ANALYSIS OF CLUSTER CONSENSUS IN CONTINUOUS-TIME NETWORKS WITH TIME DELAYS

by

Ümit Develer

B.S., Electrical and Electronics Engineering, Boğaziçi University, 2015

Submitted to the Institute for Graduate Studies in
Science and Engineering in partial fulfillment of
the requirements for the degree of
Master of Science

Graduate Program in Electrical and Electronics Engineering
Boğaziçi University
2018

ACKNOWLEDGEMENTS

I would like to begin by expressing my sincere gratitude to my supervisor Prof. Mehmet Akar for his support, guidance and providing me with all the necessary facilities to complete this thesis. The door to Prof. Akar office was always open whenever I ran into a trouble spot or had a question about my research or writing.

I am also grateful to the members of my committee Prof. Kemal Cılız and Assist. Prof. Onur Cihan for their insightful comments, advice and patience throughout my thesis study.

I would like to thank to my friends and colleagues in the Networked and Embedded Systems Laboratory (NECSLab), Özlem Feyza Erkan, Halil Yiğit Öksüz and Günay Turan for their support and assistance at each step of this thesis. Morever, I am also grateful to my other friends and colleagues at Boğaziçi University for their help and support.

A very special gratitude goes to my parents Ahmet Develer, Fatma Develer and my siblings for providing me with unfailing support and continuous encouragement throughout my years of study and through the process of researching and writing this thesis.

This thesis was sponsored by Scientific and Technical Research Council of Turkey (TUBITAK) under Grant 114E613.

ABSTRACT

ANALYSIS OF CLUSTER CONSENSUS IN CONTINUOUS-TIME NETWORKS WITH TIME DELAYS

In recent years, distributed coordination of multi-agent systems has the attention of many researchers due to its potential applications in various fields including group robots, distributed sensor networks, bird flocks, the attitude alignment in multiple spacecraft settings and control in communication systems. All of these applications indicate the importance of design and analysis of consensus protocols with which the agents agree on some particular variable of interest by exchanging information among themselves. Complete consensus where all agents in a system achieve a common objective is one of the most popular studies. A network using distributed consensus protocol may be divided into different groups which are called *clusters* depending on the interaction topology of the network. In this thesis, we investigate the cluster consensus problem for a multi-agent system with or without delay in continuous-time. Contrary to existing studies on cluster consensus in the literature, we do not assume that the clusters are pre-determined. The main contributions of the dissertation are to determine the number of clusters for a multi-agent system and to state delay conditions that does not affect the number of clusters and convergence properties of the system. The results are obtained by using primary and secondary layer subgraphs and sub-Laplacian matrices which are explained for the first time in the literature. The upper bound of input delay that does not lead to instability is determined. It is also shown that bounded communication delay does not adversely affect system stability properties, which is also supported by computer simulations.

ÖZET

GECİKMELİ SÜREKLİ ZAMAN AĞLARDA KÜMELENME ONAYLAŞIM ANALİZİ

Son yıllarda, çok etmenli sistemlerin dağıtık koordinasyonu, grup robotları, dağıtık sensör ağları, kuş sürüleri, çoklu uzay araçlarında durum hizalaması ve iletişim sistemlerinde kontrol gibi çeşitli alanlardaki potansiyel uygulamaları nedeniyle birçok araştırmacının dikkatini çekmektedir. Bu uygulamaların hepsi, etmenlerin aralarında bilgi alışverişi yaparak, belirli bir ilgi alanı değişkeninin üzerindeki onaylaşım protokollerin tasarım ve analizinin önemini göstermektedir. Bir sistemdeki tüm etmenlerin ortak bir hedefe ulaştığı tam onaylaşım en popüler çalışmalardan biridir. Dağıtık onaylaşım protokolünü kullanan bir ağ, ağın etkileşim topolojisine bağlı olarak kümeler olarak adlandırılan farklı gruplara ayrılabilir. Bu tez çalışmasında, sürekli-zaman içinde gecikmeli veya gecikmesiz bir çoklu etmen sistemi için kümelenme onaylaşım problemi incelenmiştir. Literatürde var olan kümelenme onaylaşım üzerine yapılan çalışmaların aksine, kümelerin önceden belirlendiği varsayılmamıştır. Tezin ana katkıları, çok etmenli bir sistem için küme sayısını belirlemek ve sistemin kümelenme sayısı ve yakınsama özelliklerini etkilemeyen gecikme koşullarını ortaya koymaktır. Sonuçlar birincil ve ikincil katman alt çizgeleri ile literatürde ilk kez açıklanmış olan sub-Laplacian matrisleri kullanılarak elde edilmiştir. Kararsızlığa neden olmayan giriş gecikmesinin üst sınırı belirlenmiştir. Sınırlı iletişim gecikmesinin sistem kararlılığını olumsuz etkilemediği de, hem kuramsal hem de benzetim çalışmalarıyla gösterilmiştir.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF SYMBOLS

a_{ij}	The (ij)-th element of the matrix A
A	The adjacency matrix
C_p	The cluster p
$\deg_{\text{in}}(v_i)$	The in-degree of node v_i
$\deg_{\text{out}}(v_i)$	The out-degree of node v_i
G	A directed graph
$G_{p,i}$	The i-th primary layer subgraph
$G_{s,i}$	The i-th secondary layer subgraph
I	The index set
I_p	The index set of the primary layer subgraphs
I_s	The index set of the secondary layer subgraphs
l_p	The number of primary layer subgraphs
l_s	The number of secondary layer subgraphs
L	The Laplacian matrix
N_i	The neighbor set of node v_i
$n_{p,i}$	The number of nodes in the i-th primary layer subgraph
$n_{s,j}$	The number of nodes in the j-th secondary layer subgraph
$\tilde{n}_p$	The total number of nodes in the primary layer subgraphs
$\tilde{n}_s$	The total number of nodes in the secondary layer subgraphs
q_i	The basis vector which has 1 as the i-th component
R_i	Reachability set of node v_i
$T(G)$	The total weight of all spanning subtrees in digraph G
$u_i(t)$	The system input of node v_i at time t
V	The set of nodes
$V_{p,i}$	The node set of i-th primary layer subgraph
$V_{s,i}$	The node set of i-th secondary layer subgraph
$w(T)$	The weight for spanning subtree T
$x_i(t)$	The state of node v_i at time t

$x_p(t)$	The state vector of the primary layer subgraphs at time t
$x_s(t)$	The state vector of the secondary layer subgraphs at time t
$\tilde{x}_p$	The equilibrium point of the primary layer dynamics
$\tilde{x}_s$	The equilibrium point of the secondary layer dynamics
Δ	Degree matrix
Δt	A time period
ε_{ij}	Edge from node v_j to v_i
$\mathcal{E}$	The set of edges
$\bar{\lambda}_i(\cdot)$	The real part of i-th smallest eigenvalue of a matrix
π_n	The column vector with all entries equal to 1 in R^n
$\rho(\cdot)$	The spectral radius of a matrix
τ_d	The fixed input delay
τ^*	Upper bound of fixed input delay
τ_{cd}	The bounded communication delay

LIST OF ACRONYMS/ABBREVIATIONS

MIMO	Multiple-Input Multiple-Output
LMI	Linear Matrix Inequality
RHP	Right Half Plane
ORHP	Open Right Half Plane

1. INTRODUCTION

Over the past two decades, the distributed consensus problem, one of the most studied in multi-agent systems, has been the center of interest due to its potential applications in different disciplines including group robots, social networks, biology, physics, control engineering, unmanned vehicles [1–6].

In the literature, there is a tremendous amount of studies on *complete consensus* in which all agents of a multi-agent system reach the same final state. However, it is also possible that agents in a multi-agent system may attain multiple steady states which are called *clusters*. Agents in a clusters achieve a consensus state, but other agents in different clusters go to different states. This phenomenon is referred to as *cluster consensus*.

In this thesis, our primary focus is to study the cluster consensus problem for a multi-agent system in continuous-time and to investigate the effect of delay that may lead to instability in the system. Some related studies on either cluster consensus or delayed multi-agent systems are briefly reviewed below.

The authors of [7] investigate the cluster consensus problem for multi-agent systems operating in discrete-time under fixed and switching topologies. They propose a condition which guarantees cluster consensus of a multi-agent system such that the number of clusters is equal to the period of the system. However, the result is only valid for a multi-agent system using strongly connected graph in which the agents do not use their own state values.

In [8], the cluster consensus problem is studied for discrete-time multi-agent systems with periodic inter-cluster nonidentical inputs to separate the system into clusters. Sufficient conditions are stated for fixed and switching topologies. Dividing a multi-agent network into clusters artificially is a significant drawback of this work.

In [9], the cluster consensus problem is discussed for nonlinearly networked multi-agent systems operating in continuous-time. Also, a practical controller is designed so that the clusters of the system achieve desired state values. Assumptions that provide the system to achieve cluster consensus are rather strict. The Laplacian system matrix is divided into sub-blocks, each of which is required to have zero row sums. Additionally, all diagonal sub-blocks have a spanning tree and off-diagonal sub-blocks have the same row.

The authors of [10] consider a continuous-time multi-agent system as an interconnection of smaller sub-networks under the assumption that the total information exchange between two sub-networks is zero, i.e., the interaction between any two sub-network is balanced. Some sufficient conditions for cluster consensus are obtained in the form of linear matrix inequalities (LMI), but it is not explicitly explained when these LMIs are solvable. In [11], the authors extend their study for multi-agent systems with switching topologies and communication delays.

Shang and Yamei [12] investigate the fixed-time cluster consensus problem for leader-following multi-agent systems operating in continuous-time. They generalize the leader-follower consensus protocol for multi-agent systems that have multiple leaders. The main goal of the study is to explicitly estimate the settling time independent of initial states. In [13], the authors extend their study to network that have unknown inherent nonlinear dynamics. They divide a multi-agent system in sub-groups, each of which has a virtual leader, and propose controllers not requiring the inter-group balance condition. However, the results are only valid for pre-determined clusters which is a main drawback of this work.

Neither of the studies discussed above considers the effect of delay. However, networks may have delays that can be caused by information process or information transmission from one agent to another. Although a great number of studies [14–16] on delayed multi-agent systems are done to deal with complete consensus problem, the effect of delay on cluster consensus is not deeply investigated in the literature. Some related studies on complete consensus or cluster consensus in delayed networks are briefly reviewed below.

In [17], Olfati-Saber and Murray study the consensus problem in networks of agents with or without delays. Multi-agent systems they investigate can be under fixed or switching topologies, directed or undirected information exchange. They derive the maximum delay that can be tolerated by a multi-agent system. The maximum is inversely proportional to the largest eigenvalue of the Laplacian system matrix. However, this result is only valid for complete consensus multi-agent systems evolving over undirected graphs.

The authors of [18] investigate the average consensus problem of multi-agent systems with constant or time-varying delay under switching topologies. A Lyapunov-Krasovskii function is used to guarantee average consensus of the agents under arbitrary switching topology. Sufficient conditions are stated in terms of LMIs for directed networks.

Chen *et al.* [19] study the delay effect on cluster consensus of second-order multi-agent systems. They consider a multi-agent network as an interconnection of two sub-groups and construct a consensus protocol for each sub-group to reach two different final values. They not only obtain consensus conditions without delays, but also delay-dependent conditions are stated by frequency-domain analysis.

In the multi-agent systems studied in [20], agents have second-order dynamics, connections of agents are directed, and coupling delay is time-varying. Lyapunov-Razumikhin functions are used to deal with convergence and stability problems of multi-agent systems with coupling delays in continuous-time. The authors of [21] study the consensus problem for directed networks with communication delay in continuous-time. They investigate the effect of delay in both leaderless and leader-following cases. In the leaderless case, the system can asymptotically reach consensus under finite communication delay. Desired final state can be obtained in the leader-following case. They generalize their result which is for linearly networked systems to nonlinearly coupled systems. However, all results depend on a strict assumption that the graph corresponding network is strongly connected.

In [11], cluster consensus for networks with switching topologies and communication delays is studied. Yu and Wang propose an algorithm such that agents in a network achieve more than one consistent state and introduce double-tree-form transformation that helps to investigate the cluster consensus problem for networks under time-varying topologies. Similar to other studies in the literature, they divide artificially networks into clusters.

1.1. Motivation of the Thesis

In this thesis, we investigate the cluster consensus problem for multi-agent systems with or without delay in continuous-time. The main goal of the thesis is to determine the number of clusters in a multi-agent network and to derive delay conditions that does not affect the number of clusters and convergence properties of the agents. The study discussed in [22] is extended to the networks operating in continuous-time. We study the effect of both input and communication delay on cluster consensus. The upper bound of input delay that does not let the system instability is explicitly determined. In the presence of communication delay, we carry out the stability analysis of the system. Unlike all mentioned papers on cluster consensus, we do not assume that we know the number of clusters a priori. On the contrary, we determine the number of clusters for any directed network by one of the main results of this thesis.

1.2. Contribution of the Thesis

The main contributions of this thesis, which have also been [23–25] can be summarized as follows:

- For a multi-agent system operating in continuous-time represented with an arbitrary directed graph, the number of clusters is determined explicitly.
- Sub-Laplacian matrices are introduced and their properties are exploited in the stability analysis for the first time in the literature.

- The upper bound of input delay that does not change the number of clusters and stability properties is determined.
- The effect of communication delay is investigated and it is shown that bounded communication delay does not change the number of clusters.

1.3. Organization of the Thesis

The rest of the thesis is organized as follows. In Chapter 2, we review some concepts from graph theory and formulate the distributed cluster consensus problem. Cahpter 3 contains primary and secondary layer subgraphs definitions, the investigation of cluster consensus in continuous-time and related numerical analysis. In Chapter 4, the effect of input delay on cluster consensus is studied and some simulation results are given. Chapter 5 provides stability and convergence analysis of networks under bounded communication delays. Moreover, theoretical results are illustrated via examples in this section. Finally, there are some concluding remarks in Chapter 6.

2. CONTINUOUS-TIME DISTRIBUTED CONSENSUS PROTOCOL AND MATHEMATICAL PRELIMINARIES

In this chapter, we review basic concepts of graph theory associated with multi-agent networks and a continuous-time distributed consensus protocol that will be used in the thesis. This chapter also includes the definition of the cluster consensus problem for multi-agents systems.

2.1. Basic Concepts of Graph Theory

Let $G = (V, \mathcal{E}, A)$ be a weighted digraph (directed graph) of order n with the set of nodes $V = \{v_1, v_2, \ldots, v_n\}$, set of edges $\mathcal{E} \subseteq V \times V$ and a weighted adjacency matrix $A = [a_{ij}] \in R^{n\times n}$ with nonnegative elements. The node indices belong to a finite index set $I = \{1, 2, \ldots, n\}$. If there is an information flow from node v_j to node v_i, edge $\varepsilon_{ij} = (v_j, v_i)$ exists. The adjacency element, a_{ij}, corresponding to the edge, ε_{ij}, is positive, i.e., $\varepsilon_{ij} \in \mathcal{E} \Leftrightarrow a_{ij} > 0\ \forall i, j \in I$. The set of neighbors of node v_i is defined by $N_i = \{v_j | \varepsilon_{ij} \in \mathcal{E}\}$. If there is a route between any two nodes, a digraph is called as *strongly connected*. If there exists at least one node, which has no parent, such that there are routes from the node to all other nodes, digraph has a *spanning tree*.

The *degree matrix* is an $n\times n$ diagonal matrix defined as $\Delta = \Delta(G) = \{\Delta_{ij}\}$. The in-degree of node v_i is the number of its inward edges and denoted by $\deg_{\text{in}}(v_i)$, and the out-degree of node v_i is the number of its outward edges and denoted by $\deg_{\text{out}}(v_i)$. $\deg_{\text{in}}(v_i)$ and $\deg_{\text{out}}(v_i)$ are defined as follows:

$$\deg_{\text{in}}(v_i) = \sum_{j=1}^{n} a_{ij}, \quad \deg_{\text{out}}(v_i) = \sum_{j=1}^{n} a_{ji}. \tag{2.1}$$

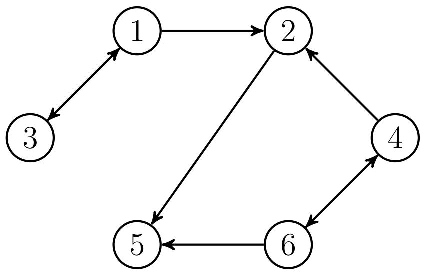

Figure 2.1. A digraph with 6 nodes and 8 edges.

The degree matrix for a digraph is defined as

$$\Delta_{ij} := \begin{cases} deg_{in}(v_i), & i = j, \\ 0, & i \neq j. \end{cases} \tag{2.2}$$

The Laplacian matrix of G is defined by

$$L = \Delta - A. \tag{2.3}$$

Example 2.1. *Consider the digraph shown in Figure 2.1. In case the digraph is not weighted the adjacency and degree matrices corresponding to given graph are as follows:*

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad \Delta = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The adjacency elements corresponding to the edges are positive and the degree matrix, Δ, is diagonal matrix. The Laplacian matrix of the digraph can be calculated by (2.3) as follows:

$$L = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}.$$

2.2. Continuous-Time Distributed Consensus Protocol

In this section, we present a continuous-time distributed consensus protocol that solves the agreement problem for multi-agent systems operating in continuous-time.

