frac{\phi_k(\bar{x})}{[H_0 - E + \mu_p + \omega(k)]^{1/2}}\frac{a(k)}{\sqrt{2\omega(k)}}. \tag{3.187}$$

Here we have $\omega(j) = \sqrt{\sigma_j^2 + m^2}$ and we introduce again $\chi = (n-1)m - E$ (we drop the μ_p for simplicity). Recall that $H_0 \geq (n-1)m$, we use this inequality, and the Cauchy-Schwartz inequality to find,

$$||\tilde{U}(E)|| < \frac{\lambda^2 n}{2}\left[\int d\mu(j)d\mu(k)\frac{|\phi_j(\bar{x})|^2}{\omega(j)[\chi + \omega(j)]} \right. \\ \left. \times \frac{1}{[\chi + \omega(j) + \omega(k)]^2}\frac{|\phi_k(\bar{x})|^2}{[\chi + \omega(k)]\omega(k)}\right]^{1/2}. \tag{3.188}$$

We use the following crude inequality,

$$[\chi + \omega(j) + \omega(k)]^2 > [\chi + \omega(j)]\,[\chi + \omega(k)]\,, \tag{3.189}$$

which implies the opposite inequality for the inverse,

$$||\tilde{U}(E)|| < \frac{\lambda^2 n}{2}\left[\int d\mu(j)d\mu(k)\frac{|\phi_j(\overline{x})|^2|\phi_k(\overline{x})|^2}{\omega(j)[\chi + \omega(j)]^2[\chi + \omega(k)]^2\omega(k)}\right]^{1/2} \\ < \frac{\lambda^2 n}{2}\left[\int d\mu(j)\frac{|\phi_j(\overline{x})|^2}{\omega(j)[\chi + \omega(j)]^2}\right]. \tag{3.190}$$

We now employ a Feynmann parametrization,

$$\frac{1}{\omega(j)[\chi+\omega(j)]^2} = \int_0^1 \frac{2\zeta d\zeta}{[\omega(j)(1-\zeta)+(\chi+\omega(j))\zeta]^3}\,. \tag{3.191}$$

To make contact with the heat kernel we employ an exponentiation and then use the subordination identity to get,

$$||\tilde{U}(E)|| < \frac{\lambda^2 n}{\sqrt{\pi}} \int_0^1 d\zeta\zeta \int_0^\infty ds\, s^3 \int_0^\infty du \left[\int d\mu(j) \frac{e^{-s^2/4u-\omega^2(j)u}}{u^{3/2}} |\phi_j(\overline{x})|^2\right] e^{-s\chi\zeta}\,. \tag{3.192}$$

Recognizing the heat kernel as,

$$e^{-m^2u} K_u(\overline{x},\overline{x}) = \int d\mu(j) |\phi(j)|^2 e^{-\omega^2(j)u}\,, \tag{3.193}$$

we can rewrite the desired inequality as,

$$||\tilde{U}(E)|| < \frac{\lambda^2 n}{\sqrt{\pi}} \int_0^1 d\zeta\zeta \int_0^\infty s^3 ds \int_0^\infty du [e^{-m^2u} K_u(\overline{x},\overline{x})] \frac{e^{-s^2/4u}}{u^{3/2}} e^{-s\chi\zeta}\,. \tag{3.194}$$

We note that for Cartan-Hadamard manifolds there is a nice inequality for the heat kernel [88],

$$K_u(\bar{x},\bar{x}) \leq \frac{C_1}{u}\,, \tag{3.195}$$

where C_1 is a positive constant related to the geometry. This in turn implies for these manifolds that

$$||\tilde{U}(E)|| < \frac{\lambda^2 n}{\sqrt{\pi}} \int_0^1 d\zeta\zeta \int_0^\infty s^3 ds \int_0^\infty du e^{-m^2u} \frac{C_1}{u} \frac{e^{-s^2/4u}}{u^{3/2}} e^{-s\chi\zeta}\,. \tag{3.196}$$

If we drop e^{-m^2u} term the integral can be easily found, we scale the u variable as s^2v and find,

$$||\tilde{U}(E)|| < \frac{\lambda^2 n}{\sqrt{\pi}} \int_0^1 d\zeta\zeta \int_0^\infty ds e^{-s\chi\zeta} \int_0^\infty dv \frac{C_1}{v^{5/2}} e^{-1/4v} = C_2 \frac{\lambda^2 n}{\chi}\,. \tag{3.197}$$

If we impose the condition, $C_2 \frac{\lambda^2 n}{\chi} < 1$, then we have no zeros for $\Phi_R(E)$, and this implies a bound on the ground state energy,

$$E_{gr}(n) > (n-1)m - 4C_2\lambda^2 n\,. \tag{3.198}$$

This shows that there is a rigorous lower bound on the ground state energy of the n-particle system.

A similar analysis can also be done for a compact manifold case. Let's assume that the manifold is a compact manifold with Ricci curvature bounded from below by $-\kappa^2$, for some constant $\kappa \geq 0$. We have the following nice heat kernel estimate [92, 93],

$$K_u(\bar{x},\bar{x}) \leq \frac{1}{V(\mathcal{M})} + \frac{A}{u}\,, \tag{3.199}$$

A being a positive constant which depends on the volume of the manifold $V(\mathcal{M})$, the lower bound κ on the Ricci curvature R, and the diameter d of the manifold $\mathcal{M}$. Moreover, this constant can explicitly be calculated. If one plugs this estimate into Eq.(3.194), then one immediately obtains the inequality below,

$$||\tilde{U}(E)|| < \frac{\lambda^2 n}{\sqrt{\pi}} \int_0^1 d\zeta\zeta \int_0^\infty s^3 ds \int_0^\infty du e^{-m^2 u} \left[\frac{1}{V(\mathcal{M})} + \frac{A}{u}\right] \frac{e^{-s^2/4u}}{u^{3/2}} e^{-s\chi\zeta}\,. \tag{3.200}$$

