NORMAL FORM APPROACH TO SOME CHAOTIC SPROTT SYSTEMS

by

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ABSTRACT

NORMAL FORM APPROACH TO SOME CHAOTIC SPROTT SYSTEMS

In this study, the normal form method which had been previously applied to three of the Sprott systems is applied to three other Sprott systems with a different eigenvalue structure of their linearized parts. It is seen that the normal form expansion can also represent these systems successfully for a longer time than expected from a perturbative method. Altough the normal form expansion is a local method it gives information about nonlocal characteristics suh as possible zero Liapunov exponents of the system via approximately conserved quantities.

ÖZET

BAZI DÜZENSİZ SPROTT SİSTEMLERİNE NORMAL FORM YAKLAŞIMI

Bu çalışmada, daha önce üç Sprott sistemine uygulanan normal form yöntemi lineerleştirilmiş bölümü farklı özdeğer yapısına sahip diğer üç Sprott sistemine de uygulanmıştır. Normal form açılımlarının bu dinamik sistemleri bir pertürbatif metottan beklenen süreden cok daha uzun süre için temsil edebildiği görülmüştür. Normal form metodu yerel bir metod olmasına rağmen yaklaşık korunan büyüklükler aracılığı ile olası sıfır Liapunov üsteli gibi sistemin yerel olmayan bir özelliği hakkında da bilgi verebilmiştir.

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LIST OF SYMBOLS/ABBREVIATIONS

A	tangent flow matrix
$\widetilde{A}$	complex diagonal tangent flow matrix
D	derivative operator or complex matrix used to transform the dynamical system into complex diagonal form
G_n	the null space
F_i	components of Taylor expansion to order n
$\widetilde{F}$	nonlinear part of dynamical system which is modified as a result of coordinate transformation
h_i	generating functions of coordinate transformation
J	Jordan canonical form
n	system dimension
T	transformation matrix
T_i^j	generating functions of near identity transformation
u	state vector of dimension n for normal form system
U_i	components of the normal from equation at ith order
w	z component in cylindrical coordinates
x	state vector of dimension n
x_i	dynamical variable
$\bar{x}_i$	dynamical variable with fixed point shifted to the origin
x_0	fixed points
y	state vector of dimension n for transformed system

α	imaginary component of a complex eigenvalue
$\alpha_{i,jkm}$	coefficient of the nonlinear part of the ith component of the dynamical system
β	real component of a complex eigenvalue
$\beta_{a,jkm}$	coefficient of the homogeneous monomials of ath component of T functions
ε	scaling parameter
ζ	complex state vector of dimension n
λ_i	ith Liapunov exponent or eigenvalue
ρ	radial component in cylindirical coordinates
φ	angular component in cylindirical coordinates

nf	normal form

INTRODUCTION

Nineteenth century physics was largely based on linear theories and models. As more and more complicated phenomena were discovered, linear theories became insufficient for a complete description. In the latter part of the twentieth century, chaotic phenomena that could not be completely explained using linear theories have gained importance in every area of science. Chaotic systems are nonlinear systems whose behavior sensitively depends on initial conditions [1,2]. They have at least one positive Liapunov exponent.

There are many numerical algorithms, used for calculation of Liapunov exponents [3,4,5]. However, their results are affected by rounding error introduced by computers. Hence, distinguishing between the ambiguity of having a small positive, a small negative or a zero exponent is crucial when determining the characteristics of a system.

The most common examples of first-order ordinary differential equations which have chaotic solutions are the Lorenz and Rössler systems. Sprott[6] made a systematic investigation on classes of autonomous ordinary differential equations and found 19 simple three dimensional autonomous systems with quadratic nonlinearities which have chaotic solutions.

The aim of this study is to apply the "Normal Form Method" to chaotic Sprott systems to gain information about characteristics of the systems [7]. This method converts the dynamical system to a simpler form by applying coordinate transformations in terms of infinite series. In practice one can expand the series up to finite terms. One of our purposes is to show that normal form equations represent dynamical system for longer times as one includes more terms of the series. Our other purpose is to investigate approximately conserved quantities which can be seen as signs of zero Liapunov exponents [7,8].

The method of normal forms which is used in this study is explained in section 2. The Sprott systems, chaotic attractors are presented in section 3. Results of the application of the normal form method to three Sprott systems and the results are given in section 4. In section 5, the importance of an approximately conserved quantity is studied and this quantity is searched in three Sprott systems. Conclusions of this study are given in section 6. Certain technical details concerning the calculation are presented in an Appendix A.

2. THE METHOD OF NORMAL FORMS

The method of normal forms [9,10,11] provides approximate solutions to systems of difference or ordinary differential equations that represent dynamical systems in $\Re^n$. The purpose is to transform the differential equations into a simpler form. A coordinate transformation (involving dependent variables) in the form of infinite series is generated so that the resulting transformed differential equations are in a normal (i.e, a simpler form using an expansion involving polynomial terms that simulates the chaotic stretching and folding plus a dynamical system which is simpler to study) form. Using this requirement coefficients in the series expansion are calculated.