Consider a multi-agent system with n agents with linear dynamics in continuous-time network. The dynamics of the node v_i are described by

$$\dot{x}_i(t) = u_i(t) \tag{2.4}$$

where $x_i(t)$ is the state of node v_i at time t, and $u_i(t)$ is a linear protocol of the system described as follows:

$$u_i(t) = \sum_{v_j \in N_i} a_{ij}(x_j(t) - x_i(t)), \quad i = 1, \ldots, n. \tag{2.5}$$

The following assumption is widely used in the analysis of multi-agent networks operating in continuous-time.

Assumption 2.1. *$a_{ij} > 0$ if $\varepsilon_{ij} \in \mathcal{E}$, and $a_{ij} = 0$ if $\varepsilon_{ij} \notin \mathcal{E}$ for $i = 1, 2, \ldots, n$.*

Assumption 2.1 implies that the information coming from a neighbor should have positive weighting.

The dynamics of the multi-agent system (2.4) with protocol (2.5) are equivalent to

$$\dot{x}(t) = -Lx(t) \tag{2.6}$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in R^n$ is the state vector of the network at time t, and $L = \Delta - A = [l_{ij}] \in R^{n\times n}$ is defined by

$$l_{ij} = \begin{cases} \sum_{k=1,k\neq i}^{n} a_{ik}, & \text{if } i = j \\ -a_{ij}, & \text{otherwise.} \end{cases} \tag{2.7}$$

Remark 2.1. *Under Assumption 2.1, we have $l_{ii} \geq 0$ and $l_{ij} \leq 0$. A crucial property of Laplacian matrix, L, is that all row sums of L are zero. Then, $\pi_n = [1, 1, \ldots, 1]^T \in R^n$ is an eigenvector of L associated with the eigenvalue $\lambda = 0$, $L\pi_n$=0.*

Lemma 2.1. *[26] Let $G = (V, \mathcal{E}, A)$ be a weighted digraph with a Laplacian matrix L whose diagonal elements are non-negative and off-diagonal elements are non-positive. Then, the following holds:*

- *(i) $L\pi_n$ = 0 where $\pi_n = [1, 1, \ldots, 1]^T \in R^n$ is an eigenvector of L and all non-zero eigenvalues have positive real parts.*
- *(ii) L has a simple zero eigenvalue if and only if G has a spanning tree.*

2.3. Cluster Consensus States

In this section, we state the cluster consensus problem that is used in this thesis.

Definition 2.1. *(Cluster Consensus) The network in (2.4) goes to K disjoint clusters, $C = \{C_1, C_2, \ldots, C_K\}$, if it satisfies following properties for any initial condition $[x_1(0), x_2(0), \ldots, x_n(0)]^T$ and any weights that satisfy Assumption 2.1,*

- $\bigcup_{p=1}^{K} C_p = V$,
- $C_p \cap Cq = \emptyset$, *for* $p \neq q$, *and* $p, q = 1, 2, \ldots, K$,

- $\lim_{t\to\infty} x_i(t) = c_p$, $\forall v_i \in C_p$, $i = 1, 2, \ldots, n$, *and* $c_p \neq c_q$ *for* $p \neq q$, $p, q = 1, 2, \ldots, K$.

Remark 2.2. *In the above definition, clusters are denoted by* C_p, $p = 1, 2, \ldots, K$ *and the consensus states of these clusters are denoted by* $\lim_{t\to\infty} x_i(t) = c_p$, $p = 1, 2, \ldots, K$. *Agents in the same cluster reach one state, but the agents in different clusters go to different distinct states. Note that union of elements of all clusters equals to vertex set* V *whereas intersection of any two clusters is empty set.*

2.4. Chapter Summary

In this chapter, we have reviewed basic graph theoretical concepts such as edge set, adjacency matrix, degree matrix and Laplacian matrix. We have given some properties of Laplacian matrices associated with the digraph. Subsequently, continuous-time distributed consensus protocol has been presented to solve agreement problem in continuous-time networks. Assumptions on the protocol have been provided. Finally, we have defined the cluster consensus problem that will be discussed in Chapters 3-5.

3. CLUSTER CONSENSUS IN CONTINUOUS-TIME

In this chapter, we investigate the conditions on the directed network topology such that the network goes to $K \geq 2$ clusters in continuous-time. First, we review some definitions that will be useful to determine the number of clusters.

3.1. Primary and Secondary Layer Subgraphs

Definition 3.1. *[22] (Primary layer subgraphs) Let G=(V,$\mathcal{E}$, A) be the weighted digraph. There exists l_p ($l_p \geq 1$) subsets in the vertex set V such that each subset $V_{p,i}$, $i = 1, 2, ..., l_p$, is the largest possible subset that has a spanning tree for its subgraph $G_{p,i}$, and for all $v_a \in V_{p,i}$ and $v_b \notin V_{p,i}$, we have $(v_b, v_a) \notin \mathcal{E}$. We say $G_{p,i}$, $i = 1, 2, .., l_p$ are the primary layer subgraphs of G where the number of primary layer subgraphs is denoted by l_p.*

Comment: For a network, the primary layer subgraphs of the weighted digraph are denoted as $G_{p,1}, G_{p,2}, \ldots G_{p,l_p}$.

Definition 3.2. *[22] (Secondary layer subgraph) Let $\bar{V}$ be the set which consists of the vertices that are not in the primary layer subgraphs, i.e., $\bar{V} = V \setminus \bigcup_{i=1}^{l_p} V_{p,i}$. Then there exists l_s subsets in $\bar{V}$ such that each subset $V_{s,i}$, $i = 1, 2, ..., l_s$, has a spanning tree for its subgraph $G_{s,i}$ and there exists exactly one vertex $v_a \in V_{s,i}$ which satisfies the following:*

(i) For all $v_b \in V_{s,i} \setminus v_a$ and $v_c \in V \setminus V_{s,i}$, we have $(v_c, v_b) \notin \mathcal{E}$.

(ii) There exist at least two nodes in two different subgraphs (either primary or secondary layer) v_d and v_e such that $(v_d, v_a) \in \mathcal{E}$ and $(v_e, v_a) \in \mathcal{E}$.

(iii) v_a is the root of a spanning tree in $V_{s,i}$.

Comment: For a network, the secondary layer subgraphs of the weighted digraph are denoted as $G_{s,1}, G_{s,2}, \ldots, G_{s,l_s}$.

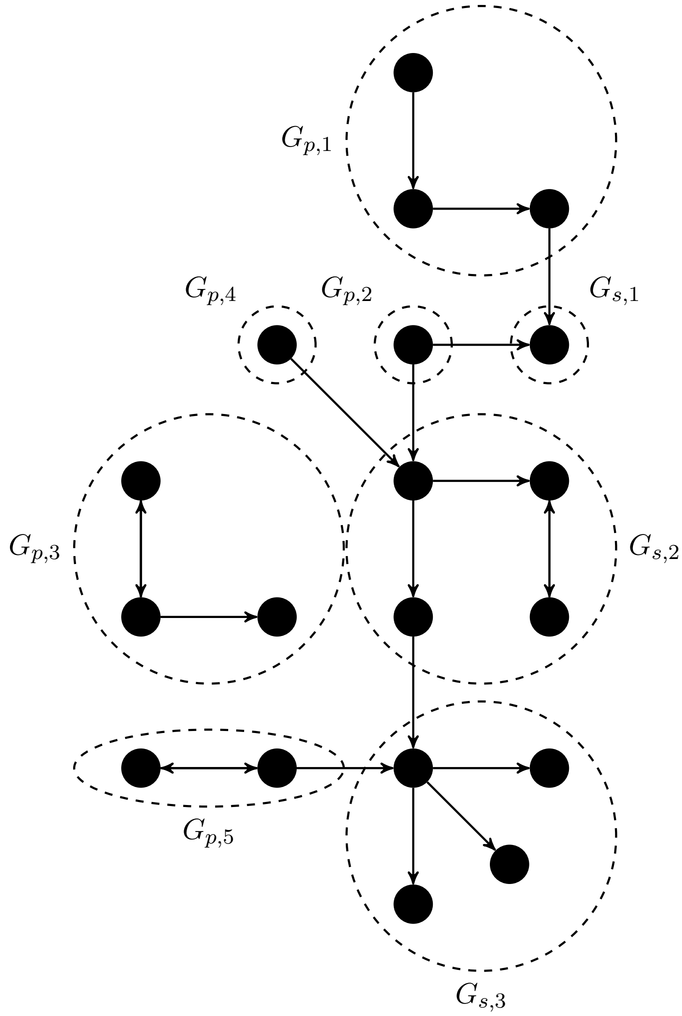

Figure 3.1. A digraph with 19 nodes and 20 edges.

The primary and secondary layer subgraphs of a given network can be determined by applying the algorithm in [22]. This concept is illustrated with the following example.

Example 3.1. *Consider the digraph with 19 nodes and 20 edges shown in Figure 3.1. Based on the primary and secondary layer subgraphs decompositions, the network is divided into 8 clusters, 5 of which correspond to primary layer subgraphs, and other 3 of which correspond to secondary layer subgraphs. This network goes to 8 distinct steady state values under Assumption 2.1. According to Definitions 3.1 and 3.2, agents of primary layer subgraphs cannot receive information from out of the subgraph and root of secondary layer subgraphs should receive information from at least two different subgraphs.*

3.2. Sub-Laplacian Matrices

In studying the system dynamics in this thesis, we consider a matrix $B = [b_{ij}]_{n\times n}$ with $b_{ij} \leq 0$ for $i \neq j$, $b_{ij} > 0$ for $i = j$, and $\sum_{j=1}^{n} b_{ij} \geq 0$ for all i. Note that B is almost like a Laplacian matrix. This matrix is in the special class of *sub-Laplacian matrices* which is defined as follows:

Definition 3.3. *(Sub-Laplacian Matrix) Let B=$[b_{ij}]$ be an $n \times n$ matrix. B is said to be a sub-Laplacian matrix, if it satisfies the following conditions:*

- $b_{ij} \leq 0$ *for* $i \neq j$,
- $\sum_{j=1}^{n} b_{ij} \geq 0$, $i = 1, 2, \ldots, n$.

In deriving the properties of sub-Laplacian matrices defined above, we will utilize Lemma 3.1-3.2 stated below.

Lemma 3.1. *[27] (Gershgorin's Disc Theorem) Let $B = [b_{ij}]$ be a complex $n \times n$ matrix. Let $\mu_i = \sum_{i\neq j} |b_{ij}|$ be the sum of the absolute values of the off-diagonal elements in the i-th row. Then, every eigenvalue of B lies in one of the Gershgorin discs, D_i, which are closed discs centered at b_{ii} with radius μ_i*

$$D_i = \{\lambda \in \mathbb{C} : |\lambda - b_{ii}| \leq \mu_i\} \quad i = 1, 2, \ldots, n. \tag{3.1}$$

Lemma 3.2. *[28] (Matrix-Tree Theorem) Let $B = [b_{ij}]_{n\times n}$ be a weighted Laplacian matrix for a digraph whose edges, ε_{ij}, corresponding to the weights, b_{ij}. Let T be a spanning subtree of digraph G. The total weight of all spanning subtrees in G is given by*

$$T(G) = \sum w(T) \tag{3.2}$$

where

$$w(T) = \prod_{\varepsilon_{ij} \in T} b_{ij} \tag{3.3}$$

is the weight for T. Let $B(i|j)$ denote the $(n-1) \times (n-1)$ sub-matrix of B obtained by deleting its i-th row and j-th column. Furthermore, we have

$$T(G) = \det(B(i|j)). \tag{3.4}$$

Remark 3.1. *The matrix-tree theorem states that the determinant of $n-1$ rows/columns of B gives the total weight of all trees. If i is the index of a root, then det$(B(i|j))$ cannot be zero.*

Lemma 3.3. *Let B be an $n \times n$ sub-Laplacian matrix that also satisfies $\sum_{j=1}^{n} b_{ij} = 0$ for all i except for $i = i^*$ where i^* is the root which has $\sum_{j=1}^{n} b_{i^*j} = \gamma > 0$. Then,*

(i) $\sigma(B) \subset ORHP$ (open right half plane).
(ii) The i^*-th column of B^{-1} is equal to $\frac{1}{\gamma}\pi_n$ where $\pi_n = [1, 1, \ldots, 1]^T \in R^n$.

Proof. Suppose B is a sub-Laplacian matrix that satisfies $\sum_{j=1}^{n} b_{ij} = 0$ for all i except for $i = i^*$ where i^* is the root which has $\sum_{j=1}^{n} b_{i^*j} = \gamma > 0$.

From Lemma 3.1, it follows that all eigenvalues of B are in the ORHP except for the possibility of eigenvalues at $\lambda = 0$. In order to show that B does not have an eigenvalue at $\lambda = 0$, we proceed as if B were a weighted Laplacian matrix. By Lemma 3.2, the determinant of B can be calculated as follows:

$$\begin{aligned} \det(B) &= \sum_{j=1}^{n} (-1)^{i+j} b_{ij} [(-1)^{i+j} T(G)] \\ &= [b_{i1} + \ldots + b_{in}] T(G). \end{aligned}$$

Note that for $i = i^*$, $T(G) > 0$ since i^* is the root of a digraph. Then, we have

$$\det(B) = [b_{i^*1} + \ldots + b_{i^*n}]T(G) \neq 0 \tag{3.5}$$

which implies that B has no eigenvalues at the origin. This concludes the proof of (i).

The second result, (ii), is on the i^*-th column of B^{-1}. Since the sum of the i^*-th row of B is $\gamma > 0$, we have

$$B\pi_n = \gamma q_i^* \tag{3.6}$$

where q_i^* is the basis vector which has 1 as the i^*-th component and the remaining elements are zero. Then, we can rewrite (3.6) as

$$\pi_n = B^{-1}\gamma q_i^*. \tag{3.7}$$

Therefore, the i^*-th column of B^{-1} is computed as $\frac{1}{\gamma}\pi_n$. □

Lemma 3.3 delineates a special class of sub-Laplacian matrices with one root whose eigenvalues are in the ORHP. We generalize this result to a broader class of sub-Laplacian matrices in Lemma 3.4. To this end, we need the following definition.

Definition 3.4. *(Reachability Set) Let I={1,2, ..., n} and B be an $n \times n$ square matrix. For every pair of distinct integers p, q with $1 \leq p, q \leq n$, if there is a sequence of distinct integers $k_1 = p, k_2, \ldots k_{m-1}, k_m = q$, $1 \leq m \leq n$, such that all of the matrix entries $b_{k_2k_1}, b_{k_3k_2}, \ldots, b_{k_mk_{m-1}}$ are nonzero, then p can reach q. The set of all $q \in I$ that can be reached from p is denoted by R_p.*

Definition 3.4 yields the following statements:

- $R_p = I$ implies that node p can reach any node of the graph, i.e., p is a root.
- If $R_i = I$, $i = 1, 2, \ldots, n$, the graph is strongly connected.

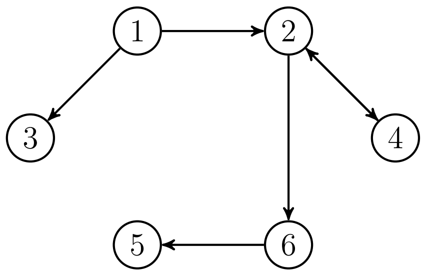

Figure 3.2. A digraph with 6 nodes and 8 edges.

- If there exists a unidirectional edge from p to q, $R_q \subset R_p$. If the edge between p and q is bidirectional, $R_q \equiv R_p$.

Example 3.2. *Consider the directed network given in Figure 3.2. According to the definition of reachability set, each node's reachability set is as follows:*

- $R_1 = \{1, 2, 3, 4, 5, 6\}$
- $R_2 = \{2, 4, 5, 6\}$
- $R_3 = \{3\}$
- $R_4 = \{2, 4, 5, 6\}$
- $R_5 = \{5\}$
- $R_6 = \{5, 6\}$

The following results can be obtained from the sets:

- $R_1 = I$*, then node 1 is the root of the network.*
- *There exists a unidirectional edge from node 1 to node 2. It implies that* $R_2 \subset R_1$*. Also, there exists bidirectional edges between node 2 and node 4 such that* $R_2 \equiv R_4$*.*

Lemma 3.4. *Let* B *be an* $n \times n$ *sub-Laplacian matrix. If there exist index sets* I_1, I_2 *from* $I = \{1, 2, \ldots, n\}$ *such that*

(i) $I = I_1 \bigcup I_2$

(ii) $I_1 = \{i : \sum_{j=1}^{n} b_{ij} > 0\}$, $I_2 = \{i : \sum_{j=1}^{n} b_{ij} = 0\}$, *and*

(iii) $I = \bigcup_{i \in I_1} R_i$,

then $-B$ *is asymptotically stable matrix, i.e.,* $\sigma(B) \subset ORHP$.