After taking the u- and s-integral successively, the inequality takes the following form,

$$||\tilde{U}(E)|| < \frac{\lambda^2 n}{\sqrt{\pi}} \int_0^1 d\zeta\zeta \left[\frac{4\sqrt{\pi}}{(m+\chi\zeta)^3 V(\mathcal{M})} + \frac{4\sqrt{\pi}(2m+\chi\zeta)A}{(m+\chi\zeta)^2}\right]. \tag{3.201}$$

Computation of the ζ-integral gives rise to

$$\begin{aligned} ||\tilde{U}(E)|| &< 2\lambda^2 n \left[\frac{1}{m(m+\chi)^2 V(\mathcal{M})} + \frac{2A}{m+\chi}\right] \\ &< 2\lambda^2 n \left[\frac{1}{mV(\mathcal{M})\chi^2} + \frac{2A}{\chi}\right]. \end{aligned} \tag{3.202}$$

If one lets $\chi/m > 1$ and replaces one of the χ's in the first denominator by m, then the right-hand side becomes bigger,

$$||\tilde{U}(E)|| < 2\lambda^2 n \left[\frac{1}{m^2 V(\mathcal{M})} + 2A \right] \frac{1}{\chi} \,. \tag{3.203}$$

If one imposes the following condition,

$$2\lambda^2 n \left[\frac{1}{m^2 V(\mathcal{M})} + 2A \right] \frac{1}{\chi} < 1 \,, \tag{3.204}$$

then $||\tilde{U}(E)|| < 1$ is guarantied. This implies a rigorous lower bound on the ground state energy of the n-particle sector,

$$E_{gr}(n) > (n-1)m - 2\lambda^2 n \left(\frac{1}{m^2 V(\mathcal{M})} + 2A \right) \,. \tag{3.205}$$

Again, one expects that these bounds are weak, that is, a better physical approximation should prove a better bound. Nevertheless the bounds that we found illustrate the power of this approach clearly.

4. CONCLUSIONS

This thesis mainly focuses on nonperturbative aspects of quantum field theory in two distinct directions. The first one is the calculation of functional determinants of differential operators, which play an indispensable role not only in quantum effective action calculations as a loop expansion or as a large N expansion but also in nonperturbative exact renormalization group approaches. The second one is the construction of an alternative formulation of the Lee model defined on general static Riemannian manifolds, which provides a rather suitable testing ground for the study of nonperturbative renormalization.

In Chapter 2, what is studied basically depends on developing an alternative and powerful method to give a semiclassical expansion for infinite dimensional determinants encountered frequently in quantum field theory applications. In order to establish this, Weyl symbol calculus, semigroup integral representation of operators, and zeta–function regularization techniques are extensively used. For Weyl-type symbol calculus, there is a one to one correspondence between the operators acting on the Hilbert space, whose determinants are needed for the quantum effective action calculations in terms of loops. Although the main idea of this method, which is finding an expansion for the arbitrary complex powers of elliptic differential operators in pseudodifferential operator language, goes back to Seeley's excellent paper [7], a slight modification is needed so as to obtain the logarithmic terms which are highly important for the quantum field theoretical calculations through the expansion for the zeta–function regularized determinants. In order to calculate the semiclassical expansion for determinants, allowing us to obtain logarithmic terms for the operators under consideration, not only the momentum part which is formerly introduced as the principal symbol in these kinds of calculations but also the so-called potential part of the symbol of the operator should be kept, that is, both the momentum and the potential term should be recognized as the principal symbol of the pseudodifferential operator. Power of this method also comes from the fact that arbitrary differential operators which contain different types of matrix operators can easily be treated in this framework, since the operators in the

Hilbert space transform into matrix valued symbols in such cases and the same recursive expansions arising from the product rule between the phase space functions are still valid as long as the matrix degrees of freedoms of the symbols are properly taken into account.

In our opinion, not only this method for calculating the regularized determinants for elliptic operators is more natural but also more systematic regarding the corrections involving higher order derivatives. The application of the renormalization group equations to the large–N Yukawa theory within this formalism is left to the future works.

In Chapter 3, the construction of the relativistic Lee model on static Riemannian manifolds is studied. This construction is, basically, based on introducing an operator, the so-called principal operator, and renormalizing it successively [48]. Moreover, it allows us to renormalize the theory nonperturbatively. This operator, which can be regarded as a kind of effective Hamiltonian of the theory, converts a divergent linear problem in the Schrödinger picture into a highly nonlinear but a well-defined problem. Since it is found through the resolvent in the Fock space, it is valid for all particle sectors of the theory. Analysis of the behavior of the principal operator in different regimes can allow us to obtain definite information about the spectrum of the theory since the zero eigenvalues of the renormalized operator implicitly determines the bound state energies. Renormalization in this construction is established in two stages. First stage is identifying the divergences in the theory, which are tamed by a cut-off at the beginning, and then curing them by redefinitions of the appropriate parameters of the model. We show that the principal operator is free of divergences when the cut-off is removed. The second stage is specifying the renormalization conditions since there remains a finite arbitrariness in the definitions of the renormalized quantities after regularization. Since the renormalized mass of the source μ_R should, intuitively, be related to the physical mass at the lowest number of particles sector, we believe that a natural choice is to impose this condition on the renormalized principal operator. So we choose μ_p as the lowest energy solution of the equation $\Phi_R(E)|0\rangle = 0$ and replace μ_R by this physical parameter.