The method is local in the sense that the coordinate transformations are generated in a neighborhood of a known solution which is a fixed point. In general, the coordinate transformations are nonlinear functions of the dependent variables, and these coordinate transformations are found by solving a sequence of linear problems. Lastly, in an appropriate space, the linear part of the vector field determines the structure of the normal form.

2.1. Normal Forms For Vector Fields

We are going to study the vector field

$$\dot{x} = G(x) \qquad , x \in \Re^n \ , \quad G \in \Re^n \to \Re^n \tag{2.1.1}$$

whose fixed point $x = x_0$ is given by G(x_0) = 0. Our purpose is to transform (2.1.1) into such a form that is easier to work, so we apply following steps

- Initially we move the fixed point to the origin by letting

$$\bar{x} = x - x_0 \qquad , \qquad \bar{x} \in \Re^n \tag{2.1.2}$$

so (2.1.1) becomes

$$\dot{\bar{x}} = G(\bar{x} + x_0) \equiv H(\bar{x}) \tag{2.1.3}$$

- Then we separate the linear part of the vector field

$$\dot{\bar{x}} = DH(0)\bar{x} + \overline{H}(\bar{x}) \tag{2.1.4}$$

where

$$\overline{H}(\bar{x}) \equiv H(\bar{x}) - DH(0)\bar{x} \tag{2.1.5}$$

- As a last step we find the matrix T that transforms the matrix $DH(0)$ into (real) Jordan canonical form. Thus

$$\bar{x} = T\tilde{x} \qquad \tilde{x} \in \Re^n \tag{2.1.6}$$

modifes (2.1.3) in the following way

$$\dot{\tilde{x}} = T^{-1}DH(0)T\tilde{x} + T^{-1}\overline{H}(T\tilde{x}) \tag{2.1.7}$$

Using definitions

$$J \equiv T^{-1}DH(0)T \tag{2.1.8}$$

and

$$F(\tilde{x}) = T^{-1}\overline{H}(T\tilde{x}) \tag{2.1.9}$$

in (2.1.7), we obtain

$$\dot{\tilde{x}} = J\tilde{x} + F(\tilde{x}), \tag{2.1.10}$$

The linearized part is diagonalized wherever possible. The Jordan canonical form is used in cases where this diagonalization is not possible.

Complex conjugate eigenvalue pairs can either be expressed in diagonal form using complex variables or in Jordan canonical form using real variables. We will call the use of diagonal complex representation wherever possible the standard form.

At this point, the linear part has been simplified as much as possible. We now proceed to transform the nonlinear part $F(\tilde{x})$ to a simpler form. By Taylor expansion of $F(\tilde{x})$ we obtain

$$\dot{\tilde{x}} = J\tilde{x} + F_2(\tilde{x}) + F_3(\tilde{x}) + \ldots + F_{r-1}(\tilde{x}) + O(|\tilde{x}|^r) \tag{2.1.11}$$

where $F_i(\tilde{x})$ is the order i terms containing homogeneous monomial terms of degree i in the Taylor expansion of $F(\tilde{x})$. Then the coordinate transformation

$$\tilde{x} = y + h_2(y) \qquad , \qquad y \in \Re^n \tag{2.1.12}$$

where $h_2(y)$ is a second degree, homogeneous polynomial in $\{y_i\}$, is first applied. Substituting (2.1.12) in (2.1.11) we have

$$\begin{aligned} \dot{\tilde{x}} &= (I + Dh_2(y))\dot{y} = Jy + Jh_2(y) + F_2(y + h_2(y)) + F_3(y + h_2(y)) + \\ &\ldots + F_{r-1}(y + h_2(y)) + O(|y|^r) \end{aligned} \tag{2.1.13}$$

Using the following expansion

$$F_k(y + h_2(y)) = F_k(y) + O(|y|^{k+1}) + \ldots + O(|y|^{2k}), \qquad 2 \le k \le r-1 \tag{2.1.14}$$

in (2.1.13) we obtain

$$(I + Dh_2(y))\dot{y} = Jy + Jh_2(y) + F_2(y) + \overline{F}_3(y) + ... + \widetilde{F}_{r-1}(y) + O(|y|^r) \qquad (2.1.15)$$

We also have

$$(I + Dh_2(y))^{-1} = I - Dh_2(y) + O(|y|^2) \qquad (2.1.16)$$

where y is sufficiently small.

Using this equation in (2.1.15) we get

$$\dot{y} = Jy + Jh_2(y) - Dh_2(y)Jy + F_2(y) + \widetilde{F}_3(y) + ... + \widetilde{F}_{r-1}(y) + O(|y|^r) \quad (2.1.17)$$

We remark that our aim is to simplify the $O(|y|^2)$ terms as much as possible so $h_2(y)$ should satisfy

$$Dh_2(y)Jy - Jh_2(y) = F_2(y) \qquad (2.1.18)$$

to remove $F_2(y)$ in (2.1.17).

We now define an appropriate linear vector space, and a linear operator on this space. Thus, equation (2.1.18), viewed as a transformation from $h_2(y)$ to $F_2(y)$, becomes a linear transformation in this vector space.