Proof. Consider the sub-Laplacian matrix $B = [b_{ij}] \in R^{n\times n}$ in the form:

$$B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{bmatrix}.$$

A Laplacian matrix $\tilde{B}$ can be obtained by adding a node whose index is 0 to the system as follows:

$$\tilde{B} = \left[\begin{array}{c:cccc} b_{00} & b_{01} & b_{02} & \dots & b_{0n} \\ \hdashline b_{10} & b_{11} & b_{12} & \dots & b_{1n} \\ b_{20} & b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n0} & b_{n1} & b_{n2} & \dots & b_{nn} \end{array}\right] = \left[\begin{array}{c:cccc} b_{00} & b_{01} & b_{02} & \dots & b_{0n} \\ \hdashline b_{10} & & & & \\ b_{20} & & & B & \\ \vdots & & & & \\ b_{n0} & & & & \end{array}\right].$$

Let $\tilde{I} = \{0\} \bigcup I$ be the augmented index set of the Laplacian matrix $\tilde{B}$. The reachability set of node 0 is $R_0 = \{0, i | b_{i0} \neq 0, i \in I_1\}$. Then, $R_0 = \{0\} \bigcup_{i \in I_1} R_i = \tilde{I}$ implies that node 0 is the root of the augmented system. From Lemma 3.2, the determinant of the root's cofactor matrix is not zero. Then, we have $\det(B) \neq 0$. This implies that B cannot have a zero eigenvalue, i.e., all eigenvalues of B are in the ORHP. □

Example 3.3. *Consider the following matrix:*

$$B = \begin{bmatrix} 3 & -2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 5 & -5 & 0 \\ 0 & 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

Index sets are $I_1 = \{1, 2\}$ and $I_2 = \{3, 4, 5\}$ based on conditions in Lemma 3.4. Reachability sets of nodes in I_1 are $R_1 = \{1, 2\}$ and $R_2 = \{1, 2\}$ such that $I \neq \bigcup_{i \in I_1} R_i$. It implies that $-B$ is not asymptotically stable matrix.

Example 3.4. *Consider the following matrix:*

$$B = \begin{bmatrix} 3 & -2 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 5 & -5 & 0 \\ 0 & 0 & -3 & 4 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

Index sets are $I_1 = \{1, 4\}$ *and* $I_2 = \{2, 3, 5\}$ *based on conditions in Lemma 3.4. Reachability sets of nodes in* I_1 *are* $R_1 = \{1, 2\}$ *and* $R_4 = \{3, 4, 5\}$ *such that* $I = \bigcup_{i \in I_1} R_i$. *It means that* $-B$ *is asymptotically stable matrix.*

3.3. Number of Clusters

Once these subgraphs are obtained, it can be considered that the Laplacian matrix, L, in (2.6) can be rewritten by using row and column operations as follows:

$$L = \begin{bmatrix} L_{1,1} & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & 0 & \vdots \\ 0 & \dots & L_{l_p,l_p} & 0 & \dots & 0 \\ L_{l_p+1,1} & \dots & \dots & L_{l_p+1,l_p+1} & \dots & L_{l_p+1,l_p+l_s} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ L_{l_p+l_s,1} & \dots & \dots & \dots & \dots & L_{l_p+l_s,l_p+l_s} \end{bmatrix}. \tag{3.8}$$

Note that in the first l_p rows of L, only the diagonal blocks are non-zero and each $L_{i,i}$, $i = 1, 2, ..., l_p$ matrix is also a Laplacian matrix.

Let $I_p = \{1, 2, \dots, l_p\}$ and $I_s = \{1, 2, \dots, l_s\}$ be index sets of the primary and secondary layer subgraphs, respectively. $n_{p,i}$, $i \in I_p$ denotes the number of nodes in the i-th primary layer subgraph and $n_{s,j}$, $j \in I_s$ denotes the j-th secondary layer subgraph. Let $\tilde{n}_p = \sum_{i=1}^{l_p} n_{p,i}$ and $\tilde{n}_s = \sum_{j=1}^{l_s} n_{s,j}$ be the total number of nodes in the primary layer subgraphs and the secondary layer subgraphs, respectively. Then, we can rewrite

(2.6) as

$$\begin{bmatrix} \dot{x}_p(t) \\ \dot{x}_s(t) \end{bmatrix} = -\begin{bmatrix} L_p & 0 \\ L_{sp} & L_s \end{bmatrix} \begin{bmatrix} x_p(t) \\ x_s(t) \end{bmatrix} \tag{3.9}$$

where $x_p(t) \in R^{\tilde{n}_p \times 1}$ and $x_s(t) \in R^{\tilde{n}_s \times 1}$ are the state vectors for the primary and secondary layer subgraphs at time t, respectively; and the system matrices are given as

$$L_p = \begin{bmatrix} L_{1,1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & L_{l_p,l_p} \end{bmatrix}_{\tilde{n}_p \times \tilde{n}_p},$$

$$L_{sp} = \begin{bmatrix} L_{l_p+1,1} & \dots & L_{l_p+1,l_p} \\ \vdots & \ddots & \vdots \\ L_{l_p+l_s,1} & \dots & L_{l_p+l_s,l_p} \end{bmatrix}_{\tilde{n}_s \times \tilde{n}_p},$$

$$L_s = \begin{bmatrix} L_{l_p+1,l_p+1} & \dots & L_{l_p+1,l_p+l_s} \\ \vdots & \ddots & \vdots \\ L_{l_p+l_s,l_p+1} & \dots & L_{l_p+l_s,l_p+l_s} \end{bmatrix}_{\tilde{n}_s \times \tilde{n}_s}.$$

One of the major contributions of this thesis is stated in Theorem 3.1 below.

Theorem 3.1. *The number of clusters for a multi-agent system described by (2.4) and (2.5) with digraph $G = (V, \mathcal{E}, A)$ can be computed as*

$$K = l_p + l_s \tag{3.10}$$

where l_p and l_s are the number of primary and secondary layer subgraphs, respectively.

Proof. From (3.9), the primary and secondary layer system dynamics can be represented in the following form:

$$\dot{x}_p(t) = -L_p x_p(t) \tag{3.11a}$$

$$\dot{x}_s(t) = -L_s x_s(t) - L_{sp} x_p(t). \tag{3.11b}$$

We investigate separately the stability and equilibrium conditions for the primary and secondary layer subgraphs in consecutive steps.

Step 1 (Consensus states for the primary layer subgraphs): Let n_i, $i = 1, 2, \ldots l_p + l_s$, denote the number of nodes in the i-th subgraph and let $\bar{x}_i \in R^{n_i \times 1}$, $i = 1, 2, \ldots l_p + l_s$, be the state vectors for the corresponding subgraphs. Due to block diagonal structure of L_p, each primary layer subgraph dynamics can be written as

$$\dot{\bar{x}}_i(t) = -L_{i,i}\bar{x}_i(t), \quad i = 1, 2, \ldots, l_p. \tag{3.12}$$

From Lemma 2.1, each Laplacian matrix, $L_{i,i}$, has a simple zero eigenvalue since $L_{i,i}$ has a spanning tree for $i = 1, 2, \ldots, l_p$. As $t \to \infty$, each primary layer subgraph with the system matrix $L_{i,i}$ achieves consensus which results in l_p clusters in l_p primary layer subgraphs.

Step 2 (Stability and equilibria of the secondary layer dynamics): In (3.11b), the system matrix L_s is a sub-Laplacian matrix which satisfies the conditions in Lemma 3.4 as per the definition of the secondary layers (i.e., Definition 3.2). Therefore, from Lemma 3.4 we have $\sigma(L_s) \subset ORHP$. Since we also have $x_p(t)$ converging to constant values from Step 1, the system in (3.11b) is stable.

After showing that the secondary layer subgraphs in the system have stable dynamics, the equilibrium point of the secondary layer dynamics in (3.11b) can be calculated as

$$\tilde{x}_s = -L_s^{-1} L_{sp} \tilde{x}_p \tag{3.13}$$

where $\tilde{x}_p$ and $\tilde{x}_s$ are the equilibrium points of the primary and secondary layer subgraphs, respectively. It can be shown that each row $L_s^{-1} L_{sp}$ adds up to -1. Hence, we conclude that the components of $\tilde{x}_s$ are convex combinations of the components of $\tilde{x}_p$.

Step 3 (Consensus states for the secondary layer subgraphs): In this step, we show that each secondary layer subgraph attains *different final values*. The secondary layer subgraph dynamics can be expressed as follows:

$$\dot{\bar{x}}_i(t) = -L_{i,i}\bar{x}_i(t) - \sum_{j=1, j\neq i}^{l_p+l_s} L_{i,j}\bar{x}_j(t), \quad i = l_p + 1, \ldots, l_p + l_s. \tag{3.14}$$

The steady state values of the secondary layer subgraphs can be calculated as

$$\bar{x}_i(\infty) = -\sum_{j=1, j\neq i}^{l_p+l_s} (L_{i,i})^{-1} L_{i,j}\bar{x}_j(\infty). \tag{3.15}$$

By using Lemma 3.3, one can obtain that $(L_{i,i})^{-1}L_{i,j}$ has identical rows. Hence, we can calculate the final values of the secondary layer subgraphs as follows:

$$\bar{x}_i(\infty) = \pi_{n_i} c_i, \quad i = l_p + 1, \ldots, l_p + l_s$$

where $\pi_{n_i} = [1, 1, \ldots, 1]^T \in R^{n_i}$, c_i are combinations of $\bar{x}_j(\infty)$. This implies that nodes in a specific second layer subgraph agree on a specific value which is not necessarily equal to a consensus value of any other layer. Consequently, the secondary layer dynamics lead to l_s distinct clusters.

Step 4 (Total number of clusters): From Step 1 and 3, the total number of clusters can be calculated as the sum of the number of clusters in the primary and secondary layer subgraphs, i.e.,

$$K = l_p + l_s. \tag{3.16}$$

□

3.4. Convergence Rate Analysis

The convergence rate of a multi-agent system utilizing continuous-time distributed consensus protocol is instrumental property in performance evaluation of the system. In [1], the authors have studied convergence of the algorithm in (2.4) for the complete consensus problem. Spectral properties of the Laplacian matrix take an important place in analysis of convergence speed. According to Lemma 3.1, if λ is an eigenvalue of the Laplacian matrix, then either $\lambda = 0$ or $Re\{\lambda\} > 0$ holds. For any digraph G which has a spanning tree, the set of eigenvalues of the Laplacian system matrix can be sorted in ascending order as

$$0 = \lambda_1 < Re\{\lambda_2\} \leq Re\{\lambda_3\} \leq \cdots \leq Re\{\lambda_n\}.$$

In this case, the second smallest eigenvalue of the Laplacian matrix, λ_2, is called the *algebraic connectivity* of a digraph which has a spanning tree. Convergence speed of a multi-agent system is related to the real part of the algebraic connectivity of the system, i.e., $Re\{\lambda_2\}$.

In the cluster consensus problem where there are l_p primary and l_s secondary layer subgraphs, the second smallest eigenvalue of L in (3.9) can be computed as

$$\begin{aligned}\bar{\lambda}_2(L) &= \min\{\bar{\lambda}_2(L_p), \bar{\lambda}_1(L_s))\} \\ &= \min_{i=1,2,\ldots,l_p}\{\bar{\lambda}_2(L_{i,i}), \bar{\lambda}_1(L_s))\}\end{aligned}$$

where $\bar{\lambda}_i(\cdot)$ is the real part of the i-th smallest eigenvalue of a matrix, i.e., $\bar{\lambda}_1(L_s)$ denotes the real part of the smallest eigenvalue of L_s and $\bar{\lambda}_2(L_{i,i})$ denotes the real part of the second smallest eigenvalue of each primary layer subgraph. Note from earlier discussions that $\bar{\lambda}_2(L_{i,i}) > 0$ for $i = 1, 2, \ldots, l_p$ and $\bar{\lambda}_1(L_s) > 0$ since L_s is a sub-Laplacian matrix which satisfies the conditions in Lemma 3.4. In this way, the convergence rate can easily be analyzed by exploiting the properties of the primary and secondary layer subgraphs.

3.5. Numerical Analysis

The theoretical results presented in this chapter are illustrated by the following two examples. In the first example, we determine the primary and secondary layer subgraphs of given network and analyze the case where one of the edges is removed. In the second example, we state a special case in which the number of clusters may be reduced.

Example 3.5. *Consider the network with 18 nodes and 26 edges in Figure 3.3. The state values are shown in Figure 3.4 for randomly selected initial state vector,* $x(0) = [x_1(0), x_2(0), \ldots, x_{18}(0)]$*, and elements of weighted adjacency matrix,* a_{ij}*, which satisfy Assumption 2.1. The network converges to 8 clusters, 4 of which correspond to primary layer subgraphs, and the other 4 of which correspond to secondary layer subgraphs. This agrees with the result of* $K = l_p + l_s$*. Clusters are marked as* $G_{p,1}, G_{p,2}, G_{p,3}, G_{p,4}$ *for the primary layer subgraphs and* $G_{s,1}, G_{s,2}, G_{s,3}, G_{s,4}$ *for the secondary layer subgraphs.*

Now consider the secondary layer subgraph's root marked as white in Figure 3.5. This root receives information from just two different subgraphs. If the edge shown as dashed line is broken, the number of clusters in the system reduces since the root of any secondary layer subgraph should have at least two inward edges from two different subgraphs. In such a case, the secondary layer subgraph, $G_{s,4}$*, joins the other subgraph,* $G_{p,3}$*, which continues to give information to the red node. The new group which is denoted by* $G^*_{p,3}$ *has spanning tree. Therefore, the number of clusters is equal to 7. The state values are shown in Figure 3.6.*

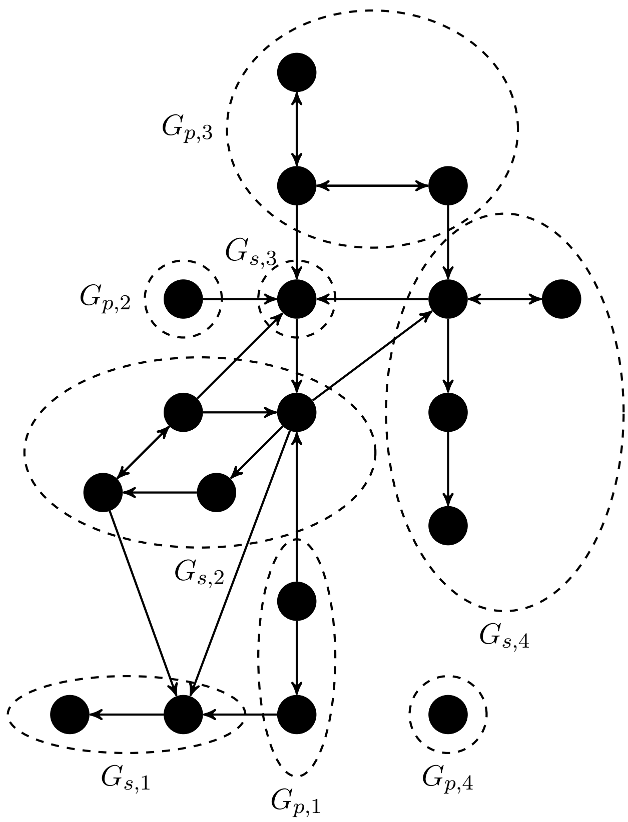

Figure 3.3. A digraph with 18 nodes and 26 edges.

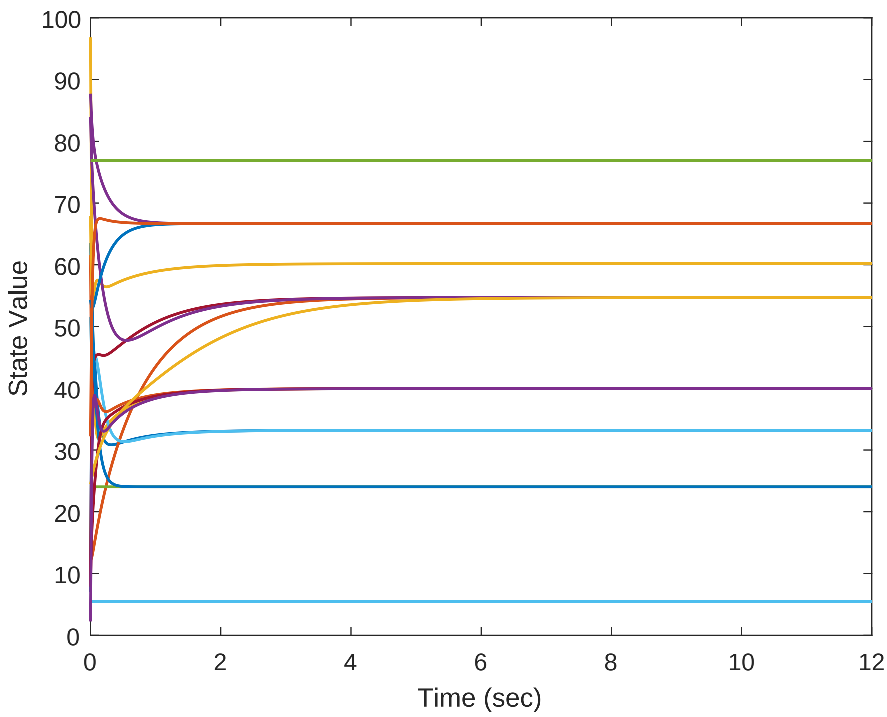

Figure 3.4. The simulation results for the system given in Figure 3.3.

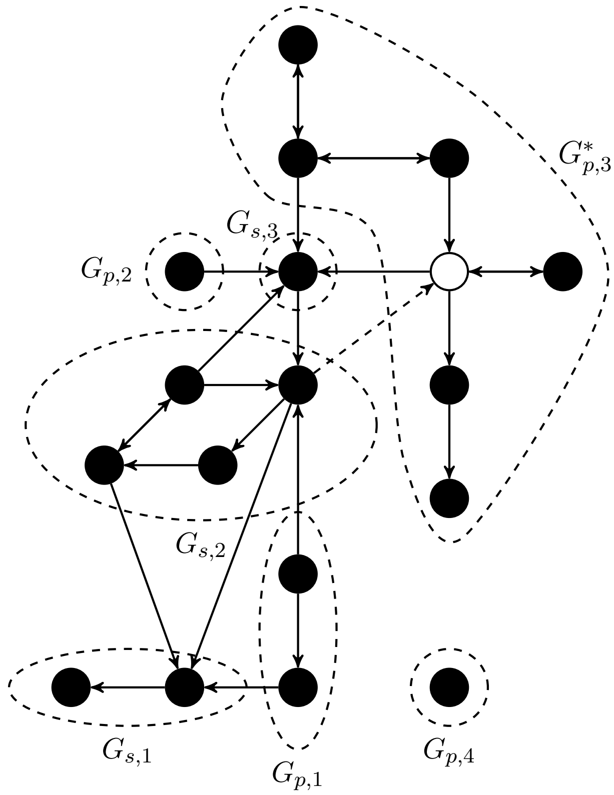

Figure 3.5. A digraph with 18 nodes and 25 edges.