As shown, renormalization in the manifold case is much more complicated than the one in the flat case. The ultra-violet divergence in the theory is identified through the short-time singularity of the heat kernel, the short-time expansion of the heat kernel allows us to determine how to renormalize the bare parameters. Only the first term in the short-time expansion contributes to the divergences and these can be absorbed in the redefinitions of mass and coupling constant in $3+1$ dimensions, and the redefinition of mass in $2+1$ dimensions. As known, mass and coupling constant renormalizations are not sufficient to let the theory be free of divergences so a wave function renormalization is needed. To fix the wave function renormalization constant, we start with a Hamiltonian in which a different normalization of two states of the system is allowed. In that way, we do not need to change the normalizations of the spin states after renormalization. The well-defined limit of a suitable combination of the cut-off dependent principal operator, coupling constant and wave function renormalization constant dictates the form of the constant $Z(\epsilon)$. The divergence structure in the manifold case is the same as the one in the flat case. This is, actually, not a surprising result and it stems from the fact the divergence in the theory is an ultra-violet type. We also analyze the model in an oblique light-front coordinate system as a case study in Appendix. Same results are obtained, which encourages us to confirm the results found in[94].

There is another unconventional alternative; where we set the wave function renormalization constant $Z(\epsilon)$ to $-\lambda_R^2/\lambda^2(\epsilon)$. This will make $Z(\epsilon)$ positive below a certain value of the cut-off ϵ, hence the lower block of the Hamiltonian multiplied by a positive divergent number. It will change the off-diagonal blocks into operators multiplied by an extra i. To make the Hamiltonian hermitian on $\mathbb{C}^2 \otimes \mathcal{F}_{\mathcal{B}}(\mathcal{H})$, we should define it through the operator,

$$H_\epsilon - E = \begin{bmatrix} H_0 - E & \sqrt{Z(\epsilon)}\lambda(\epsilon)\phi_\epsilon^{(-)}(0) \\ \sqrt{Z(\epsilon)}\lambda^*(\epsilon)\phi_\epsilon^{(+)}(0) & Z(\epsilon)\left[H_0 - E + \mu(\epsilon)\right] \end{bmatrix} . \tag{4.1}$$

It is an intresting alternative to study.

In Section 3.6, working with the $3+1$ dimensional model, we calculate, first, the exact principal operator in the flat case, and then analyze the asymptotic behavior of it in the large number of bosons limit. The analysis shows that the renormalization process changes the leading term distinctively with respect to the free Hamiltonian and it takes the form $-H_0 \ln H_0$. This seems to change the dynamics of the model drastically. Therefore one should be very careful how to define the quantum Hamiltonian from the constructed resolvent. Another astonishing characteristic of this result is the sign of the leading term, which is negative. Since the normal ordered interaction term has also a negative-definite sign, the total operator is negative-definite. This implies that the ground state energy is positive. In [52] it is shown that the quantum effective action of the large-N Yukawa theory also takes a similar multiplicative contribution to the kinetic term. We, therefore, believe these results call our attentions to the point that the quantum field theoretical models should be examined in much more detail at the functional level.

In Section 3.7, to show the power of this approach, we look at the $2+1$ dimensional model, which only requires a mass renormalization and simpler. The model seems to have no ghosts. The cut-off Hamiltonian is well-defined. The renormalized resolvent allows us to give a rigorous bound on the ground state. The existance of the quantum Hamiltonian can be proved by the methods by Dimock *et al.*[71] in $2+1$ dimensions.

How to construct the relativistic Lee model on a general static Riemannian manifold is addressed so far in this paper. However, the present analysis does not give adequate information how the spectrum of the theory can be build up. Although naïve scaling arguments for the normal-ordered interaction term in $3+1$ dimensions suggest that it gives a contribution of order n, a scrutiny of this contribution around the vicinity of the source hints at a stronger dependence of n. In light of these, it is possible that the actual contribution of the interaction term is of order $n \ln n$, that is comparable to the term generated as a result of the renormalization process. The detailed analysis of the principal operator, and hence the spectrum, requires developing new approximation methods. These questions are postponed to the future works.

APPENDIX A: DETERMINANT VIA STAR-EXPONENTIAL

In this section we will give an alternative approach in order to calculate the determinant of a bosonic operator. This can be established in two stages. The first stage is to define a new heat kernel in terms of the symbol of an operator via its star-exponential in the phase space. Let $\tilde{A}(p,x)$ be the symbol of the operator A and it satisfies the scalar heat equation,

$$-\frac{\partial e^{-\tilde{A}t}}{\partial t} = \tilde{A}e^{-\tilde{A}t}\,. \tag{A.1}$$

Given a star product, one can construct a star-exponential as the formal power series via the twisted multiplication defined by this product rather than the usual pointwise multiplication, and it is given by

$$\tilde{G} = e_{\circ}^{-\tilde{A}t} = \sum_{n=0}^{\infty}\frac{1}{n!}(\tilde{A}\circ)^n\,, \tag{A.2}$$

where $(\tilde{A}\circ)^n = \tilde{A}\circ\ldots\circ\tilde{A}$ (n times). Let's now assume that this star-exponential also satisfies the following heat equation,

$$-\frac{\partial\tilde{G}}{\partial t} = \tilde{A}\circ\tilde{G}\,, \tag{A.3}$$

then the star-exponential becomes the solution of this differential equation. The second stage is to obtain an approximate solution of this equation so that by using the solution we can compute the zeta function of the operator under consideration, which allows us to attain the regularized determinant. This can be accomplished by making an anzast such that the heat kernel defined in terms of the star product admit the following semiclassical expansion,

$$\tilde{G} = e^{-\tilde{A}t}\tilde{B}\,, \tag{A.4}$$

in which the function $\tilde{B}$ has the following power series expansion in $\hbar$,

$$\tilde{B} = \tilde{B}(p, x, t) = \sum_{n=0}^{\infty} \tilde{B}_n \hbar^n \,. \tag{A.5}$$

It turns out that this anzast allows us to reduce the problem of obtaining the approximate solution of Eq. (A.3) to finding a systematic way to compute the semiclassical corrections to the classical solution of Eq. (A.3), which is nothing but merely Eq. (A.1). After placing Eq.(A.5) into Eq.(A.3), one can obtain the following differential equation which actually induces a recursion relation for $\tilde{B}_n$,

$$\frac{\partial \tilde{B}}{\partial t} = \tilde{A}\tilde{B} - e^{\tilde{A}t}\tilde{A} \circ \left(e^{-\tilde{A}t}\tilde{B}\right) \,. \tag{A.6}$$