The definition of the appropriate linear vector space, the linear operator on this vector space and the equation to be solved in this linear vector space are obtained by applying the following procedure:

1. Let $\{s_1,...,s_n\}$ be a basis of $\Re^n$, and let $y = \{y_1,..., y_n\}$ be coordinates with respect to this basis. Then we define "vector-valued homogeneous polynomials of degree

k” as basis elements with coefficients consisting of homogeneous polynomials of degree k, i.e.,

$$(y_1^{m_1} y_2^{m_2} \ldots y_n^{m_n}) s_i, \quad \sum_{j=1}^{n} m_j = k,\ m_j \in Z^+ \tag{2.1.19}$$

A linear vector space H_k consists of the set of all vector-valued homogeneous polynomials of degree k.

2. Then $h_2(y)$ can be seen as a component of H_2. Let us examine the operator

$$Dh_2(y)Jy - Jh_2(y)\,. \tag{2.1.20}$$

The second term involves the multiplication of $h_2(y)$ with J, this is a linear transformation on s_i and can be written as a linear superposition of $\{s_i\}$. The first term involves the multiplication of the Jacobian of h_2, J_y and s_i, which is a superposition of terms of the form (2.1.19). Thus, the operation can be written as a linear transformation in H_2. The map

$$h_k(y) \to Dh_k(y)Jy - Jh_k(y) \tag{2.1.21}$$

is a linear map of H_k into H_k. In Lie algebra notation, the Lie bracket operation on the vector fields $h_k(y)$ and Jy can be defined as

$$L_J^{(k)}(h_k(y)) \equiv -(Dh_k(y)Jy - Jh_k(y))$$

$$L_J^{(k)} h_k \equiv \left[h_k(y), Jy\right]\,. \tag{2.1.22}$$

3. Now $F_2(y)$ can be considered as an element of H_2. In elementary linear algebra H_2 can be written as

$$H_2 = L_j^{(2)}(H_2) \oplus G_{2,} \tag{2.1.23}$$

where G_2 is a space complementary to $L_j^{(2)}(H_2)$. When $F_2(y)$ is in the range of $L_j^{(2)}(\bullet)$, all $O(|y|^2)$ terms can be removed from (2.1.18). In any case, $h_2(y)$ can be chosen so that only $O(|y|^2)$ terms that are in G_2 remain. These terms are called "resonance" terms;

$$F_2^r(y) \in G_2 \tag{2.1.24}$$

Using these ideas let's write (2.1.17) again

$$\dot{y} = Jy + F_2^r(y) + \widetilde{F}_3(y) + ... + \widetilde{F}_{r-1}(y) + O(|y|^r). \tag{2.1.25}$$

The next step "Simplification of the Third Order Terms" is similar to the previous simplification. We introduce the coordinate transformation

$$y \to y + h_3(y), \qquad h_3(y) = O(|y|^3)\,. \tag{2.1.26}$$

Then following the same steps as in the second-order terms (2.1.25) modifies as

$$\dot{y} = Jy + F_2^r(y) + Jh_3(y) - Dh_3(y)Jy + \widetilde{F}_3(y) + \widetilde{\widetilde{F}}_4(y) + ... + \widetilde{\widetilde{F}}_{r-1}(y) + O(|y|^r). \tag{2.1.27}$$

Next we have the requirement

$$Dh_3(y)Jy - Jh_3(y) = \widetilde{F}_3(y) \tag{2.1.28}$$

to determine $h_3(y)$ that transforms the third-order terms into a simpler form. Then it is clear that

$$h_3(y) \to Dh_3(y)Jy - Jh_3(y) \equiv -L_j^{(3)}(h_3(y)) \qquad (2.1.29)$$

is a linear map of H_3 into itself. Hence, the third-order terms can be reduced to

$$F_3^r(y) \in G_3 \qquad (2.1.30)$$

where

$$H_3 = L_j^{(3)}(H_3) \oplus G_3. \qquad (2.1.31)$$

When $L_j^{(3)}(H_3) = H_3$ the third order terms can be removed.

2.1.1. The Normal Form Theorem

Quoting from the definition in Wiggins[9], we now state the Poincaré-Dulac normal form theorem which includes the resonant terms. By a sequence of near identity coordinate changes, (2.1.11) can be transformed into

$$\dot{y} = F_2^r(y) + ... + \widetilde{F}_{r-1}(y) + O(|y|^r), \qquad (2.1.1.1)$$

where $F_k^r(y) \in G_k$, $2 \le k \le r-1$, and G_k is a space complementary to $L_j^{(k)}(H_k)$. Equation (2.1.1.1) is said to be in normal form through order $r-1$. Several comments are now in order

1. The terms $F_k^r(y)$, $2 \le k \le r-1$, are referred to as resonance terms. Let the linear part of the vector field have eigenvalues λ_i, then the resonance equation is

$$\lambda_i = \sum_{j=1}^{n} m_j \lambda_j, \quad m_j \ge 0 \text{ are integers}, \sum m_j \ge 2. \qquad (2.1.1.2)$$

The order of resonance is the integer, $\sum_{j=1}^{n} m_j$.

2. The structure of the nonlinear terms in (2.1.1.1) is absolutely determined by J, the linear part of the vector field.

3. It should be clear that simplifying the terms at order k does not modify any lower order terms. However, terms of order higher then k are modified. This happens at each step of the application of the method. If one wanted to actually calculate the coefficients on each term of the normal form in terms of the original vector field, it would be necessary to keep track of how the higher order terms are modified by successive coordinate transformations.