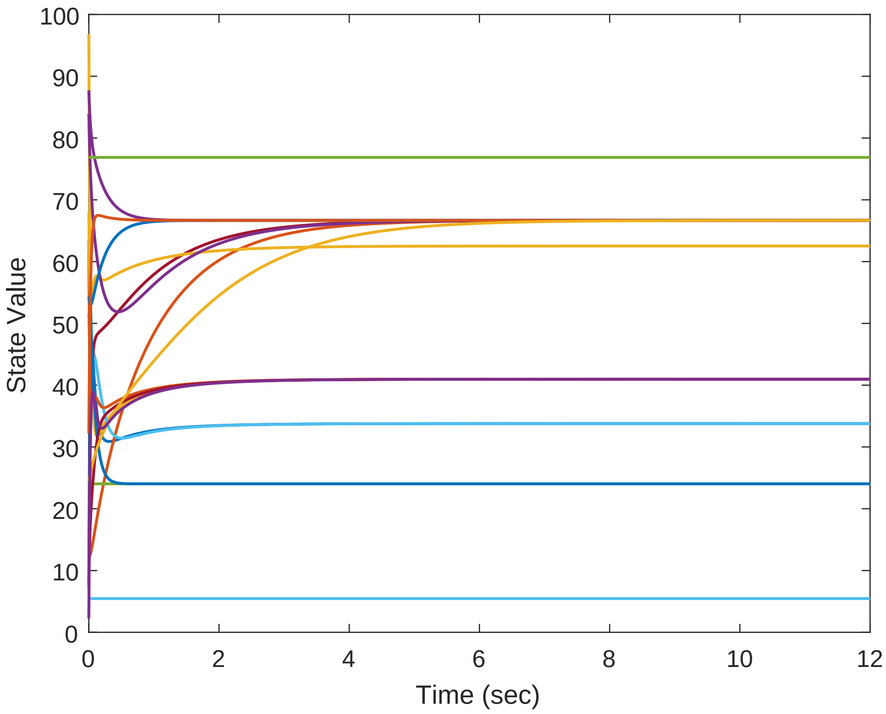

Figure 3.6. The simulation results for the system given in Figure 3.5.

Example 3.6. *The result of $K = l_p + l_s$ is valid for the general case in the sense of Definition 2.1. However, exceptional cases reducing the number of clusters may arise. To illustrate this, let C_1, C_2, and C_3 be primary layer or secondary layer subgraphs, and C_4 and C_5 be secondary layer subgraphs with the following Laplacian:*

$$L = \begin{array}{c} \begin{array}{ccccc} \overbrace{\quad\quad}^{C_1} & \overbrace{\quad\quad}^{C_2} & \overbrace{\quad\quad}^{C_3} & \overbrace{\quad\quad}^{C_4} & \overbrace{\quad\quad}^{C_5} \end{array} \\ \left[\begin{array}{cc:cc:cc:cc:cc} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hdashline a & 0 & b & 0 & c & 0 & \zeta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline d & 0 & e & 0 & f & 0 & 0 & 0 & \zeta_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}\right] \end{array}. \tag{3.17}$$

Let ζ_1 and ζ_2 be the roots of the secondary layer subgraphs of C_4 and C_5, respectively. From the Laplacian matrix property, we have

$$\zeta_1 = -(a + b + c)$$
$$\zeta_2 = -(d + e + f).$$

Let $\overline{C_1}$, $\overline{C_2}$, $\overline{C_3}$, $\overline{C_4}$ and $\overline{C_5}$ be the consensus values of the subgraphs C_1, C_2, C_3, C_4, and C_5, respectively. At steady state, the consensus values of C_4 and C_5 can be calculated from the following equations:

$$-a\overline{C_1} - b\overline{C_2} - c\overline{C_3} = \zeta_1\overline{C_4}$$
$$-d\overline{C_1} - d\overline{C_2} - f\overline{C_3} = \zeta_2\overline{C_5}.$$

If $\frac{a}{\zeta_1} = \frac{d}{\zeta_2}$, $\frac{b}{\zeta_1} = \frac{e}{\zeta_2}$, and $\frac{c}{\zeta_1} = \frac{f}{\zeta_2}$, then we have $\frac{a}{d} = \frac{b}{e} = \frac{c}{f} = \frac{\zeta_1}{\zeta_2}$ which implies that the nodes in subgraphs C_4 and C_5 reach on the same consensus value. Under this condition, the number of cluster decreases.

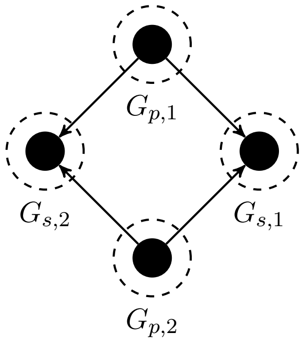

Figure 3.7. A digraph with 4 nodes and 4 edges.

To illustrate the above special case, consider the network with 4 nodes and 4 edges in Figure 3.7. Suppose the Laplacian matrix is chosen as follows:

$$L = \left[\begin{array}{c:c:c:c} 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \\ \hdashline -0.3 & -0.4 & 0.7 & 0 \\ \hdashline -0.6 & -0.8 & 0 & 1.4 \end{array}\right].$$

Note that we have

$$\frac{-0.3}{-0.6} = \frac{-0.4}{-0.8} = \frac{0.7}{1.4} \tag{3.18}$$

which results in the special case. The state values are shown in Figure 3.8 which demonstrates that we have 3 clusters.

For all other weights that do not satisfy the above ratio (3.18), we have 4 clusters as predicted by the result of $K = l_p + l_s$. *Such an example is shown in Figure 3.9 for the Laplacian matrix*

$$L = \left[\begin{array}{c:c:c:c} 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \\ \hdashline -0.3 & -0.4 & 0.7 & 0 \\ \hdashline -0.1 & -1.1 & 0 & 1.2 \end{array}\right].$$

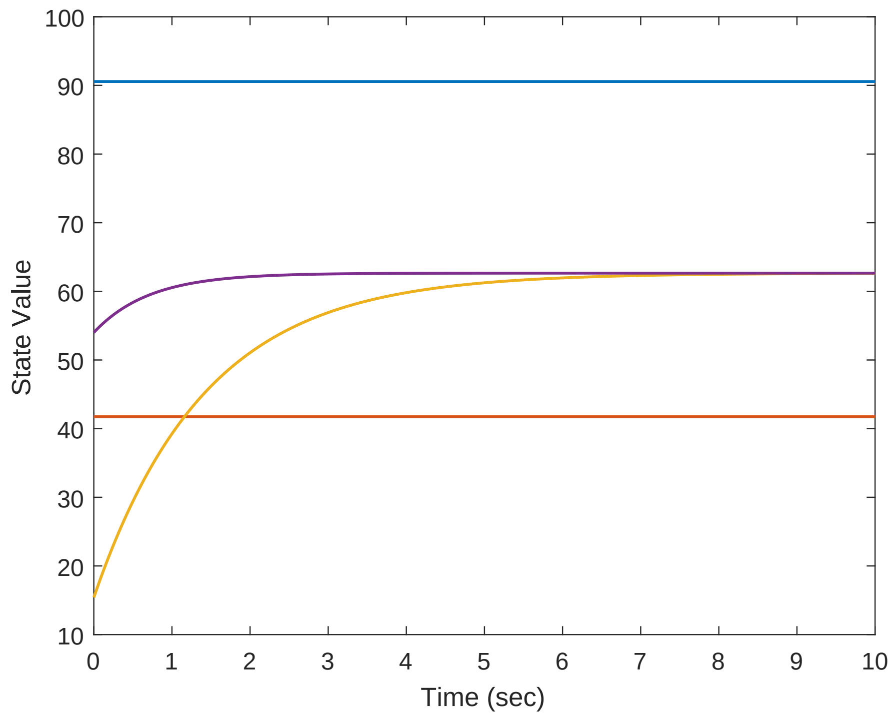

Figure 3.8. The simulation results for the system given in Figure 3.7 with special case.

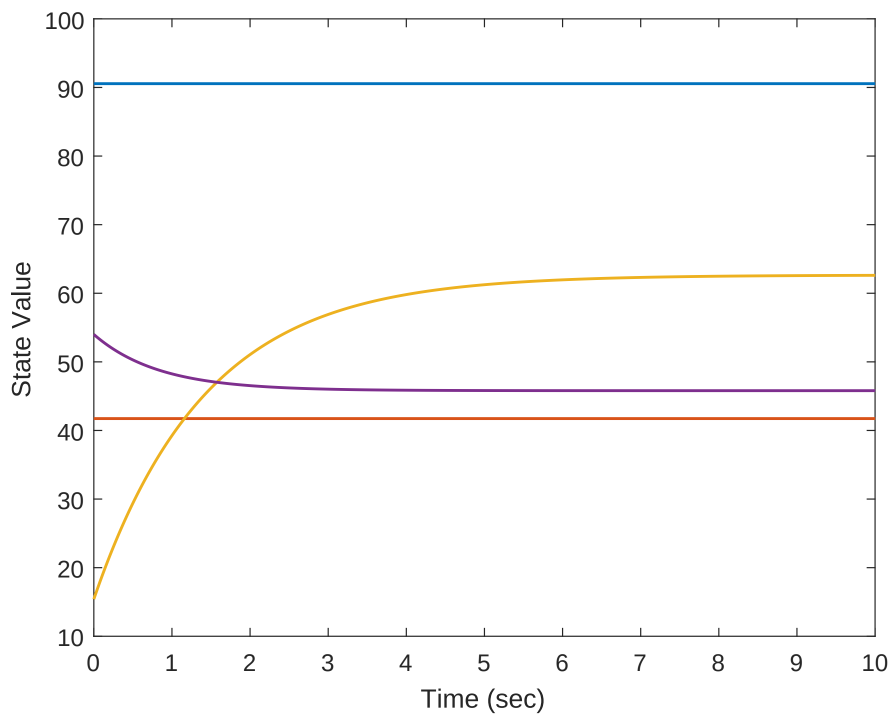

Figure 3.9. The simulation results for the system given in Figure 3.7 without special case.

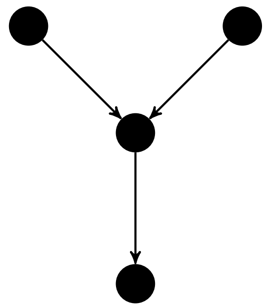

Figure 3.10. A digraph with 4 nodes and 3 edges.

Remark 3.2. *The above example is a special instance of degenerate cases excluded by Definition 2.1. Note that such degenerate cases are also excluded in single consensus problems. To illustrate what we mean, recall that necessary and sufficient conditions is that there exists a spanning tree and consider the following simple network which does not have a spanning tree in Figure 3.10. If initial values of all nodes are equal to 1, consensus values are the same for all nodes although there is not a spanning tree in the system.*

3.6. Extension to Switching Systems

In this section, we discuss possible extentions where the weights or topology of a given multi-agent network change over time.

3.6.1. Weight-Varying Systems

In a multi-agent system, the weights of edges can be altered according to requirements of the system. This system is called *Weight-Varying Systems*. Consider the network with 11 nodes and 10 edges shown in Figure 3.11. The given network can be layered as shown in Figure 3.11, i.e., there are 3 primary layer subgraphs and 2 secondary layer subgraphs. Suppose that the weights of edges in the system switch among two randomly formed edge sets, i.e., $\mathcal{E}_1$ and $\mathcal{E}_2$, periodically with period Δt.

We note that Δt has effect on the equilibrium of the system. If Δt is small enough i.e., the weights change fast, the system reaches cluster consensus without any change

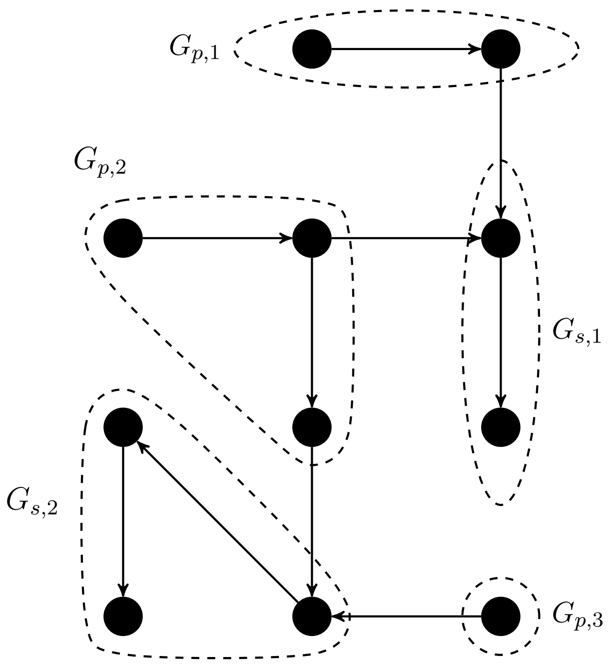

Figure 3.11. A digraph with 11 nodes and 10 edges.

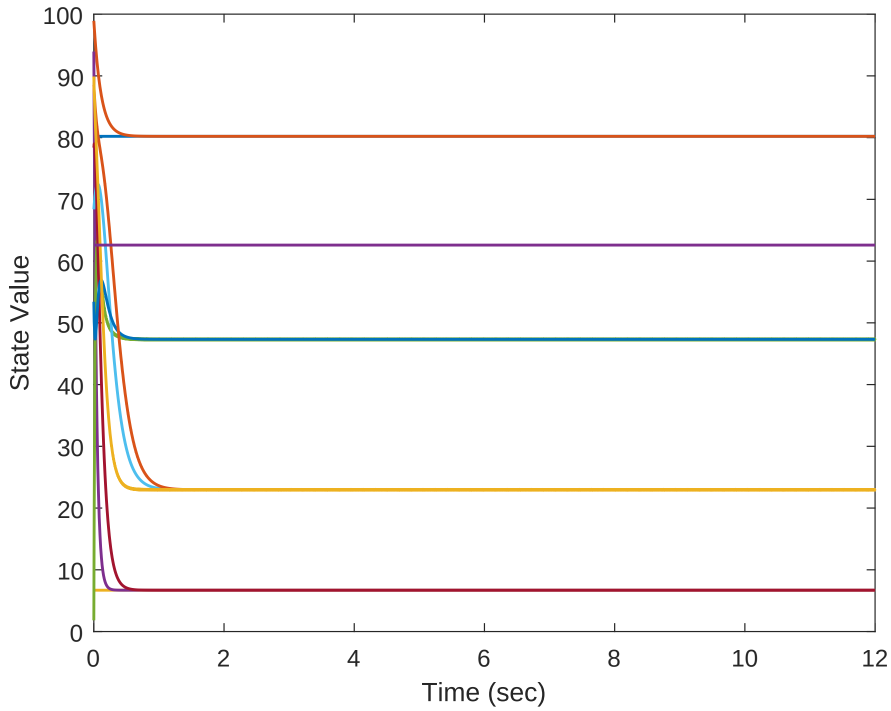

Figure 3.12. The simulation results for the system given in Figure 3.11 with $\Delta t = 1$ ms.

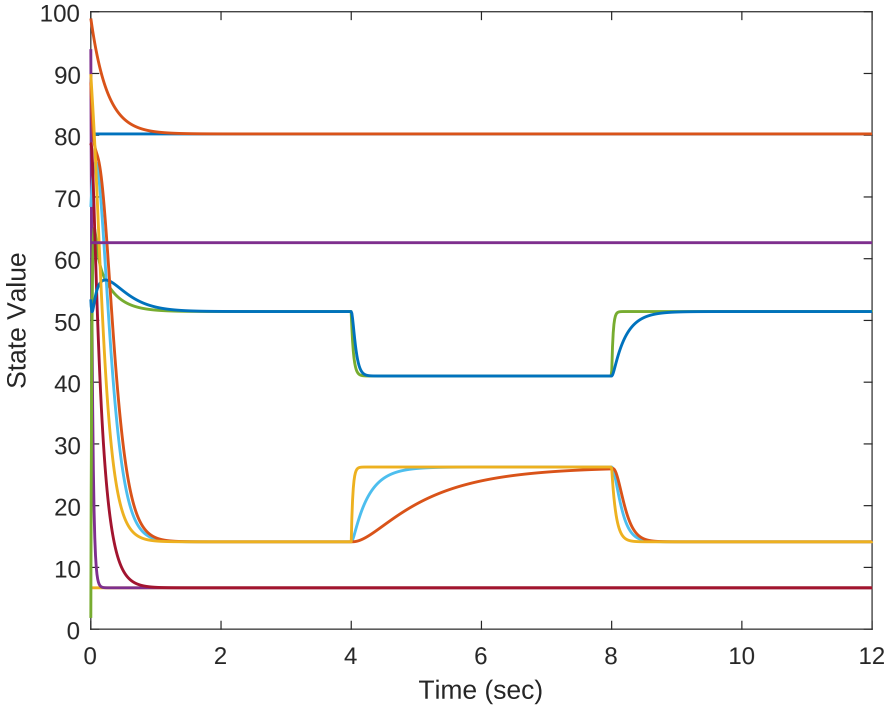

Figure 3.13. The simulation results for the system given in Figure 3.11 with $\Delta t = 4$ s.

in the equilibrium. Figure 3.12 shows the state values of the system where Δt equals 1 millisecond.