Let's now consider the bosonic operator $-\partial^2 + V(x)$ whose symbol is given by

$$\tilde{A} = p^2 + \tilde{V}(x) \,. \tag{A.7}$$

What we would like to do is to compute $\tilde{G}$ up to $\hbar^2$-order. So, in the first place, we have to compute the second term on the right hand side of Eq. (A.6) up to $\hbar^2$-order, which is as follows:

$$\tilde{A} \circ \left(e^{-\tilde{A}t}\tilde{B}\right) \simeq \tilde{A}e^{-\tilde{A}t}\tilde{B} + \frac{i\hbar}{2}\left\{\tilde{A}, e^{-\tilde{A}t}\tilde{B}\right\}_{(1)} - \frac{\hbar^2}{8}\left\{\tilde{A}, e^{-\tilde{A}t}\tilde{B}\right\}_{(2)} + O\left(\hbar^3\right) \,. \tag{A.8}$$

The corrections are the first and the second generalized Poisson brackets. The former is given by

$$\left\{\tilde{A}, e^{-\tilde{A}t}\tilde{B}\right\}_{(1)} = \frac{\partial \tilde{A}}{\partial x^\mu}\frac{\partial}{\partial p_\mu}\left(e^{-\tilde{A}t}\tilde{B}\right) - \frac{\partial \tilde{A}}{\partial p_\mu}\frac{\partial}{\partial x^\mu}\left(e^{-\tilde{A}t}\tilde{B}\right) , \tag{A.9}$$

in which the derivatives to be calculated in the first and second term are given respectively by

$$\frac{\partial \tilde{A}}{\partial x^\mu} = \frac{\partial \tilde{V}}{\partial x^\mu}\,,$$
$$\frac{\partial}{\partial p_\mu}\left(e^{-\tilde{A}t}\tilde{B}\right) = -2tp^\mu e^{-\tilde{A}t}\tilde{B} + e^{-\tilde{A}t}\frac{\partial \tilde{B}}{\partial p_\mu}\,, \tag{A.10}$$

and

$$\frac{\partial \tilde{A}}{\partial p_\mu} = 2p^\mu\,,$$
$$\frac{\partial}{\partial x^\mu}\left(e^{-\tilde{A}t}\tilde{B}\right) = -t\frac{\partial \tilde{V}}{\partial x^\mu}e^{-\tilde{A}t}\tilde{B} + e^{-\tilde{A}t}\frac{\partial \tilde{B}}{\partial x^\mu}\,. \tag{A.11}$$

So the first correction consists of the following terms,

$$\left\{\tilde{A}, e^{-\tilde{A}t}\tilde{B}\right\}_{(1)} = \frac{\partial \tilde{V}}{\partial x^\mu}e^{-\tilde{A}t}\frac{\partial \tilde{B}}{\partial p_\mu} - 2p^\mu e^{-\tilde{A}t}\frac{\partial \tilde{B}}{\partial x^\mu}\,. \tag{A.12}$$

The latter is a bit complicated since it has more terms than the former does. It contains double derivative terms, and it takes the following form,

$$\left\{\tilde{A}, e^{-\tilde{A}t}\tilde{B}\right\}_{(2)} = \frac{\partial^2 \tilde{A}}{\partial x^\mu \partial x^\nu}\frac{\partial^2}{\partial p_\mu \partial p_\nu}\left(e^{-\tilde{A}t}\tilde{B}\right) - \frac{\partial^2 \tilde{A}}{\partial x^\mu \partial p_\mu}\frac{\partial^2}{\partial x^\nu \partial p_\mu}\left(e^{-\tilde{A}t}\tilde{B}\right)$$
$$+ \frac{\partial^2 \tilde{A}}{\partial p_\mu \partial p_\nu}\frac{\partial^2}{\partial x^\mu \partial x^\nu}\left(e^{-\tilde{A}t}\tilde{B}\right)\,. \tag{A.13}$$

The double derivatives of $\tilde{A}$ are given by

$$\frac{\partial^2 \tilde{A}}{\partial x^\mu \partial x^\nu} = \frac{\partial^2 \tilde{V}}{\partial x^\mu \partial x^\nu}\,,$$
$$\frac{\partial^2 \tilde{A}}{\partial x^\mu \partial p_\nu} = 0\,,$$
$$\frac{\partial^2 \tilde{A}}{\partial p_\mu \partial p_\nu} = 2g^{\mu\nu}\,. \tag{A.14}$$

On the other hand, the double derivatives of the combination of $e^{-\tilde{A}t}\tilde{B}$ are equal to

$$\frac{\partial^2}{\partial x^\mu \partial x^\nu}\left(e^{-\tilde{A}t}\tilde{B}\right) = -t\frac{\partial^2 \tilde{V}}{\partial x^\mu \partial x^\nu} e^{-\tilde{A}t}\tilde{B} + t^2 \frac{\partial \tilde{V}}{\partial x^\mu}\frac{\partial \tilde{V}}{\partial x^\nu} e^{-\tilde{A}t}\tilde{B} - t\frac{\partial \tilde{V}}{\partial x^\nu} e^{-\tilde{A}t}\frac{\partial \tilde{B}}{\partial x^\mu} - t\frac{\partial \tilde{V}}{\partial x^\mu} e^{-\tilde{A}t}\frac{\partial \tilde{B}}{\partial x^\nu} + e^{-\tilde{A}t}\frac{\partial^2 \tilde{B}}{\partial x^\mu \partial x^\nu}, \tag{A.15}$$

$$\frac{\partial^2}{\partial p_\mu \partial p_\nu}\left(e^{-\tilde{A}t}\tilde{B}\right) = -2tg^{\mu\nu} e^{-\tilde{A}t}\tilde{B} + 4t^2 p^\mu p^\nu e^{-\tilde{A}t}\tilde{B} - 2tp^\nu e^{-\tilde{A}t}\frac{\partial \tilde{B}}{\partial p_\mu} - 2tp^\mu e^{-\tilde{A}t}\frac{\partial \tilde{B}}{\partial p_\nu} + e^{-\tilde{A}t}\frac{\partial^2 \tilde{B}}{\partial p_\mu \partial p_\nu}. \tag{A.16}$$