2.1.2 Calculation of the Linear Part

If all of the eigenvalues of the linearized part can be placed to one side of a straight line through the origin, the eigenvalues are said to be in a "Poincare domain". Otherwise, they lie in a "Siegel domain". A finite number of resonance relations can be seen in the Poincare domain, whereas an infinite number of resonant conditions occur in the Siegel domain.

We will concentrate our studies on three dimensional autonomous systems that show chaotic behavior. Sprott [6] has proposed a set of nineteen such so called simplest systems. Most of them have a linear part whose eigenvalues consist of a complex conjugate pair and a real eigenvalue. Thus they are in the Siegel domain.

The first step will consist of converting the linear part in to a standart form[7]. The eigenvalues will be of the form of a complex pair $\lambda_1, \lambda_2 = \lambda_1^*$ and $\lambda_3 = \lambda_3^*$. The eigenvectors will be of the form $\mathbf{v_1} \pm i\mathbf{v_2}$ and $\mathbf{v_3}$. We can now construct the invertible transformation matrix

$$T = [\, \mathbf{v_2}, \mathbf{v_1}, \mathbf{v_3} \,] \tag{2.1.1.3}$$

which will convert the linearized part to the Jordan canonical form, if the system is diagonalizable.

$$T^{-1}AT = \begin{bmatrix} \mathrm{Re}(\lambda_1) & -\mathrm{Im}(\lambda_1) & 0 \\ \mathrm{Im}(\lambda_1) & \mathrm{Re}(\lambda_1) & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{2.1.1.4}$$

A second transformation will convert the linearized part into the complex diagonal (standard) form.

$$D^{-1}T^{-1}ATD = \widetilde{A} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{2.1.1.5}$$

where

$$D = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ -i & i & 0 \\ 0 & 0 & 2 \end{bmatrix}, \qquad D^{-1} = \begin{bmatrix} 1 & i & 0 \\ 1 & -i & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.1.1.6}$$

$$\dot{\zeta} = \widetilde{A}\zeta + \widetilde{F}(\zeta) \tag{2.1.1.7}$$

where

$$\zeta = D^{-1}T^{-1}\overline{x} \tag{2.1.1.8}$$

and the nonlinear part is

$$\widetilde{F} = D^{-1}T^{-1}F \tag{2.1.1.9}$$

2.2. The Near Identity Transformation

A slightly modified notation than that of the last subsection will be used here. Following Zarmi [13], we split the ordinary nonlinear, autonomous differential equation into a linear part and a nonlinear part so it can be written as

$$\frac{d}{dt}x = Ax + \varepsilon F(\varepsilon, x) \tag{2.2.1}$$

where A is a $n \times n$ constant matrix of either diagonal or Jordan canonical form. The last term is the perturbation with $|\varepsilon| << 1$ and F is an n-dimensional vector field whose terms are polynomials or functions with Taylor series expansion in x of degree at least one.

For application let's rewrite our system as a three-dimensional nonlinear system

$$\frac{dx_i}{dt} = \lambda_i x_i + \sum_{n=1}^{N} \varepsilon^n \sum_{j,k,m} \alpha^n_{i,jkm} x_1^j x_2^k x_3^m \tag{2.2.2}$$

where $i = 1,2,3, \quad j + k + m \geq 2, \quad |\varepsilon| << 1, \quad \lambda_1 \neq \lambda_2 \neq \lambda_3$

A "near identity transformation (NIT)" [12,13] is written as

$$x_i = u_i + \sum_n \varepsilon^n T_n^i(u_1, u_2, u_3) \tag{2.2.3}$$

where εT_n correspond to the h_n 's

This transforms the variables (x_i) to new set (u_i). The task of the functions T is to eliminate as many terms in (2.2.2) as possible. The new variables obey the following relation

$$\frac{d}{dt}u_i = U_0^i + \sum_n \varepsilon^n U_n^i, \qquad U_0^i = \lambda_i u_i \tag{2.2.4}$$

Let’s expand the equation in powers of ε

$$\frac{dx_1}{dt} = \lambda_1 x_1 + \sum_{n=1}^{N} \varepsilon^n \sum_{j,k,m} \alpha_{1,jkm}^n x_1^j x_2^k x_3^m, \quad j+k+m \geq 2, \quad j,k,m \geq 0 \tag{2.2.5}$$

and perform a parallel expansion

$$x_1 = u_1 + \varepsilon T_1^1 + \varepsilon^2 T_2^1 \tag{2.2.6}$$

we can now expand equation (2.2.4)

$$\frac{d}{dt}u_1 = U_0^1 + \varepsilon U_1^1 + \varepsilon^2 U_2^1 \tag{2.2.7}$$

In (2.2.5) subscripts refer to components of the vectors. In (2.2.6) and (2.2.7) components are shown by superscripts and the order of the perturbation is shown by subscripts.