When Δt is selected as 4 s, the simulation results are as shown in Figure 3.13. We note that the number of clusters and the elements of the clusters do not change and the system achieves cluster consensus since the system has 5 clusters at the end of each period. However, note also that the equilibrium points of the secondary layer subgraphs keep changing whereas the equilibria of the primary layer subgraphs do not change over time.

In the special case when the rows corresponding to the roots of the secondary layer subgraphs do not change or if they vary in the same proportion is equal, then the equilibria of the whole system seems to be independent of Δt. The simulation results for this special case are shown in Figures 3.14 and 3.15 for $\Delta t = 1$ millisecond and $\Delta t = 4$ seconds, respectively.

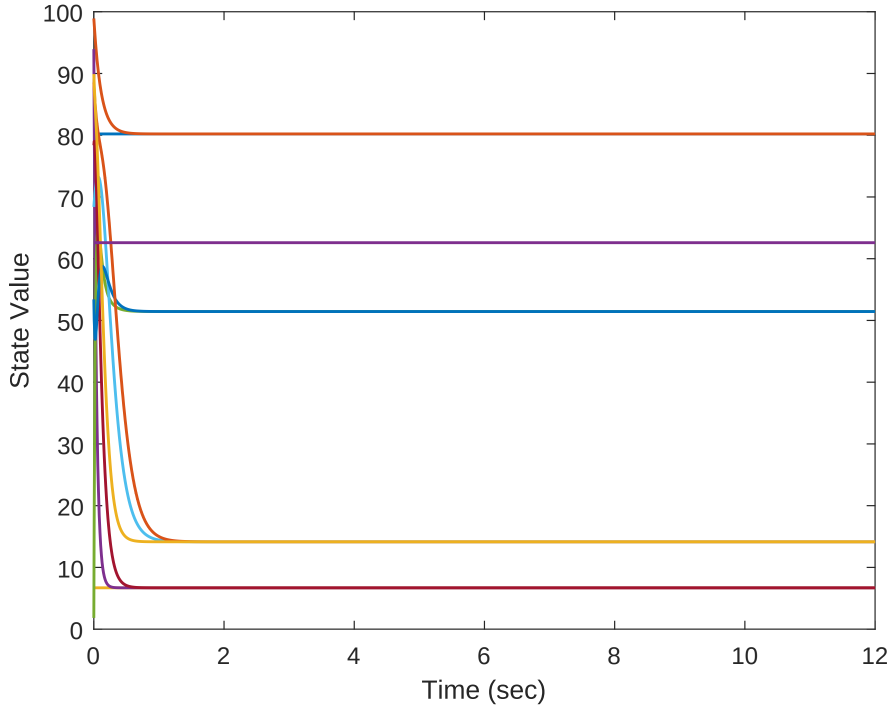

Figure 3.14. The simulation results for the system in Figure 3.11 with proportioned root weight variations and $\Delta t = 1$ ms.

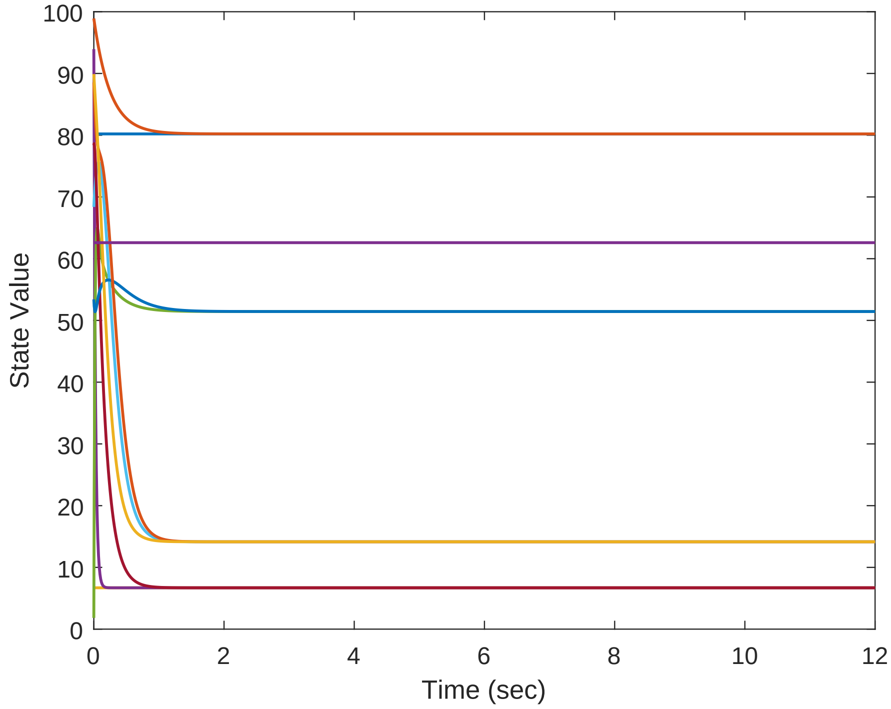

Figure 3.15. The simulation results for the system in Figure 3.11 with proportioned root weight variations and $\Delta t = 4$ s.

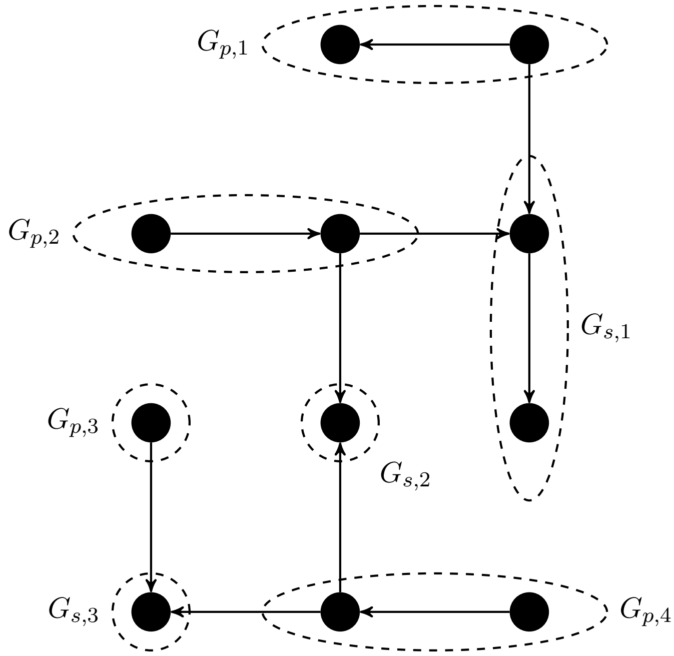

Figure 3.16. A digraph with 11 nodes and 10 edges.

3.6.2. Topology-Varying Systems

In this subsection, we present numerical examples for *Topology-Varying Systems* where the topology of a multi-agent system may change over time. Consider the network with 11 nodes and 10 edges in Figure 3.16 in addition to the network shown in Figure 3.11. The two networks have the same nodes but different topologies. Based on the primary and secondary layer detection algorithms, there are 5 clusters in the system under the first topology, and 7 clusters in the second one. Suppose that the system interaction switches periodically among given two topologies at each period Δt.

When Δt is selected small enough, i.e., 1 millisecond, the system attains 6 clusters. In this case, we note the number of clusters which the system attains is the same as that of clusters in the union digraph of these two digraphs shown in Figure 3.19.

The simulation result of the switched system for $\Delta t = 4$ seconds is as shown in Figure 3.18. The system reaches 5 clusters during the time when it uses the first topology, and 7 clusters during the second one.

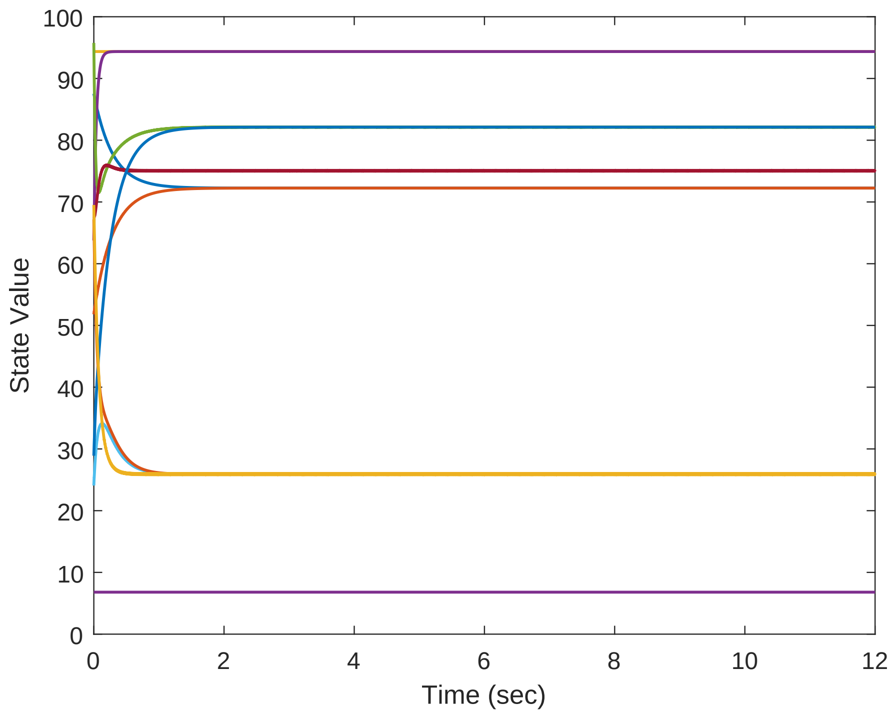

Figure 3.17. The simulation result for the switched systems given in Figures 3.11 and 3.16 with $\Delta t = 1$ ms.

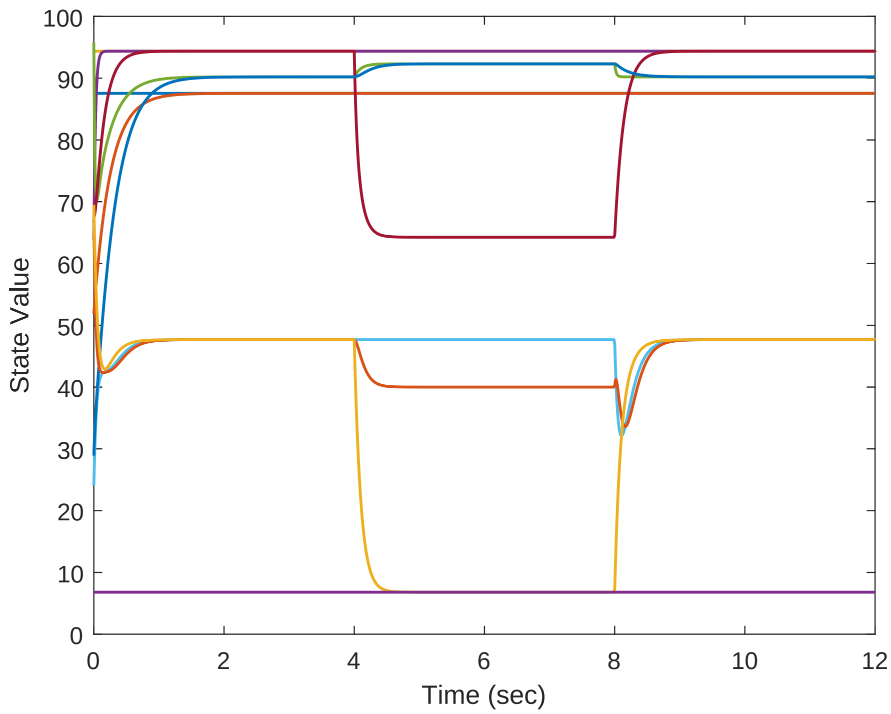

Figure 3.18. The simulation result for the switched systems given in Figures 3.11 and 3.16 with $\Delta t = 4$ s.

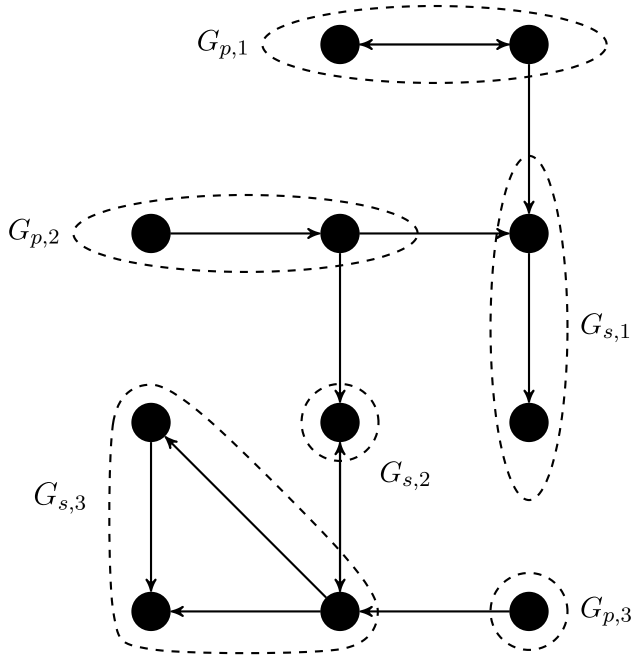

Figure 3.19. The union digraph of two digraphs shown in Figure 3.11 and Figure 3.16.

3.7. Chapter Summary

This chapter has investigated the cluster consensus problem for multi-agent systems that are expressed as digraph in continuous-time. We have given the properties of sub-Laplacian matrices which are exploited for the first time in the literature. Moreover, the stability conditions of the sub-Laplacian matrices are illustrated with several numerical examples. We have theoretically shown that the total number of primary and secondary layer subgraphs is equal to the number of clusters of a given network. Finally, some simulations are provided to demonstrate our theoretical results.

The concepts of primary and secondary layer subgraphs are not only considered as the algorithms that provide how many clusters a system has, but also make easier to analyze the behavior of networks using linear consensus dynamics under different topologies. For example, estimating the number of votes each party will get from social networks such as twitter, facebook etc. can be possible by using the primary and secondary layer subgraphs. Another example is to create a directed network among mobile robot clusters such that each cluster is doing different assignments.

Networks investigated in this chapter are considered as without any delay. However, due to the finite speed of communication and processing, time-delay is unavoidable in the networks. Chapter 4 and 5 contain cases where networks have input or communication delay.

4. CLUSTER CONSENSUS UNDER INPUT DELAY

In Chapter 3, we have investigated the cluster consensus problem for multi-agent systems in absence of delay. However, delays are inevitable in real-time systems and may lead to instability. Delays in networked systems can roughly be considered as input delay and communication delay. In this chapter, we analyze the effect of input delay on cluster consensus and provide stability analysis of delayed multi-agent systems with fixed input delay whereas the effect of bounded communication delay is examined in Chapter 5.

4.1. Model with Input Delay

If there exists input delay, (2.4) becomes

$$\dot{x}_i(t) = u_i(t - \tau_d) \tag{4.1}$$

where

$$u_i(t - \tau_d) = \sum_{v_j \in N_i} a_{ij}(x_j(t - \tau_d) - x_i(t - \tau_d)), \quad i \in I, \tag{4.2}$$

and τ_d is the fixed input delay.

By taking the Laplace transform of the above equation, we obtain

$$\begin{aligned} sX_i(s) - x_i(0) &= U_i(s)e^{-\tau_d s} \\ &= \sum_{v_j \in N_i} a_{ij}(X_j(s) - X_i(s))e^{-\tau_d s} \end{aligned} \tag{4.3}$$

where $X_i(s)$ and $U_i(s)$ are the Laplace transforms of $x_i(t)$ and $u_i(t)$, respectively.

Equation (4.3) can also be rewritten in matrix form as

$$X(s) = (sI + e^{-\tau_d s} L)^{-1} x(0). \tag{4.4}$$

4.2. Analysis of Input Delay

Now, the analysis of cluster consensus for a multi-agent system with input delay becomes a stability problem of the following MIMO (multiple-input multiple-output) transfer function:

$$H(s) = (sI + e^{-\tau_d s} L)^{-1}. \tag{4.5}$$

Theorem 4.1. *A multi-agent network with fixed input delay $\tau_d > 0$ is stable if and only if the following condition is satisfied:*

- $\tau_d \in (0, \tau^*)$ *with* $\tau^* = \min_k \left[\frac{1}{|\lambda_k|} \arctan \left(\left| \frac{Re\{\lambda_k\}}{Im\{\lambda_k\}} \right| \right) \right]$ *where λ_k is a nonzero eigenvalue of L.*

Furthermore, the network results in $l_p + l_s$ clusters where l_p and l_s are the number of primary and secondary layer subgraphs, respectively.

Proof. There are two parts of the proof which are stability of a multi-agent system with fixed input delay and the effect of the delay on the number of clusters.