If one uses them to calculate the second generalized Poisson bracket and plugs it with the first Poisson bracket into Eq. (A.8) then the second term on the right handside of Eq. (A.6) follows that

$$\begin{aligned}\tilde{A}\circ\left(e^{-\tilde{A}t}\tilde{B}\right) = \tilde{A}\tilde{B}e^{-\tilde{A}t} &+ \frac{i\hbar}{2}\left(\frac{\partial \tilde{V}}{\partial x^\mu}\frac{\partial \tilde{B}}{\partial p_\mu} - 2p^\mu \frac{\partial \tilde{B}}{\partial x^\mu}\right)e^{-\tilde{A}t} \\ &- \frac{\hbar^2}{8}\left(-4t\frac{\partial^2 \tilde{V}}{\partial x^\mu \partial x_\mu}\tilde{B} + 4t^2 \frac{\partial^2 \tilde{V}}{\partial x^\mu \partial x^\nu}p^\mu p^\nu \tilde{B}\right. \\ &- 4t\frac{\partial^2 \tilde{V}}{\partial x^\mu x_\nu}p^\mu \frac{\partial \tilde{B}}{\partial p_\nu} + \frac{\partial^2 \tilde{V}}{\partial x^\mu \partial x^\nu}\frac{\partial^2 \tilde{B}}{\partial p_\mu \partial p_\nu} \\ &\left. + 2t^2 \frac{\partial \tilde{V}}{\partial x^\mu}\frac{\partial \tilde{V}}{\partial x_\nu}\tilde{B} - 4t\frac{\partial \tilde{V}}{\partial x^\mu}\frac{\partial \tilde{B}}{\partial x_\mu} + 2\frac{\partial^2 \tilde{B}}{\partial x^\mu \partial x_\mu}\right)e^{-\tilde{A}t}.\end{aligned} \tag{A.17}$$

Therefore, $\tilde{B}$ should satisfy the differential equation below,

$$\begin{aligned}\frac{\partial \tilde{B}}{\partial t} = &-\frac{i\hbar}{2}\left(\frac{\partial \tilde{V}}{\partial x^\mu}\frac{\partial \tilde{B}}{\partial p_\mu} - 2p^\mu \frac{\partial \tilde{B}}{\partial x^\mu}\right) \\ &\frac{\hbar^2}{8}\left(-4t\frac{\partial^2 \tilde{V}}{\partial x^\mu \partial x_\nu}\tilde{B} + 4t^2 \frac{\partial^2 \tilde{V}}{\partial x^\mu \partial x^\nu}p^\mu p^\nu \tilde{B}\right. \\ &- 4t\frac{\partial^2 \tilde{V}}{\partial x^\mu \partial x_\nu}p^\mu \frac{\partial \tilde{B}}{\partial p_\nu} + \frac{\partial^2 \tilde{V}}{\partial x^\mu \partial x^\nu}\frac{\partial^2 \tilde{B}}{\partial p_\mu \partial p_\nu} \\ &\left. + 2t^2 \frac{\partial \tilde{V}}{\partial x^\mu}\frac{\partial \tilde{V}}{\partial x_\mu}\tilde{B} - 4t\frac{\partial \tilde{V}}{\partial x^\mu}\frac{\partial \tilde{B}}{\partial x_\mu} + 2\frac{\partial^2 \tilde{B}}{\partial x^\mu \partial x_\mu}\right).\end{aligned} \tag{A.18}$$

Using the formal $\hbar$-expansion of $\tilde{B}$ on the both handsides and comparing the coefficients of the same power of $\hbar$ imply the following equalities,

$$\frac{\partial \tilde{B}_0}{\partial t} = 0\,, \quad \tilde{B}_0 = 1\,, \tag{A.19}$$

$$\frac{\partial \tilde{B}_1}{\partial t} = 0\,, \quad \tilde{B}_1 = 0\,, \tag{A.20}$$

and

$$\begin{aligned} \frac{\partial \tilde{B}_2}{\partial t} &= \frac{1}{8}\left(-4t\frac{\partial^2 \tilde{V}}{\partial x^\mu \partial x_\mu} + 4t^2 \frac{\partial^2 \tilde{V}}{\partial x^\mu \partial x^\nu} p^\mu p^\nu + 2t^2 \frac{\partial \tilde{V}}{\partial x^\mu}\frac{\partial \tilde{V}}{\partial x_\mu}\right)\,, \\ \tilde{B}_2 &= -\frac{t^2}{4}\frac{\partial^2 \tilde{V}}{\partial x^\mu \partial x_\mu} + \frac{t^3}{6}\frac{\partial^2 \tilde{V}}{\partial x^\mu \partial x^\nu} p^\mu p^\nu + \frac{t^3}{12}\frac{\partial \tilde{V}}{\partial x^\mu}\frac{\partial \tilde{V}}{\partial x_\mu}\,. \end{aligned} \tag{A.21}$$

Thus, the semiclassical expansion of the heat kernel via star-exponential up to $\hbar^2$-order takes the following form

$$\tilde{G}(p,x,t) \simeq e^{-t(p^2+\tilde{V})}\left(1 - \frac{\hbar^2 t^2}{4}\frac{\partial^2 \tilde{V}}{\partial x^\mu \partial x_\mu} + \frac{\hbar^2 t^3}{6}\frac{\partial^2 \tilde{V}}{\partial x^\mu \partial x^\nu} p^\mu p^\nu + \frac{\hbar^2 t^3}{12}\frac{\partial \tilde{V}}{\partial x^\mu}\frac{\partial \tilde{V}}{\partial x_\mu}\right) + O\left(\hbar^3\right)\,. \tag{A.22}$$