Using (2.2.6) and (2.2.7) in (2.2.5) at the left-hand side we have;

$$\frac{dx_1}{dt} = U_0^1 + \varepsilon U_1^1 + \varepsilon^2 U_2^1 + \varepsilon (U_0^1 \frac{\partial T_1}{\partial u_1} + U_0^2 \frac{\partial T_1^1}{\partial u_2} + U_0^3 \frac{\partial T_1^1}{\partial u_3})$$

$$+ \varepsilon^2 (U_0^1 \frac{\partial T_2^1}{\partial u_1} + U_0^2 \frac{\partial T_2^1}{\partial u_2} + U_0^3 \frac{\partial T_2^1}{\partial u_3}) \tag{2.2.8}$$

$$+ \varepsilon^2 (U_1^1 \frac{\partial T_1^1}{\partial u_1} + U_1^2 \frac{\partial T_1^1}{\partial u_2} + U_1^3 \frac{\partial T_1^1}{\partial u_3})$$

and on the right- hand side we have;

$$\lambda_1(u_1+\varepsilon T_1^1+\varepsilon^2 T_2^1)+\varepsilon\sum_{j+k+m\geq 2}\alpha_{1,jkm}^1 u_1^j u_2^k u_3^m$$

$$+\varepsilon^2\sum_{j+k+m\geq 2}\alpha_{1,jkm}^1\{jT_1^1 u_1^{j-1}u_2^k u_3^m+ku_1^j T_1^2 u_2^{k-1}u_3^m+mu_1^j u_2^k u_3^{m-1}T_1^3\} \tag{2.2.9}$$

$$+\varepsilon^2\sum_{j+k+m\geq 2}\alpha_{1,jkm}^1 u_1^j u_2^k u_3^m$$

taking the difference of (2.2.8) and (2.2.9) and collecting the coefficients of ε order by order we get

$$\varepsilon^0;\quad U_0^1=\lambda_1 u_1 \tag{2.2.10}$$

$$\varepsilon^1;\quad U_1^1=\lambda_1 T_1^1-(U_0^1\frac{\partial T_1^1}{\partial u_1}+U_0^2\frac{\partial T_1^1}{\partial u_2}+U_0^3\frac{\partial T_1^1}{\partial u_3})+\sum_{j+k+m\geq 2}a_{1,jkm}^1 u_1^j u_2^k u_3^m \tag{2.2.11}$$

Here the "Lie bracket" (LB) is defined as

$$[U_0,T_1]^1\equiv\sum_{i=1}^{3}(T_1^i\frac{\partial U_0^1}{\partial u_i}-U_0^i\frac{\partial T_1^1}{\partial u_i}) \tag{2.2.12}$$

Substituting (2.2.10) in(2.2.12) we have

$$[U_0,T_1]^1=\lambda_1 T_1^1-(U_0^1\frac{\partial T_1^1}{\partial u_1}+U_0^2\frac{\partial T_1^1}{\partial u_2}+U_0^3\frac{\partial T_1^1}{\partial u_3}) \tag{2.2.13}$$

Thus equation (2.2.11) becomes

$$U_1^1=[U_0,T_1]^1+\sum_{j+k+m\geq 2}\alpha_{1,jkm}^1 u_1^j u_2^k u_3^m \tag{2.2.14}$$

The unperturbed linear part of the initial differential equation gives the zero-order term U_0^1. The superscript of the square bracket refers to the component of each term.

In general, we construct the functions T_i^a as a polynomial whose terms are homogeneous monomials of order i, the superscript a indicates the component :

$$T_i^a = \sum_{j+k+m\geq 2} \beta_{a,jkm} u^j v^k w^m \quad , \quad a = 1,2,...n \tag{2.2.15}$$

The Lie bracket can easily be calculated. For example:

$$[U_0, T_1]^1 = \sum_{j+k+m\geq 2} \beta_{jkm}^1 u_1^j u_2^k u_3^m \{\lambda_1 - (\lambda_1 j + \lambda_2 k + \lambda_3 m)\} \tag{2.2.16}$$

Thus

$$U_1^1 = \sum_{j+k+m\geq 2} \beta_{jkm}^1 u_1^j u_2^k u_3^m \{\lambda_1 - (\lambda_1 j + \lambda_2 k + \lambda_3 m)\} + \sum_{j+k+m\geq 2} \alpha_{1,j,k,m}^1 u_1^j u_2^k u_3^m \tag{2.2.17}$$

We remark that our purpose is to simplify the 2nd order terms as much as possible. Hence we choose T_1^1 such that it eliminates U_1^1. However we know from the normal form theorem this can not always be possible. Indeed the last equation imply that this elimination fails for terms such that

$$\lambda_1 = j\lambda_1 + k\lambda_2 + m\lambda_3 \tag{2.2.18}$$

where λ ' s are the eigenvalues of the linearized part, j,k,m are positive integers. This is our “resonance relation”. For cases there is no resonance, we remove the terms $u_1^j u_2^k u_3^m$ by choosing

$$\beta^{1}_{jkm} = -\frac{\alpha^{1}_{1\,jkm}}{\{\lambda_1 - (\lambda_1 j + \lambda_2 k + \lambda_3 m)\}} \qquad (2.2.19)$$

Thus normal form (2.2.4) consists of the remaining terms which occur at the resonance level.