Part (i) (Stability of a multi-agent system with fixed input delay):

From [17], the condition for stability of the MIMO transfer function, $H(s)$, is

$$s = 0 \text{ or } s + \lambda_k e^{-\tau_d s} = 0 \text{ for } s \neq 0. \tag{4.6}$$

Unlike the analysis by [17], eigenvalues of L can be complex since the graph of the system is directed, i.e. $\lambda_k = Re\{\lambda_k\} + jIm\{\lambda_k\}$. In order to determine the upper bound of input delay τ_d, we need to calculate the smallest value of τ_d that is larger than zero such that (4.6) has a zero on the imaginary axis. Then, (4.6) can be rewritten as

$$jw + (Re\{\lambda_k\} + jIm\{\lambda_k\})e^{-j\tau_d w} = 0. \tag{4.7}$$

Expanding (4.7) yields

$$\begin{gathered} jw + (Re\{\lambda_k\} + jIm\{\lambda_k\})(cos(\tau_d w) - jsin(\tau_d w)) = 0 \\ \\ jw + Re\{\lambda_k\}cos(\tau_d w) - jRe\{\lambda_k\}sin(\tau_d w) + \\ jIm\{\lambda_k\}cos(\tau_d w) + Im\{\lambda_k\}sin(\tau_d w) = 0. \end{gathered} \tag{4.8}$$

Real and imaginary parts of (4.8) must be zero individually:

$$j(w - Re\{\lambda_k\}sin(\tau_d w) + Im\{\lambda_k\}cos(\tau_d w)) = 0. \tag{4.9}$$

$$Re\{\lambda_k\}cos(\tau_d w) + Im\{\lambda_k\}sin(\tau_d w) = 0. \tag{4.10}$$

From (4.10), we have

$$\frac{Re\{\lambda_k\}}{Im\{\lambda_k\}} = -\frac{sin(\tau_d w)}{cos(\tau_d w)} = -tan(\tau_d w). \tag{4.11}$$

Substituting (4.11) into (4.9), we obtain

$$
\begin{aligned}
& w = Re\{\lambda_k\}sin(\tau_d w) - Im\{\lambda_k\}cos(\tau_d w) \\
& Re\{\lambda_k\} = -wcos(\tau_d w) \\
& Im\{\lambda_k\} = wsin(\tau_d w) \\
& w^2 = Re\{\lambda_k\}^2 + Im\{\lambda_k\}^2 \\
& w = \pm|\lambda_k|.
\end{aligned}
\tag{4.12}
$$

After finding w in terms of λ_k, from (4.11) we get

$$
\begin{aligned}
\tau_d w &= \arctan\Big(-\frac{Re\{\lambda_k\}}{Im\{\lambda_k\}}\Big) \\
\tau_d &= \frac{1}{w}\arctan\Big(-\frac{Re\{\lambda_k\}}{Im\{\lambda_k\}}\Big) \\
\tau_d &= \frac{1}{\pm|\lambda_k|}\arctan\Big(-\frac{Re\{\lambda_k\}}{Im\{\lambda_k\}}\Big).
\end{aligned}
\tag{4.13}
$$

We know that non-zero eigenvalues of the Laplacian matrix have positive real-parts and $\tau_d > 0$. So, if $Im\{\lambda_k\} > 0$, $w = -|\lambda_k|$, otherwise $w = |\lambda_k|$. Then, the smallest value of $\tau^* > 0$ satisfies the following equation:

$$
\tau^* = \min_k \left[\frac{1}{|\lambda_k|}\arctan\Big(\Big|\frac{Re\{\lambda_k\}}{Im\{\lambda_k\}}\Big|\Big)\right]. \tag{4.14}
$$

Moreover, if $\tau_d = \tau^*$ holds, the response of the system becomes oscillatory.

Part (ii) (The effect of the input delay on the number of clusters):

In the previous part, the interval of input delay that can be tolerated by a multi-agent system is given. We now investigate the number of clusters in the multi-agent system with fixed input delay.

Based on the primary and secondary layer decompositions of the network, the delayed consensus system can be represented as

$$\dot{x}_p(t) = -L_p x_p(t-\tau_d) \tag{4.15a}$$

$$\dot{x}_s(t) = -L_s x_s(t-\tau_d) - L_{sp} x_p(t-\tau_d) \tag{4.15b}$$

We investigate separately the primary and secondary layer dynamics.

L_p contains l_p Laplacian matrices corresponding to l_p primary layer subgraphs. Each primary layer subgraph dynamics can be expressed as

$$\dot{x}_{p,i}(t) = -L_{i,i} x_{p,i}(t-\tau_d) \quad i = 1, 2, \ldots, l_p$$

where $x_{p,i}$ is the state vector for the i-th primary layer subgraph. Each primary layer subgraphs with time-delay reaches consensus which results in l_p clusters in l_p primary layer subgraphs since each primary layer subgraph is independent of each other.

The equilibrium point of the secondary layer dynamics in (4.15b) can be calculated as

$$\tilde{x}_s = -L_s^{-1} L_{sp} \tilde{x}_p \tag{4.16}$$

where $\tilde{x}_p$ and $\tilde{x}_s$ are the equilibrium points of the delayed primary and secondary layer subgraphs, respectively. The result is the same with (3.13).

The number of clusters in the delayed secondary layer subgraphs is l_s since the number of the delayed primary layer subgraphs does not change.

Consequently, a multi-agent system with fixed input delay is stable if and only if $\tau_d \in (0, \tau^*)$ and the time-delay does not affect the number of clusters in the system. $\square$

Remark 4.1. *In case all eigenvalues of L are real, i.e., $Im\{\lambda_k\} = 0$ for all k, from (4.14) we have $\tau^* = \min\limits_{k} \frac{\pi}{2\lambda_k}$ which agrees with [17].*

4.3. Numerical Analysis of Input Delay

Theoretical results given in this chapter are demonstrated via several numerical examples for different input delays. In the first case, we provide simulation result for that input delay in the interval. In the second case, we investigate the situation where input delay equals the upper bound that is determined by the theorem. In the third case, the situation where input delay is greater than the upper bound is simulated. In the last case, system behavior which is to be theoretically justified is given for the situation where input delay is time-varying.

Consider the network with 12 nodes and 19 edges in Figure 4.1. The state values for the network without any delay are shown in Figure 4.2 for randomly selected initial state values, and the following Laplacian matrix:

$$L = \left[\begin{array}{c|cc|cc|c|c|c|cccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -10 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -13 & 13 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & -13 & 33 & 0 & 0 & -12 & 0 & 0 & -8 \\ \hline
-13 & -2 & 0 & 0 & 0 & 0 & 40 & -10 & 0 & 0 & -5 & 0 \\ \hline
0 & 0 & -1 & 0 & 0 & 0 & 0 & 17 & -16 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & -14 & 0 & 0 & -11 & 0 & 42 & 0 & -17 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -10 & 10 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 17 & -17 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -17 & -6 & 23
\end{array}\right] \tag{4.17}$$

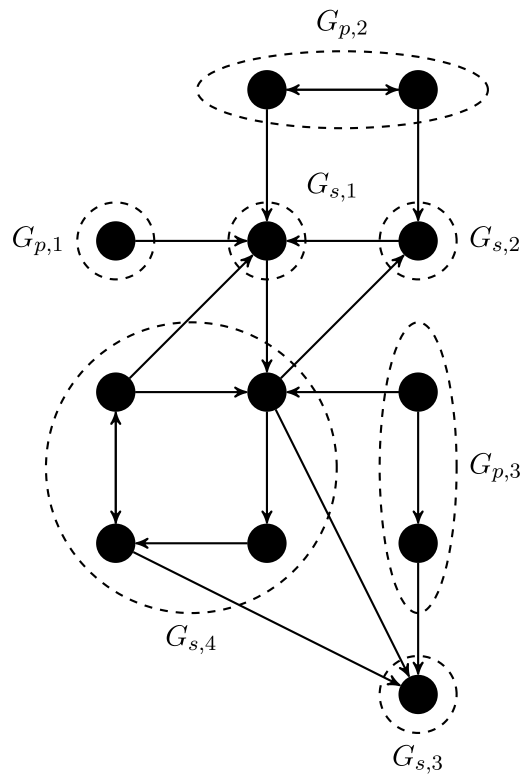

Figure 4.1. A digraph with 12 nodes and 19 edges.

The above Laplacian matrix is randomly formed from the elements of the weighted adjacency matrix, a_{ij}, which satisfy Assumption 2.1. As shown in Figure 4.2, the network converges to 7 clusters, 3 of which are from primary layer subgraphs and the other 4 of which are from secondary layer subgraphs. Clusters are labelled as $G_{p,1}, G_{p,2}, G_{p,3}$ for the primary layer subgraphs and $G_{s,1}, G_{s,2}, G_{s,3}, G_{s,4}$ for the secondary layer subgraphs.

4.3.1. Case with $\tau_d < \tau^*$

The eigenvalues of the Laplacian matrix used for the network in Figure 4.1 are (33, 43.1021 + j7.0639, 43.1021 - j7.0639, 2.4858, 12, 19.9808, 20.1647 + j5.4505, 20.1647 - j5.4505, 13, 0, 0, 0). From Theorem 4.1, we can obtain the smallest $\tau_d = \tau^*$ that is larger than zero as 0.03224 for $\lambda_k = 43.1021 - j7.0639$, $w = |\lambda_k|$ or $\lambda_k = 43.1021 + j7.0639$, $w = -|\lambda_k|$.

In the case where τ_d is selected as 0.02500 seconds, i.e., τ_d is below the upper bound, the state values are shown in Figure 4.3. As it can be seen, the system reaches cluster consensus without any change in the clusters.

Figure 4.4 shows the state values of the network with input delay, τ_d, that is equal to 0.0300 seconds. Still, there is no change in any cluster. It implies that if $\tau_d < \tau^*$ holds, the cluster agreement is achieved.

4.3.2. Case with $\tau_d = \tau^*$

For the upper bound $\tau_d = \tau^* = 0.03224$ seconds, the simulation results are depicted in Figure 4.5 which shows the oscillatory behaviour as expected.

4.3.3. Case with $\tau_d > \tau^*$

The result is illustrated in Figure 4.6 for the network with $\tau_d = 0.03500 > \tau^* = 0.03224$ seconds. Exceeding the limit, τ^*, brings instability to the network.

4.3.4. Case with Time-Varying Delay

Now, consider the case where the input delay is time varying but does not exceed the limit, τ^*, e.g. $\tau_d(t) = 0.016(1 + sin(2\pi t)) < \tau^*$ where $\tau_d(t)$ is shown in Figure 4.7. As shown in Figure 4.8, the system exhibits stable behaviour, which is to be theoretically justified.

4.4. Chapter Summary

In this chapter, we have discussed the effect of input delay on cluster consensus for a given multi-agent system operating in continuous-time. We have presented a theorem which gives the interval of input delay that can be tolerated by a multi-agent system before it becomes unstable. The theorem yields the upper bound of input delay that does not affect the number of clusters in the multi-agent system based on the decompositions of primary and secondary layer subgraphs given in Chapter 3. While input delay is studied in this chapter, the effect of communication delay is considered next in Chapter 5.

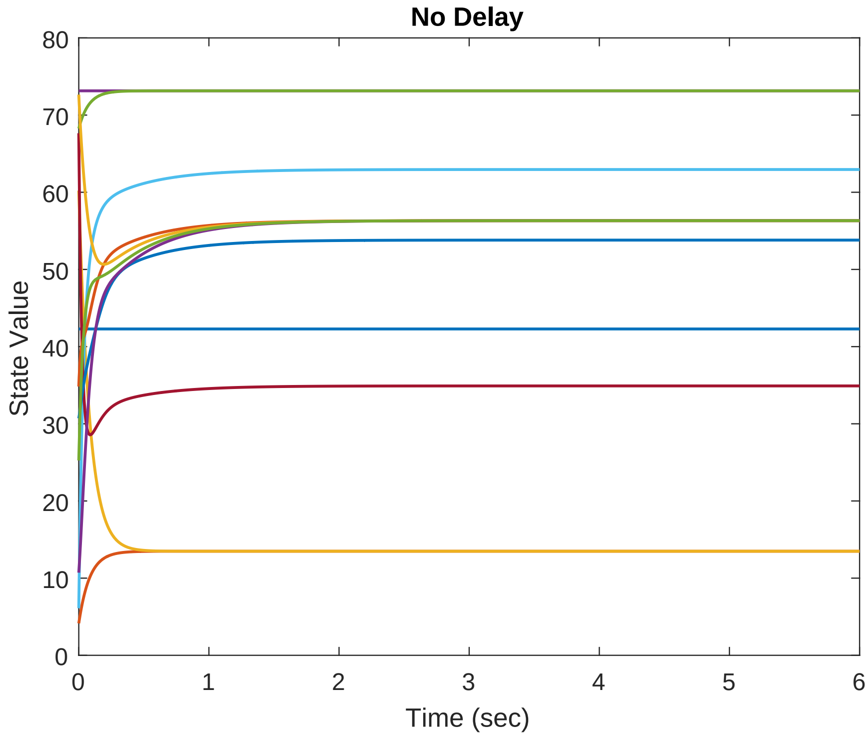

Figure 4.2. Simulation results for the network in Figure 4.1 without any delay.

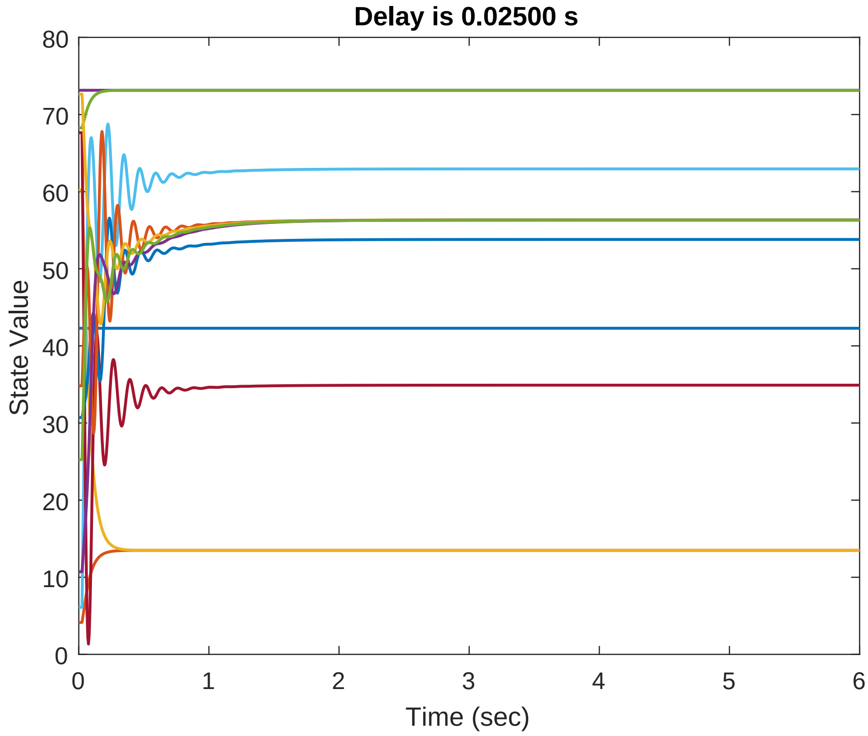

Figure 4.3. Simulation results for the network in Figure 4.1 with input delay $\tau_d =$ 0.02500 s.

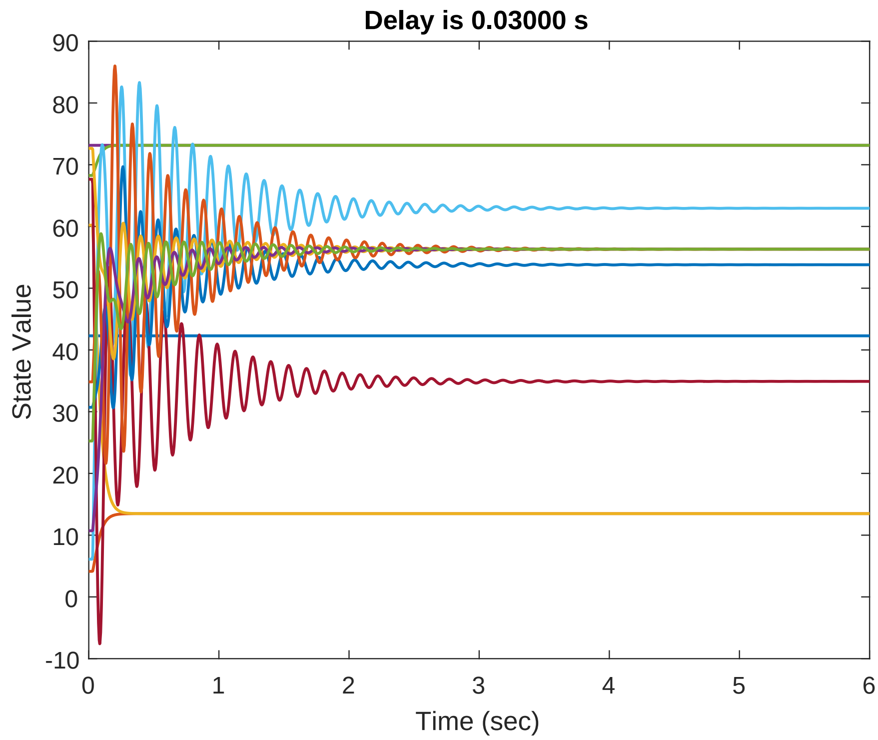

Figure 4.4. Simulation results for the network in Figure 4.1 with input delay $\tau_d =$ 0.03000 s.