Since the trace of the heat kernel $\mathrm{Tr}\tilde{G}$ is needed in order to compute the zeta function of the operator, one should take the p- and x-integral of $\tilde{G}$, subsequently. It follows that

$$\begin{aligned} \mathrm{Tr}\tilde{G} &\simeq \int d^4x e^{-t\tilde{V}} \int \frac{d^4p}{(2\pi)^4}\frac{e^{-p^2 t}}{t^2}\left(1 - \frac{\hbar^2 t^2}{4}\frac{\partial^2 \tilde{V}}{\partial x^\mu \partial x_\mu} + \frac{\hbar^2 t^3}{6}\frac{\partial^2 \tilde{V}}{\partial x^\mu \partial x^\nu} p^\mu p^\nu \right. \\ &\quad \left. + \frac{\hbar^2 t^3}{12}\frac{\partial \tilde{V}}{\partial x^\mu}\frac{\partial \tilde{V}}{\partial x_\mu}\right) + O\left(\hbar^3\right) \\ &= \int d^4x \frac{e^{-t\tilde{V}}}{16\pi^2 t^2}\left(1 - \frac{t^2}{4}\frac{\partial^2 \tilde{V}}{\partial x^\mu \partial x_\mu} + \frac{t^3}{12}\frac{\partial \tilde{V}}{\partial x^\mu}\frac{\partial \tilde{V}}{\partial x_\mu} + \frac{t^2}{12}\frac{\partial^2 \tilde{V}}{\partial x^\mu \partial x_\mu}\right) \\ \mathrm{Tr}G &\simeq \int d^4x \frac{e^{-tV}}{16\pi^2 t^2}\left(1 - \frac{t^2}{6}\frac{\partial^2 V}{\partial x^\mu \partial x_\mu} + \frac{t^3}{12}\frac{\partial V}{\partial x^\mu}\frac{\partial V}{\partial x_\mu}\right) + O\left(\hbar^3\right)\,. \end{aligned} \tag{A.23}$$

Now, we are ready to compute the zeta function of the operator. What we know is the fact that the zeta function of an operator is the Mellin transform of its heat kernel such that it is given by

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \mathrm{Tr}\tilde{G}\,. \tag{A.24}$$

Thus, the zeta function of the operator $-\partial^2 + V(x)$, which is calculated throughout its heat kernel defined by the star-exponential of its symbol $p^2 + \tilde{V}(x)$, up to $\hbar^2$-order is given by

$$\begin{aligned}
\zeta(s) &= \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \int d^4x \frac{e^{-tV}}{16\pi^2 t^2} \left(1 - \frac{t^2}{6} \frac{\partial^2 V}{\partial x^\mu x_\mu} + \frac{t^3}{12} \frac{\partial V}{\partial x^\mu} \frac{\partial V}{\partial x_\mu}\right) + \cdots \\
&= \frac{1}{16\pi^2} \int d^4x \left[\frac{\Gamma(s-2)}{\Gamma(s)} \tilde{V}^{2-s} - \frac{V^{-s}}{6} \frac{\partial^2 V}{\partial x^\mu \partial x_\mu} + \frac{\Gamma(s+1)}{\Gamma(s)} \frac{V^{-s-1}}{12} \frac{\partial V}{\partial x^\mu} \frac{\partial V}{\partial x_\mu}\right] + \cdots \\
\zeta(s) &= \frac{1}{16\pi^2} \int d^4x \left[\frac{V^{2-s}}{(s-1)(s-2)} - \frac{V^{-s}}{6} \frac{\partial^2 V}{\partial x^\mu \partial x_\mu} + \frac{sV^{-s-1}}{12} \frac{\partial V}{\partial x^\mu} \frac{\partial V}{\partial x_\mu}\right] + \cdots\,,
\end{aligned} \tag{A.25}$$

which is exactly the same result that we found in Section 2.5. I noticed that a similar approach to the determinant calculation of a bosonic operator via moyal product was also studied in [95] after I had constructed this method.

APPENDIX B: THE LEE MODEL IN THE LIGHT-FRONT COORDINATES

In this section, we will give a brief sketch of the construction of the Lee model and the calculation of the principal operator in the light-front coordinate system, and will show that the theory in this coordinate system has the same divergence structure. The following oblique coordinate system is chosen,

$$u = t + x \, , \tag{B.1}$$

where u is the light-front time coordinate. The infinitesimal invariant distance element, the metric tensor and its inverse are also given by

$$ds^2 = du^2 - 2dudx - dy^2 - dz^2 \, , \tag{B.2}$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \, , \quad g^{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \, . \tag{B.3}$$

The scalar product of the coordinates and the conjugate momenta is

$$p_\mu x^\mu = p_u u + px + \mathbf{p}_\perp \cdot \mathbf{x}^\perp \tag{B.4}$$

where x and $\mathbf{x}^\perp$ are the longitudinal and the transverse coordinates, on the other hand p_u, p and $\mathbf{p}_\perp$ are the light-front energy, the longitudinal and the transverse momenta, respectively. In the equal-time formulation, the bosonic field operator is given by

$$\phi(x, \mathbf{x}^\perp) = \int_0^\infty \frac{dp}{2\pi} \int \frac{d^2 p_\perp}{(2\pi)^2} \frac{1}{\sqrt{2p}} \left[a(p, \mathbf{p}_\perp) e^{-ipx - i\mathbf{p}_\perp \cdot \mathbf{x}^\perp} + a^\dagger(p, \mathbf{p}_\perp) e^{ipx + i\mathbf{p}_\perp \cdot \mathbf{x}^\perp} \right] \, . \tag{B.5}$$