3. SPROTT SYSTEMS

It is well known that so-called chaotic systems reprersented by ordinary differential equations are either two dimensional non-autonomous or three dimensional autonomous system. The Poincare-Bendixson theorem[1] is usually quoted in this connection, although there is a more heuristic way to see this. A two dimensional autonomous system, $\frac{dx_1}{dt} = f_1(x_1, x_2), \frac{dx_2}{dt} = f_2(x_1, x_2)$ can be rewritten as the first order system $f_2(x_1, x_2)dx_1 - f_1(x_1, x_2)dx_2$ which always posesses an integrating factor. Hence, either a two dimensional nonautonomous system or a three dimensional autonomous system of ordinary differential equations are needed. Exponential divergence of nearby trajectories also requires the presence of at least one nonlinearity [6]. Most common examples are Lorenz and Rössler systems. The Lorenz equations are

$$\dot{x} = -\sigma x + \sigma y \tag{3.1}$$

$$\dot{y} = -xz + rx - y \tag{3.2}$$

$$\dot{z} = xy - bz \tag{3.3}$$

have seven terms on the right hand side two of them are quadratic nonlinear. The Rössler equations

$$\dot{x} = -y - z \tag{3.4}$$

$$\dot{y} = x + ay \tag{3.5}$$

$$\dot{z} = b + xz - cz \tag{3.6}$$

have one quadratic nonlinear term and seven terms on the right hand side. Sprott investigated the existence of systems of autonomous ordinary differential equations with one or two quadratic nonlinearities and fewer than seven terms which have chaotic solution. He found several thousand cases. Only 19 of these systems are independent in the sense that there is no obvious transformation of one to another, all others can be transfomed to one of these. These equations consist of six or fewer terms with at most two independent control parameters.

He also found other systems with six terms and two nonlinearities but they appear to be more complicated than the previous ones. System A is a volume-conserving system, and the others are dissipative systems with strange attractors. In Table (3.1) these 19 systems with their fixed points, eigenvalues of the the linearized Jacobian matrix and Liapunov exponents, as quoted by Sprott, [6] are given.

The Liapunov exponents have been obtained by numeric calculations. For chaotic behavior the largest exponent is positive; this is seen in the Sprott systems. Using the special case of the Kaplan-Yorke relation

$$D_L = 2 - \frac{\lambda_1}{\lambda_3} \tag{3.7}$$

(λ_i donetes Liapunov exponent) the Liapunov dimensions are also calculated for each system. Except for system A all of the Liapunov dimensions are between 2.00 and 2.19.

Table 3.1. Sprott Systems

Case	System	Fixed Points	Liapunov Exponents	Eigenvalues of A
A	$\dot{x}_1 = x_2$ $\dot{x}_2 = -x_1 + x_2 x_3$ $\dot{x}_3 = 1 - x_2^2$	none	0,±0.014	(0,±i)
B	$\dot{x}_1 = x_2 x_3$ $\dot{x}_2 = x_1 - x_2$ $\dot{x}_3 = 1 - x_1 x_2$	(±1, ±1,0)	0.210,0,-1.210	(-1.35,0.18±1.20i)
C	$\dot{x}_1 = x_2 x_3$ $\dot{x}_2 = x_1 - x_2$ $\dot{x}_3 = 1 - x_1^2$	(±1, ±1,0)	0.163,0,-1.163	(-1,±1.41i)
D	$\dot{x}_1 = -x_2$ $\dot{x}_2 = x_1 + x_3$ $\dot{x}_3 = x_1 x_3 + 3x_2^2$	(0,0,0)	0.103,0,-1.320	(0,±i)
E	$\dot{x}_1 = x_2 x_3$ $\dot{x}_2 = x_1{}^2 - x_2$ $\dot{x}_3 = 1 - 4x_1$	(0.25,0.063,0)	0.078,0,-1.078	(-1,±0.5i)
F	$\dot{x}_1 = x_2 + x_3$ $\dot{x}_2 = -x_1 + 0.5x_2$ $\dot{x}_3 = x_1^2 - x_3$	(0,0,0) (-2,-4,4)	0.117,0,-0.617	(-1,0.25±0.97i) (0.21,-0.35±2.12i)
G	$\dot{x}_1 = 0.4x_1 + x_3$ $\dot{x}_2 = x_1 x_3 - x_2$ $\dot{x}_3 = -x_1 + x_2$	(0,0,0) (-2.5,-2.5,1)	0.034,0,-0.634	(-1,0.20±0.97i) (0.29,-0.44±1.78i)
H	$\dot{x}_1 = -x_2 + x_3^2$ $\dot{x}_2 = x_1 + 0.5x_2$ $\dot{x}_3 = x_1 - x_3$	(0,0,0) (-2,4,-2)	0.117,0,-0.617	(-1,0.25±0.97i) (-2.50,0.96±0.54i)
I	$\dot{x}_1 = -0.2x_2$ $\dot{x}_2 = x_1 + x_3$ $\dot{x}_3 = x_1 + x_2^2 - x_3$	(0,0,0)	0.012,0,-1.012	(0.57,-0.78±0.28i)