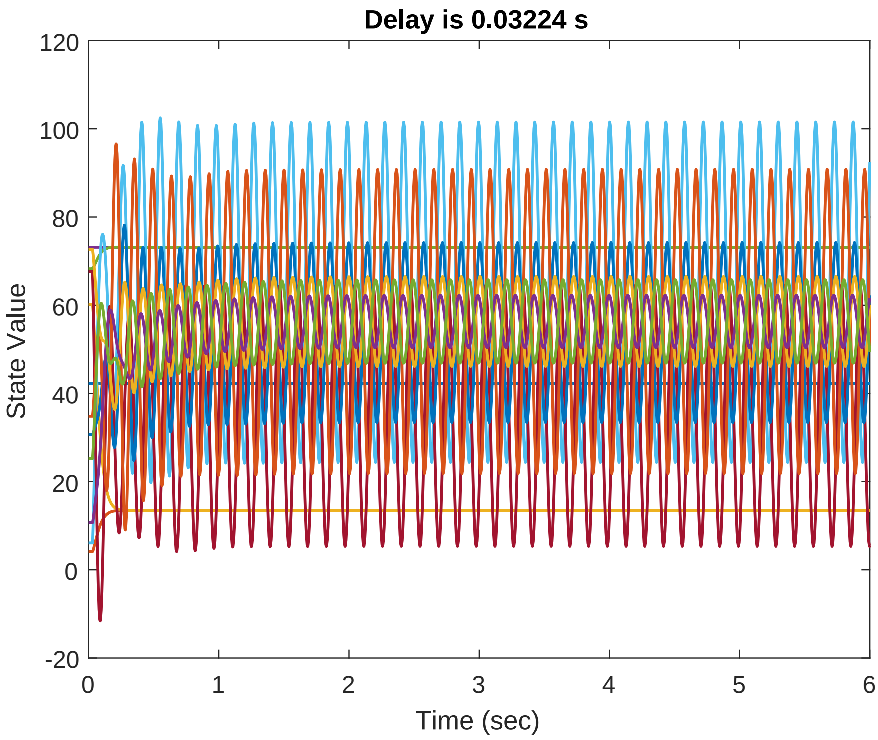

Figure 4.5. Simulation results for the network in Figure 4.1 with input delay $\tau_d =$ 0.03224 s.

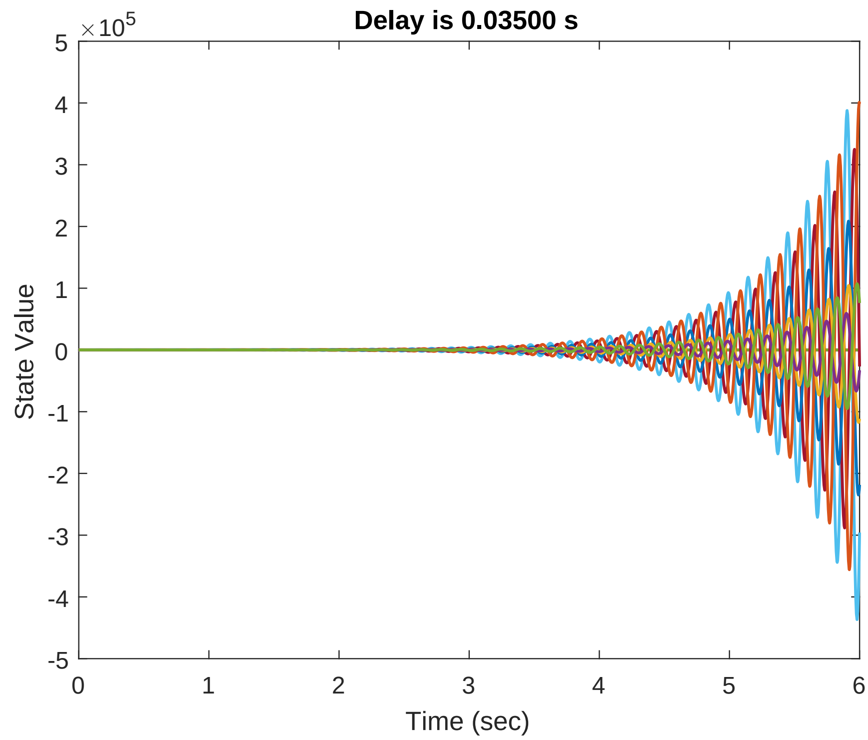

Figure 4.6. Simulation results for the network in Figure 4.1 with input delay $\tau_d = 0.0350$ s.

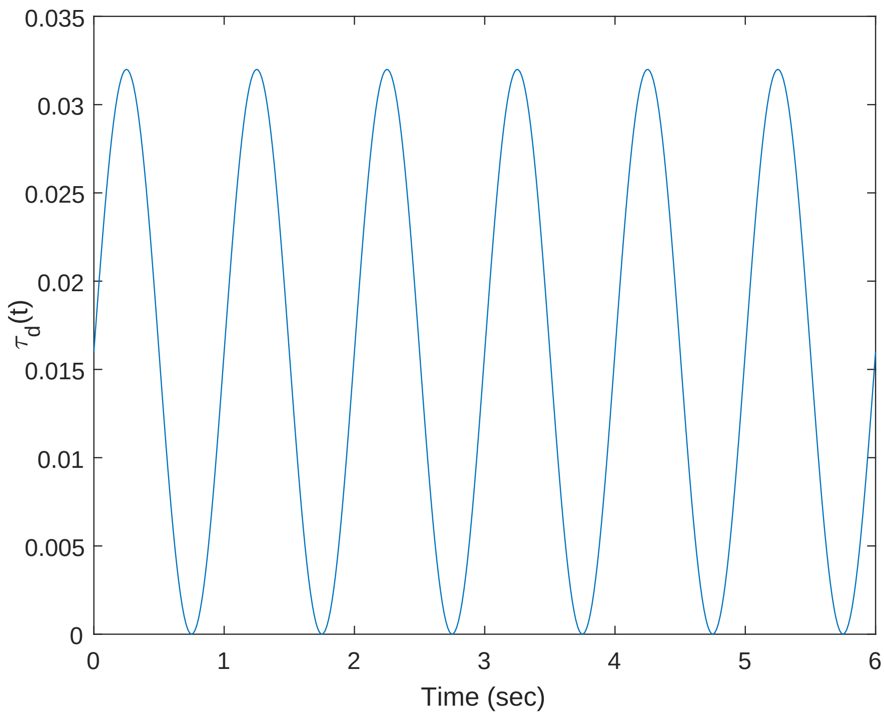

Figure 4.7. Input delay $\tau_d(t) = 0.016(1 + sin(2\pi t))$.

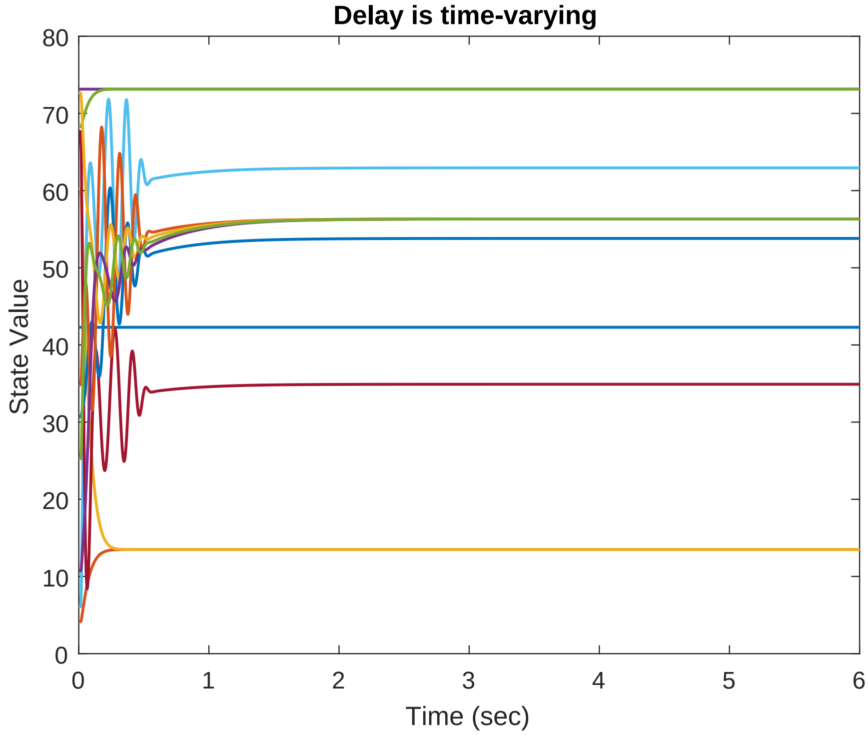

Figure 4.8. Simulation results for the network in Figure 4.1 with input delay $\tau_d(t) = 0.016(1 + sin(2\pi t))$.

5. CLUSTER CONSENSUS UNDER COMMUNICATION DELAY

In this chapter, the effect of bounded communication delay on cluster consensus is investigated and its convergence properties are analyzed.

5.1. Model with Communication Delay

If there exists a fixed communication delay, (2.4) and (2.5) become

$$\dot{x}_i(t) = \sum_{v_j \in N_i} a_{ij}(x_j(t-\tau_{cd}) - x_i(t)), \quad \tau_{cd} > 0, \quad i \in I. \tag{5.1}$$

The system dynamics can be expressed in compact form as follows:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau_{cd}), \quad \tau_{cd} > 0 \tag{5.2}$$

where A_0 is a diagonal matrix whose diagonal elements are non-positive, A_1 is a matrix whose diagonal elements are zero and off-diagonal elements are non-negative.

5.2. Analysis of Bounded Communication Delay

Now, we present Lemmas 5.1 and 5.2 that will be used to prove our main result on communication delay.

Lemma 5.1. *[29] Consider the system (5.2) with a fixed delay, τ_{cd}. If there exists $\bar{\tau} \geq \tau_{cd}$ such that:*

(i) $1 + \lambda_i(A_1)\frac{1-e^{-s\bar{\tau}}}{s} \neq 0$, for all $s \in \mathbb{C}^+$ and $i = 1, 2, \ldots, n$ and

(ii) there exist symmetric and positive-definite matrices: P, S such that the following

LMI holds:

$$\begin{bmatrix} (A_0+A_1)^T P + P(A_0+A_1) + \bar{\tau} S & (A_0+A_1)^T P A_1 \\ A_1^T P (A_0+A_1)^T & -\bar{\tau} S \end{bmatrix} < 0$$

then all elements of the system (5.2) converge asymptotically to a common value.

Lemma 5.2. *[30] (Frequency Sweeping Test) The system (5.2) is stable independent of delay if and only if*

(i) A_0 *is asymptotically stable,*

(ii) $A_0 + A_1$ *is asymptotically stable, and*

(iii) $\rho((jwI - A_0)^{-1}A_1) < 1$, $\forall w > 0$,

where $\rho(\cdot)$ *denotes the spectral radius of a matrix.*

Theorem 5.1. *The system (5.2) with bounded communication delay,* τ_{cd} *is stable. Furthermore, the network results in* $l_p + l_s$ *clusters where* l_p *and* l_s *are the number of primary and secondary layer subgraphs, respectively.*

Proof. In presence of communication delay, the primary layer dynamics in (4.15) can be rewritten as follows:

$$\dot{x}_p(t) = D_p x_p(t) + E_p x_p(t - \tau_{cd}) \tag{5.3}$$

where D_p is $\tilde{n}_1 \times \tilde{n}_1$ diagonal matrix whose diagonal elements are non-positive, E_p is a matrix whose diagonal elements are zero and off-diagonal elements are non-negative. Note that $D_p + E_p = -L_p$. Secondary layer dynamics in (4.15) can be expressed as

$$\dot{x}_s(t) = D_s x_s(t) + E_s x_s(t - \tau_{cd}) - L_{sp} x_p(t - \tau_{cd}) \tag{5.4}$$

where D_s is $\tilde{n}_2 \times \tilde{n}_2$ diagonal matrix whose diagonal elements are negative, E_s is a matrix whose diagonal elements are zero and off-diagonal elements are non-negative. Note that $D_s + E_s = -L_s$.

We investigate separately the stability of primary layer and secondary layer dynamics in consecutive steps.

Step 1 (Consensus states for the primary layer subgraphs): L_p contains l_p Laplacian matrices corresponding to l_p primary layer subgraphs. Each primary layer subgraph dynamics can be expressed as

$$\dot{x}_{p,i}(t) = D_{p,i}x_{p,i}(t) + E_{p,i}x_{p,i}(t - \tau_{cd}), \quad i = 1, 2, \ldots, l_p$$

where $D_{p,i}$ is diagonal matrix whose diagonal elements are non-positive, $E_{p,i}$ is a matrix whose diagonal elements are zero and off-diagonal elements are non-negative, $x_{p,i}$ is the state vector for the i-th primary layer subgraph.

From Lemma 5.1, each primary layer dynamics with bounded communication delay is stable. Then, each primary layer subgraph with the system matrix $L_{i,i}$ achieves consensus which results in l_p clusters in l_p primary layer subgraph.

Step 2 (Consensus states for the secondary layer subgraphs): Characteristic quasipolynomial of (5.4) equals characteristic quasipolynomial of the following dynamics without external input:

$$\dot{x}_s(t) = D_s x_s(t) + E_s x_s(t - \tau_{cd}). \tag{5.5}$$

We can check the stability conditions of the above system from Lemma 5.2:

(i) $A_0 = D_s$ is always asymptotically stable since D_s is a diagonal matrix whose diagonal elements are negative.

(ii) $A_0 + A_1 = D_s + E_s$ is always asymptotically stable since $-(E_s + D_s) = L_s$ is sub-Laplacian matrix defined in Lemma 3.4 such that $-L_s$ is asymptotically stable.

(iii)

$$D_s = \begin{bmatrix} d_{11} & 0 & 0 & \dots \\ 0 & d_{22} & 0 & \dots \\ 0 & 0 & d_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, E_s = \begin{bmatrix} 0 & e_{12} & e_{13} & \dots \\ e_{21} & 0 & e_{23} & \dots \\ e_{31} & e_{32} & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

where d_{ii} is negative and e_{ij} is non-negative for i, $j = 1, 2, \dots, \tilde{n}_2$, $i \neq j$. Note that $d_{ii} + \sum_{j=1}^{n} e_{ij} \leq 0$ since $-(E_s + D_s)$ is a sub-Laplacian matrix.

$$(jwI - D_s)^{-1}E_s = \begin{bmatrix} 0 & \frac{e_{12}}{jw-d_{11}} & \frac{e_{13}}{jw-d_{11}} & \dots \\ \frac{e_{21}}{jw-d_{22}} & 0 & \frac{e_{23}}{jw-d_{22}} & \dots \\ \frac{e_{31}}{jw-d_{33}} & \frac{e_{32}}{jw-d_{33}} & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \quad (5.6)$$

From Lemma 3.1, we obtain circles whose radii are smaller than 1 for all $w > 0$. Then, the condition (iii) always holds.

Hence, the secondary layer subgraphs are stable independent of delay such that the secondary layers subgraphs reaches l_s clusters in l_s secondary layer subgraphs.

Step 3 (Total number of clusters): From Step 1 and 2, the total number of clusters can be calculated as follows:

$$K = l_p + l_s. \quad (5.7)$$

□

5.3. Numerical Analysis of Communication Delay

Consider the network with 12 nodes and 17 edges shown in Figure 5.1. The state values for the network without any delay are shown in Figure 5.2 for randomly selected

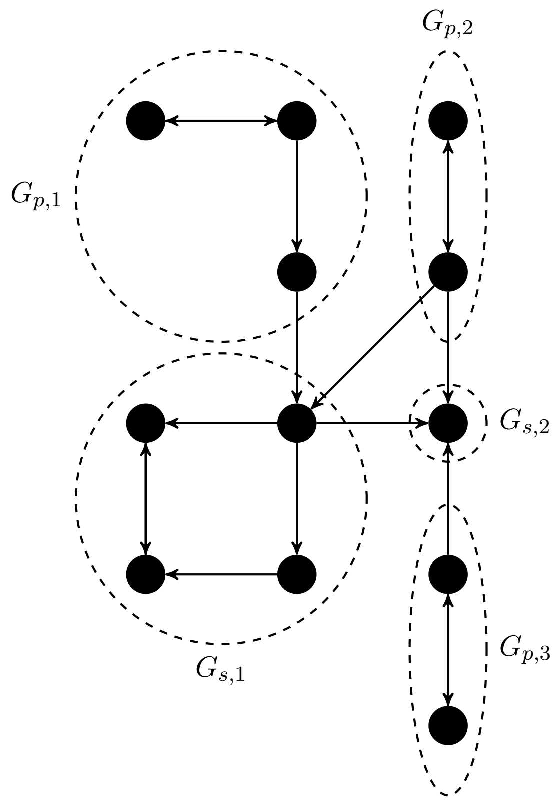

Figure 5.1. A digraph with 12 nodes and 17 edges.

initial state and weighted adjacency matrix whose elements satisfy Assumption 2.1. The network converges to 5 clusters, 3 of which are from primary layer subgraphs and the other 2 of which are from secondary layer subgraphs. Clusters are labelled as $G_{p,1}$, $G_{p,2}$, $G_{p,3}$ for the primary layer subgraphs and $G_{s,1}$, $G_{s,2}$ for the secondary layer subgraphs.

Figure 5.3 shows the state values of the network with communication delay, τ_{cd}, that is equal to 0.4 seconds. As it can be seen, the system reaches cluster consensus without any change in the clusters.

In the case where communication delay, τ_{cd}, is selected as 2 seconds, the state values are shown in Figure 5.4. Still, there is no change in any cluster.

Consider the case where the communication delay, τ_{cd}, is equal to 5 seconds. The delayed network converges to the same number of clusters as depicted in Figure 5.5.