The equal-time commutation relations both for fields and for creation and annihilation operators are, respectively, given by

$$\big[\phi(u,x,\mathbf{x}^{\perp}),\phi(u,y,\mathbf{y}^{\perp})\big] = \frac{1}{4}\mathrm{sgn}(x-y)\delta^{(2)}(\mathbf{x}^{\perp}-\mathbf{y}^{\perp}) \,, \tag{B.6}$$

$$\big[a(p,\mathbf{p}_{\perp}),a^{\dagger}(q,\mathbf{q}_{\perp})\big] = (2\pi)^3\delta(p-q)\delta^{(2)}(\mathbf{p}_{\perp}-\mathbf{q}_{\perp}) \,. \tag{B.7}$$

The free Hamiltonian of the bosonic sector is

$$H_0 = \int_0^{\infty}\frac{dp}{2\pi}\int\frac{d^2p_{\perp}}{(2\pi)^2}\,\omega(p,\mathbf{p}_{\perp})a^{\dagger}(p,\mathbf{p}_{\perp})a(p,\mathbf{p}_{\perp}) \,, \tag{B.8}$$

where $\omega(p,\mathbf{p}_{\perp}) = \frac{m^2+p^2+\mathbf{p}_{\perp}^2}{2p}$. The positive and the negative frequency parts of the fields evaluated at the point zero are given by

$$\phi_{\epsilon}^{(+)}(0) = \int_0^{\infty}\frac{dp}{2\pi}\int\frac{d^2p_{\perp}}{(2\pi)^2}\,\frac{a(p,\mathbf{p}_{\perp})}{\sqrt{2p}} \,, \tag{B.9}$$

$$\phi_{\epsilon}^{(-)}(0) = \int_0^{\infty}\frac{dp}{2\pi}\int\frac{d^2p_{\perp}}{(2\pi)^2}\,\frac{a^{\dagger}(p,\mathbf{p}_{\perp})}{\sqrt{2p}} \,. \tag{B.10}$$

After normal-ordering the creation and annihilation operators, the principal operator takes the form

$$\begin{aligned}
\frac{\Phi_{\epsilon}(E)}{\lambda^2(\epsilon)} &= Z(\epsilon)\Bigg\{\frac{(H_0-E)}{\lambda^2(\epsilon)} + \frac{\mu(\epsilon)}{\lambda^2(\epsilon)} \\
&\quad - \frac{1}{2}\int_0^{\infty}\frac{dp}{2\pi}\int\frac{d^2p_{\perp}}{(2\pi)^2}\int_0^{\infty}\frac{dq}{2\pi}\int\frac{d^2q_{\perp}}{(2\pi)^2}\,\frac{1}{\sqrt{pq}}a(p,\mathbf{p}_{\perp})\frac{1}{H_0-E}a^{\dagger}(q,\mathbf{q}_{\perp})\Bigg\} \\
&= Z(\epsilon)\Bigg\{\frac{(H_0-E)}{\lambda^2(\epsilon)} + \frac{\mu(\epsilon)}{\lambda^2(\epsilon)} - \int_0^{\infty}\frac{dp}{2\pi}\int\frac{d^2p_{\perp}}{(2\pi)^2}\,\frac{1}{2p}\frac{1}{H_0-E+\omega(p,\mathbf{p}_{\perp})} \\
&\quad - \int_0^{\infty}\frac{dp}{2\pi}\int\frac{d^2p_{\perp}}{(2\pi)^2}\int_0^{\infty}\frac{dq}{2\pi}\int\frac{d^2q_{\perp}}{(2\pi)^2}\,\frac{1}{2\sqrt{pq}} \\
&\quad \times a^{\dagger}(q,\mathbf{q}_{\perp})\frac{1}{H_0-E+\omega(q,\mathbf{q}_{\perp})+\omega(p,\mathbf{p}_{\perp})}a(p,\mathbf{p}_{\perp})\Bigg\} \,.
\end{aligned} \tag{B.11}$$

We do not need to use any Feynman parametrizations here and only an exponentiation is enough to complete the calculations, so the momentum integral in the fourth term in Eq. (B.11) is, just,

$$\int_0^\infty \frac{dp}{2\pi} \int \frac{d^2p_\perp}{(2\pi)^2} \frac{1}{2p} \frac{1}{H_0 - E + \omega(p, \mathbf{p}_\perp)}$$
$$= \int_\epsilon^\infty du \int_0^\infty \frac{dp}{2\pi} \int \frac{d^2p_\perp}{(2\pi)^2}\, e^{-2u(H_0-E)p - u(m^2+p^2+\mathbf{p}_\perp^2)}\,. \tag{B.12}$$

At this stage, we should be careful about the limits of the angular part of the momentum integral. Since we work in a coordinate system which covers either the future-cone or the past-cone, after the following change of variables,

$$p^2 + \mathbf{p}_\perp^2 = s^2 \quad \Rightarrow \quad p = s\cos\theta\,, \quad \mathbf{p}_\perp = s\sin\theta\,, \tag{B.13}$$

the integration interval of the θ-integral becomes $[0, \frac{\pi}{2}]$. Equation (B.12) is, then,

$$\frac{1}{(2\pi)^3} \int_\epsilon^\infty du\, e^{-m^2u} \int_0^\infty ds\, s^2 \int_0^{\pi/2} d\theta\, \sin\theta \int_0^{2\pi} d\phi\, e^{-2u(H_0-E)s\cos\theta - us^2} =$$
$$\frac{1}{8(2\pi)^2} \int_\epsilon^\infty du\, \frac{e^{-m^2u}}{u^{3/2}} \int_0^\infty ds\, se^{-s^2/4} \frac{1}{\sqrt{u}(H_0-E)} \left[1 - e^{-s\sqrt{u}(H_0-E)}\right]\,. \tag{B.14}$$