Case	System	Fixed Points	Liapunov Exponents	Eigenvalues of A
J	$\dot{x}_1 = 2x_3$ $\dot{x}_2 = -2x_2 + x_3$ $\dot{x}_3 = -x_1 + x_2 + x_2^2$	(0,0,0)	0.076,0,-2.076	(-2.31,0.16±1.31i)
K	$\dot{x}_1 = x_1 x_2 - x_3$ $\dot{x}_2 = x_1 - x_2$ $\dot{x}_3 = x_1 + 0.3x_3$	(0,0,0) (-3.33,-3.33,11.11)	0.038,0,-0.890	(-1,0.15±0.99i) (0.15,-2.09±1.61i)
L	$\dot{x}_1 = x_2 + 3.9x_3$ $\dot{x}_2 = 0.9{x_1}^2 - x_2$ $\dot{x}_3 = 1 - x_1$	(1,0.9,0.23)	0.061,0,-1.061	(-1.43,0.22±1.64i)
M	$\dot{x}_1 = -x_3$ $\dot{x}_2 = -x_1^2 - x_2$ $\dot{x}_3 = 1.7 + 1.7x_1 + x_2$	(2.406,-5.791,0) (-0.706,-0.499,0)	0.044,0,-1.044	(-1.38,0.19±1.48i) (0.91,-0.95±1.59i)
N	$\dot{x}_1 = -2x_2$ $\dot{x}_2 = x_1 + {x_3}^2$ $\dot{x}_3 = 1 + x_2 - 2x_3$	(-0.25,0,0.5)	0.076,0,-2.076	(-2.31,0.16±1.31i)
O	$\dot{x}_1 = x_2$ $\dot{x}_2 = x_1 - x_3$ $\dot{x}_3 = x_1 + x_1 x_3 + 2.7x_2$	(0,0,0) (-1,0,-1)	0.049,0,-0.319	(-0.51,0.39±1.15i) (0.29,-0.65±1.46i)
P	$\dot{x}_1 = 2.7x_2 + x_3$ $\dot{x}_2 = -x_1 + x_2^2$ $\dot{x}_3 = x_1 + x_2$	(0,0,0) (1,-1,2.7)	0.087,0,-0.481	(-0.51,0.39±1.15i) (0.27,-1.13±1.19i)
Q	$\dot{x}_1 = -x_3$ $\dot{x}_2 = x_1 - x_2$ $\dot{x}_3 = 3.1x_1 + x_2^2 + 0.5x_3$	(0,0,0) (-3.1,-3.1,0)	0.109,0,-0.609	(-1,0.25±1.74i) (0.83,-0.66±1.81i)
R	$\dot{x}_1 = 0.9 - x_2$ $\dot{x}_2 = 0.4 + x_3$ $\dot{x}_3 = x_1 x_2 - x_3$	(-0.44,0.90,-0.40)	0.062,0,-1.062	(-1.23,0.12±0.85i)
S	$\dot{x}_1 = -x_1 - 4x_2$ $\dot{x}_2 = x_1 + x_3^2$ $\dot{x}_3 = 1 + x_1$	(-1,0.25, ±1)	0.188,0,-1.188	(-1.61,0.30±2.21i) (1.20,-1.10±2.33i)

4. APPLICATION OF NORMAL FORM THEORY TO THE SPROTT SYSTEMS

The normal form analysis has been applied [7] to three of the Sprott systems, namely systems C,D,E with a linearized eigenvalue spectrum of $\pm i\alpha$, β. They have resonance terms at all odd orders, namely 3,5,7,... It has been shown that the normal form (henceforth referred to as nf) solution of these chaotic systems represents the dynamical system for a longer time interval than one would expect for a perturbative solution. The Sprott F system has also been analyzed [7] up to order 6 and a comparable, although a less accurate result (based on one rather than three resonance terms) has been reported.

We have extended the Sprott F calculation from the sixth order onwards and repeated the calculation to the Sprott Q and H systems so that up to four resonance terms are calculated. All four systems have a linearized eigenvalue spectrum of $\beta/4\pm i\alpha$, $-\beta$ (with β having either sign) and resonance terms occur at 6,11,16,21,... orders. These calculations are handled by the REDUCE symbolic programming system[14]. These three systems have two fixed points, one at (0,0,0) and other one at (a_i, b_i, c_i) where the specific values are quoted in Table 3.1. When we move the first fixed point to the origin, it causes resonance terms in the nf expansion whereas using the nonzero fixed point does not cause resonance terms. Theoretically we can apply the nf method near both fixed points by using the symbolic program we developed. However limits of the symbolic programming system used in this work, R3.8 Personal edition has not been able to complete the calculations near the nonzero fixed point due to memory overflows. Thus we used the fixed point at the origin.