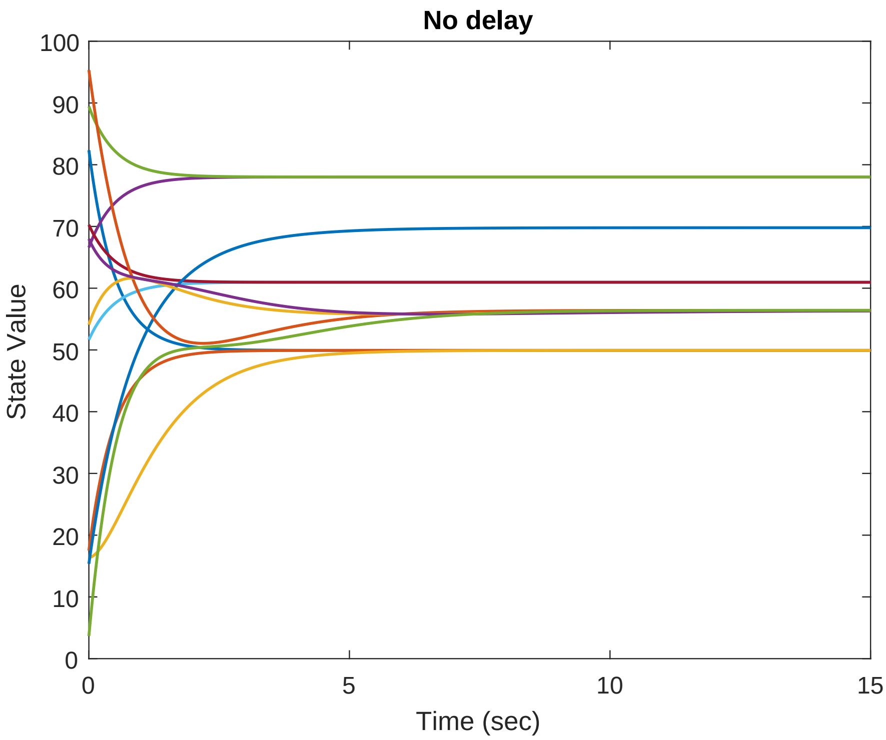

Figure 5.2. Simulation results for the network given in Figure 5.1 without any delay.

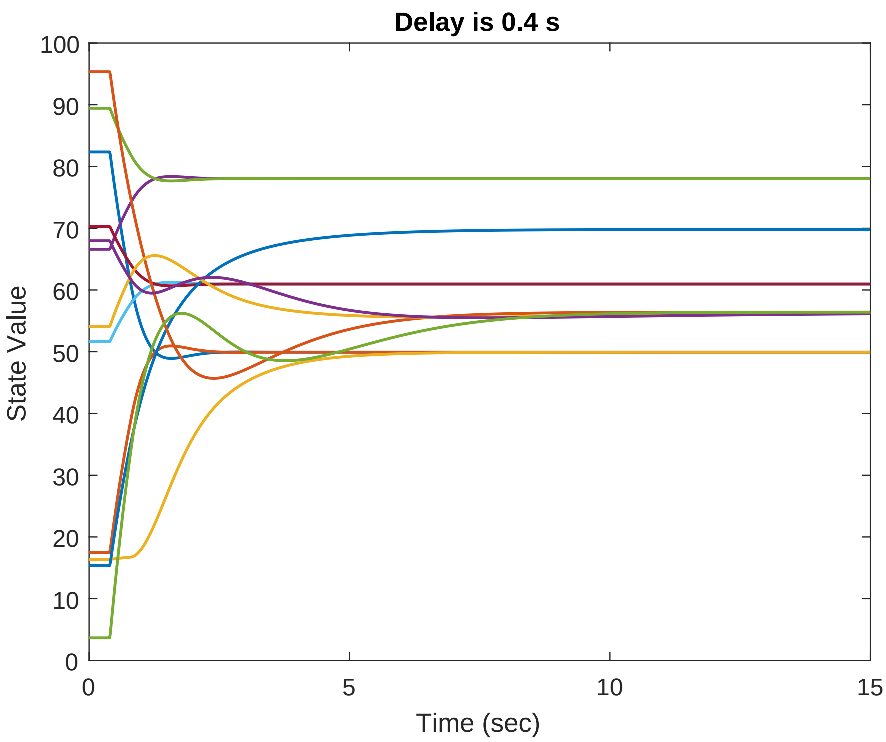

Figure 5.3. Simulation results for the network given in Figure 5.1 with communication delay $\tau_{cd} = 0.4$ s.

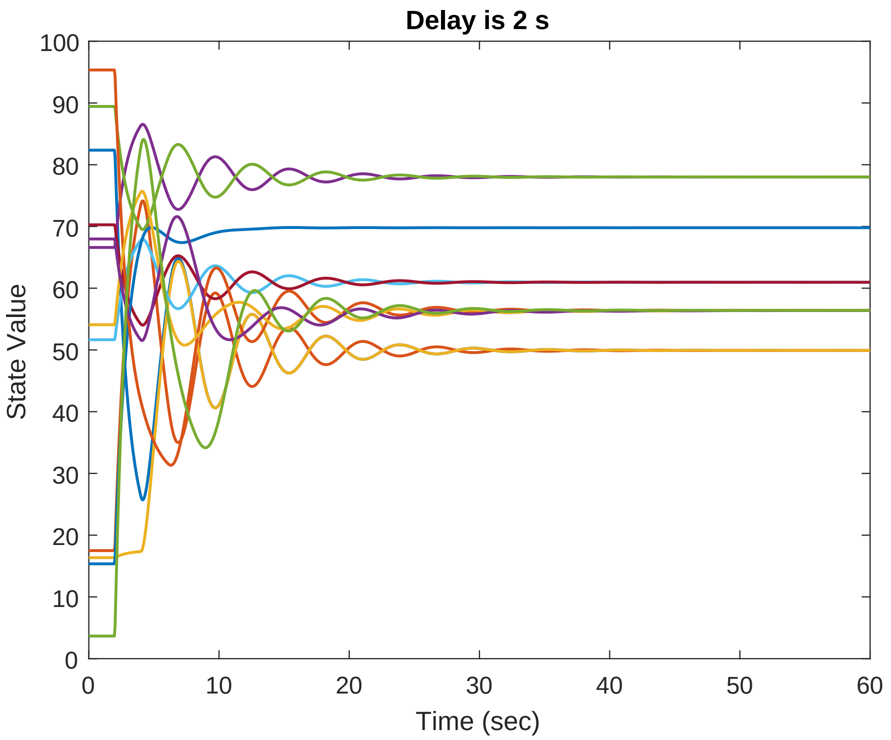

Figure 5.4. Simulation results for the network given in Figure 5.1 with communication delay $\tau_{cd} = 2$ s.

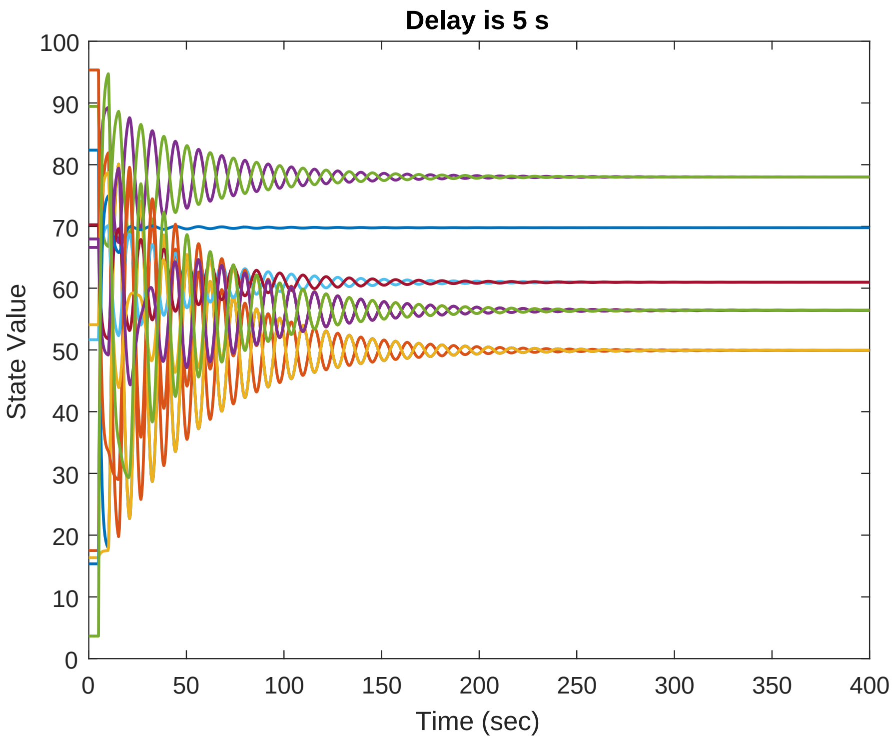

Figure 5.5. Simulation results for the network given in Figure 5.1 with communication delay $\tau_{cd} = 5$ s.

5.4. Chapter Summary

In this chapter, we have investigated the effect of communication delay on multi-agent networks achieving cluster consensus in continuous-time. Delayed system protocols are stated and the assumption imposed on the protocol is given. We have shown that bounded communication delay does not change the number of clusters in a given multi-agent system. One can also calculate the equilibrium for the primary layer subgraphs based on the additional assumption that was analyzed in [31]:

Assumption 5.1. *Weighted elements of the adjacency matrix A satisfy the following conditions:*

(i) $a_{ii} = 0$, $\forall i \in I$.

(ii) $a_{ij} > 0$ *if* $\varepsilon_{ij} \in \mathcal{E}$, *and* $a_{ij} = 0$ *if* $\varepsilon_{ij} \notin \mathcal{E}$ *for all* $i, j \in I$ *and* $i \neq j$.

(iii) $\sum_{j=1}^{n} a_{ij} = \eta$, $\forall i \in I$ *and* η *is a positive constant.*

The result in [31], can be applied to each primary layer subgraph individually to determine the equilibrium points of the primary layer subgraphs.

Finally, some simulation results are included to evaluate the effect of communication delay.

6. CONCLUSIONS AND FUTURE WORK

In this thesis, we have discussed the cluster consensus problem for multi-agent networks using linear dynamics in continuous-time. The contribution of the thesis is to provide the stability analysis and convergence properties of the networks with or without delay.

In the first part of this thesis, we have reviewed basic graph theoretical concepts related with the networks and given consensus protocol to deal with our problem. The assumption on the protocol is stated clearly.

In the second part, we have shown that the number of clusters depends on the structure of the system and equals the total number of primary and secondary layer subgraphs. We have not only presented a theorem which gives how many clusters a system has, but also studied a special case in which the number of clusters may be reduced. We have introduced *sub-Laplacian matrices* for the first time in the literature and presented their mathematical properties.

In the third part of this thesis, we have investigated the stability of multi-agent networks with input delay. The analysis yields the upper bound of input delay that does not change the number of clusters and convergence properties of the agents.

In the last part, the effect of bounded communication delay on a system is studied and a theorem is presented which gives the stability conditions in communication delayed-networks.

6.1. Future Work

There are extensions of this thesis that are interesting and challenging for future studies. These are sorted as follows:

- The results presented in the thesis can be applied to systems using second or higher order dynamics.
- It is interesting to apply the proposed algorithm to real systems that are nonlinear.
- The result on delayed system can be extended to cases where input and communication delays are time-varying.
- It is worthwhile to a design controller that allows clusters of a given network to attain desired final states.
- The work presented in this thesis can be generalized to networks under switching topologies.

REFERENCES

1. Olfati-Saber, R., J. A. Fax and R. M. Murray, "Consensus and cooperation in networked multi-agent systems", *Proceedings of the IEEE*, Vol. 95, No. 1, pp. 215–233, 2007.

2. Ren, W. and R. W. Beard, "Consensus seeking in multiagent systems under dynamically changing interaction topologies", *IEEE Transactions on Automatic Control*, Vol. 50, No. 5, pp. 655–661, 2005.

3. Jadbabaie, A., J. Lin and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules", *IEEE Transactions on Automatic Control*, Vol. 48, No. 6, pp. 988–1001, 2003.

4. Akar, M. and R. Shorten, "Distributed probabilistic synchronization algorithms for communication networks", *IEEE Transactions on Automatic Control*, Vol. 53, No. 1, pp. 389–393, 2008.

5. Wang, A. and Y. Zhao, "Event-triggered consensus control for leader-following multi-agent systems with time-varying delays", *Journal of the Franklin Institute*, Vol. 353, No. 17, pp. 4754–4771, 2016.

6. Hu, H.-x., L. Yu, W.-A. Zhang and H. Song, "Group consensus in multi-agent systems with hybrid protocol", *Journal of the Franklin Institute*, Vol. 350, No. 3, pp. 575–597, 2013.

7. Chen, Y., J. Lü, F. Han and X. Yu, "On the cluster consensus of discrete-time multi-agent systems", *Systems & Control Letters*, Vol. 60, No. 7, pp. 517–523, 2011.

8. Han, Y., W. Lu and T. Chen, "Cluster consensus in discrete-time networks of multiagents with inter-cluster nonidentical inputs", *IEEE Transactions on Neural*

Networks and Learning systems, Vol. 24, No. 4, pp. 566–578, 2013.

9. Li, J., Z.-H. Guan and G. Chen, “Multi-consensus of nonlinearly networked multi-agent systems”, *Asian Journal of Control*, Vol. 17, No. 1, pp. 157–164, 2015.

10. Yu, J. and L. Wang, “Group consensus of multi-agent systems with directed information exchange”, *International Journal of Systems Science*, Vol. 43, No. 2, pp. 334–348, 2012.

11. Yu, J. and L. Wang, “Group consensus in multi-agent systems with switching topologies and communication delays”, *Systems & Control Letters*, Vol. 59, No. 6, pp. 340–348, 2010.

12. Shang, Y. and Y. Ye, “Fixed-time group tracking control with unknown inherent nonlinear dynamics”, *IEEE Access*, Vol. 5, pp. 12833–12842, 2017.

13. Shang, Y. and Y. Ye, “Leader-follower fixed-time group consensus control of multiagent systems under directed topology”, *Complexity*, Vol. 2017, 2017.

14. Xiao, F. and L. Wang, “State consensus for multi-agent systems with switching topologies and time-varying delays”, *International Journal of Control*, Vol. 79, No. 10, pp. 1277–1284, 2006.

15. Lin, P., K. Qin, H. Zhao and M. Sun, “A new approach to average consensus problems with multiple time-delays and jointly-connected topologies”, *Journal of the Franklin Institute*, Vol. 349, No. 1, pp. 293–304, 2012.

16. Sun, Y., “Average consensus in networks of dynamic agents with uncertain topologies and time-varying delays”, *Journal of the Franklin Institute*, Vol. 349, No. 3, pp. 1061–1073, 2012.

17. Olfati-Saber, R. and R. M. Murray, “Consensus problems in networks of agents with switching topology and time-delays”, *IEEE Transactions on Automatic Con-*

trol, Vol. 49, No. 9, pp. 1520–1533, 2004.

18. Lin, P. and Y. Jia, "Average consensus in networks of multi-agents with both switching topology and coupling time-delay", *Physica A: Statistical Mechanics and its Applications*, Vol. 387, No. 1, pp. 303–313, 2008.

19. Chen, S., C. Liu and F. Liu, "Delay effect on group consensus seeking of second-order multi-agent systems", *Proceedings of the 29th Chinese Control And Decision Conference*, pp. 1190–1195, 2017.

20. Hu, J. and Y. Hong, "Leader-following coordination of multi-agent systems with coupling time delays", *Physica A: Statistical Mechanics and its Applications*, Vol. 374, No. 2, pp. 853–863, 2007.

21. Lu, J., D. W. Ho and J. Kurths, "Consensus over directed static networks with arbitrary finite communication delays", *Physical Review E*, Vol. 80, No. 6, p. 066121, 2009.

22. Erkan, Ö. F., O. Cihan and M. Akar, "Analysis of distributed consensus protocols with multi-equilibria under time-delays", *Journal of the Franklin Institute*, Vol. 355, No. 1, pp. 332–360, 2018.

23. Develer, Ü. and M. Akar, "Analysis of cluster consensus in continuous-time networks", *Proceedings of the 2018 American Control Conference*, pp. 448–453, 2018.

24. Develer, Ü. and M. Akar, "Input delay effect on cluster consensus in continuous-time networks", *to appear in Proceedings of the 14th IFAC Workshop on Time Delay Systems*, 2018.

25. Develer, Ü. and M. Akar, "Cluster consensus in continuous-time networks with input and communication delays", *submitted to Journal of the Franklin Institute*, 2018.

26. Shang, Y., “On consensus of nonlinear multi-agent systems under directed interaction topologies”, *International Conference on Cyber-Enabled Distributed Computing and Knowledge Discovery*, pp. 128–131, 2009.

27. Gershgorin, S. A., “Uber die abgrenzung der eigenwerte einer matrix”, *Izv. Akad. Nauk SSSR Ser. Fiz.-Mat.*, No. 6, pp. 749–754, 1931.

28. Tutte, W. T., *"The matrix-tree theorem" in Graph theory*, Addison-Wesley Publishing Company Advanced Book Program, Reading, MA, 1984.

29. Niculescu, S.-I., *"Neutral model transformations" in Delay effects on stability: a robust control approach*, Vol. 269, Springer Science & Business Media, 2001.

30. Gu, K., J. Chen and V. L. Kharitonov, *"Frequency sweeping tests" in Stability of time-delay systems*, Springer Science & Business Media, 2003.

31. Seuret, A., D. V. Dimarogonas and K. H. Johansson, “Consensus under communication delays”, *Proceedings of the 47th IEEE Conference on Decision and Control*, pp. 4922–4927, 2008.