By using the exponential representation of the fractions in the fifth term in Eq. (B.11), the principal operator is given by

$$\frac{\Phi_\epsilon(E)}{\lambda^2(\epsilon)} = Z(\epsilon)\Bigg\{ \frac{(H_0 - E)}{\lambda^2(\epsilon)} + \frac{\mu(\epsilon)}{\lambda^2(\epsilon)}$$
$$- \frac{1}{32\pi^2} \int_\epsilon^\infty du\, \frac{e^{-m^2u}}{u^{3/2}} \int_0^\infty ds\, se^{-s^2/4} \frac{\left[1 - e^{-s\sqrt{u}(H_0-E)}\right]}{\sqrt{u}(H_0-E)}$$
$$- \frac{2}{\pi} \int_0^\infty ds \int_0^\infty d\alpha \int_0^\infty d\beta \int_0^\infty \frac{dp}{2\pi} \int \frac{d^2p_\perp}{(2\pi)^2} \int_0^\infty \frac{dq}{2\pi} \int \frac{d^2q_\perp}{(2\pi)^2}$$
$$\times\, e^{-q\alpha^2 - s\omega(q,\mathbf{q}_\perp)} e^{-p\beta^2 - s\omega(p,\mathbf{p}_\perp)} a^\dagger(q, \mathbf{q}_\perp) e^{-s(H_0-E)} a(p, \mathbf{p}_\perp) \Bigg\}\,. \tag{B.15}$$

With the help of the redefinitions of the mass and the coupling constant below

$$\frac{\mu(\epsilon)}{\lambda^2(\epsilon)} = \frac{\mu_R}{\lambda_R^2} + \frac{1}{32\pi^2}\int_\epsilon^\infty du\, \frac{e^{-um^2}}{u^{3/2}} \int_0^\infty ds\, s^2 e^{-s^2/4}\,, \tag{B.16}$$

$$\frac{1}{\lambda^2(\epsilon)} = \frac{1}{\lambda_R^2} - \frac{1}{64\pi^2}\int_\epsilon^\infty du\, \frac{e^{-um^2}}{u} \int_0^\infty ds\, s^3 e^{-s^2/4}\,, \tag{B.17}$$

one can take the limit $\epsilon \to 0^+$ after dividing both sides by $Z(\epsilon)$ and hence the renormalized principal operator takes the form,

$$\begin{aligned}
\frac{\Phi_R(E)}{\lambda_R^2} = & \frac{(H_0 - E)}{\lambda_R^2} + \frac{\mu_R}{\lambda_R^2} - \frac{1}{32\pi^2}\int_0^\infty du\, \frac{e^{-m^2 u}}{u^{3/2}} \int_0^\infty ds\, s e^{-s^2/4} \frac{1}{\sqrt{u}(H_0 - E)} \\
& \times \left[1 - s\sqrt{u}(H_0 - E) + \frac{1}{2}s^2 u (H_0 - E) - e^{-s\sqrt{u}(H_0 - E)}\right] \\
& - \frac{2}{\pi}\int_0^\infty ds \int_0^\infty d\alpha \int_0^\infty d\beta \int_0^\infty \frac{dp}{2\pi} \int \frac{d^2 p_\perp}{(2\pi)^2} \int_0^\infty \frac{dq}{2\pi} \int \frac{d^2 q_\perp}{(2\pi)^2} \\
& \times e^{-q\alpha^2 - s\omega(q, \mathbf{q}_\perp)} e^{-p\beta^2 - s\omega(p, \mathbf{p}_\perp)} a^\dagger(q, \mathbf{q}_\perp) e^{-s(H_0 - E)} a(p, \mathbf{p}_\perp)\,.
\end{aligned} \tag{B.18}$$

Now to see the divergence patterns, we can again calculate the bare mass and the bare coupling constant asymptotically in ϵ, as a result we find the following,

$$\frac{\mu(\epsilon)}{\lambda^2(\epsilon)} \simeq \frac{\mu_R}{\lambda_R^2} + \frac{1}{8\pi^{3/2}}\frac{1}{\sqrt{\epsilon}} \quad \text{as} \quad \epsilon \to 0^+\,, \tag{B.19}$$

$$\frac{1}{\lambda^2(\epsilon)} \simeq \frac{1}{\lambda_R^2} + \frac{1}{8\pi^2}\ln\epsilon \quad \text{as} \quad \epsilon \to 0^+\,. \tag{B.20}$$

We note that the divergences are controlled by the cut-off parameters in exactly the same way as in the previous cases. We believe, this is in a cord with the discussion presented by the authors in [94] about the equivalence of the covariant perturbation theory and the light-front perturbation theory. This may be seen as another verification of this equivalence at a nonperturbative level.

The asymptotic limit of the renormalized principal operator can, of course, be analyzed in this case, as well. For the calculations to be done for this analysis repeat themselves, we will not continue further in this direction.

APPENDIX C: HYPERBOLIC SPACE $\mathbb{H}^3$

Three dimensional hyperbolic space in the half space model can be defined by, [86],

$$\mathbb{H}^3 = \{z = (x, y) : x \in \mathbb{R}^2 \text{ and } 0 < y < \infty\}\,. \tag{C.1}$$

The metric of this manifold can be given by

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2)\,. \tag{C.2}$$

The top volume form is

$$\mathrm{dvol} = \frac{1}{y^3} dx\, dy\,, \tag{C.3}$$

dx being the Euclidean volume element in $\mathbb{R}^2$. The Laplace operator is given by

$$\Delta = y^2 \left(\Delta_x + \frac{\partial^2}{\partial y^2} - \frac{1}{y}\frac{\partial}{\partial y} \right)\,, \tag{C.4}$$

where Δ_x is the Euclidean Laplace operator on $\mathbb{R}^2$. The geodesic distance $d = d(z_1, z_2)$ between two points z_1 and z_2 on $\mathbb{H}^3$ can be given by

$$\cosh d(z_1, z_2) = 1 + \frac{|x_1 - x_2|^2 + (y_1 - y_2)^2}{2y_1 y_2}\,, \tag{C.5}$$

in which $|x_1 - x_2|$ denotes the Euclidean geodesic distance in $\mathbb{R}^2$.

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