4.1. The Sprott Q System

The Sprott Q system is represented by the following set of equations:

$$\frac{dx_1}{dt} = -x_3 \tag{4.1.1}$$

$$\frac{dx_2}{dt} = x_1 - x_2 \tag{4.1.2}$$

$$\frac{dx_3}{dt} = 3.1x_1 + x_2^2 + 0.5x_3 \tag{4.1.3}$$

For the initial points $x_1 = -0.9983341882 \times 10^{-9}$, $x_2 = 0.595885367 \times 10^{-8}$, $x_3 = -0.1709177196 \times 10^{-7}$, the Sprott Q system behaves for the first 1000 seconds as shown in the following figure

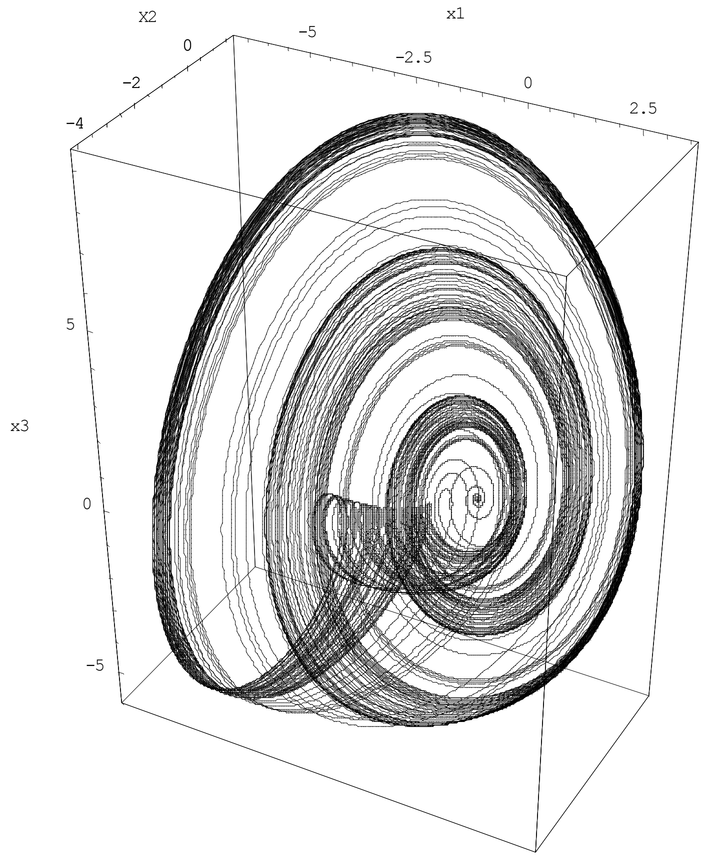

Figure 4.1.1 The Sprott Q system in a phase space

The standard (complex diagonal) form of the Sprott Q system is given as

$$\begin{aligned}\frac{dz}{dt} &= 0.155917203954irz - 0.155917203954ir\bar{z} - 0.0295368620038iz^2 \\ &+ 1.74284250579iz + 0.0295368620038i\bar{z}^2 - 0.573775310549r^2 \\ &+ 0.217391304348rz + 0.217391304348r\bar{z} - 0.00999903807963z^2 \\ &- 0.0623668815814z\bar{z} + 0.25z - 0.00999903807963\bar{z}^2\end{aligned} \tag{4.1.4}$$

$$\begin{aligned}\frac{dr}{dt} &= 0.0590737240076irz - 0.0590737240076ir\bar{z} - 0.0111908909974i2^2 \\ &+ 0.0111908909974i\bar{z}^2 - 0.217391304348r^2 + 0.0823649577407rz \\ &+ 0.0823649577407r\bar{z} - r - 0.00378842360483z^2 - 0.023629489603z\bar{z}\end{aligned} \tag{4.1.5}$$

The 21$^{\text{th}}$ order nf expansion of this system is,

$$\begin{aligned}\frac{du}{dt} &= -0.000000000695199066281iu^9\bar{u}^8w^4 + 0.000000226743237833iu^7\bar{u}^6w^3 \\ &- 0.00000898599024676iu^5\bar{u}^4w^2 + 0.000576451865303iu^3\bar{u}^2w \\ &+ 1.74284250579iu - 0.00000000133154703656u^9\bar{u}^8w^4 \\ &+ 0.000000414334286634u^7\bar{u}^6w^3 - 0.0000154627282699u^5\bar{u}^4w^2 \\ &+ 0.000892924730024u^3\bar{u}^2w + 0.25u\end{aligned} \tag{4.1.6}$$

$$\begin{aligned}\frac{dw}{dt} &= 0.0000000506060029783u^8\bar{u}^8w^5 - 0.00000149326594996u^6\bar{u}^6w^4 \\ &+ 0.0000519977279321u^4\bar{u}^4w^3 - 0.00267877419007u^2\bar{u}^2w^2 - w\end{aligned} \tag{4.1.7}$$

Then we apply the transformation $u = \rho e^{-i\varphi}$ to the nf expansion and we obtain

$$\begin{aligned}\dot{\rho} &= -0.00000000133154703656\rho^{17}w^4 + 0.000000414334286634\rho^{13}w^3 \\ &- 0.0000154627282699\rho^9w^2 + 0.00089292473002424\rho^5w + 0.25\rho\end{aligned} \tag{4.1.8}$$

$$\begin{aligned}\dot{\varphi} &= 0.000000000695199066281\rho^{16}w^4 - 0.000000226743237833\rho^{12}w^3 \\ &+ 0.00000898599024676\rho^8w^2 - 0.000576451865303\rho^4w - 1.74284250579\end{aligned} \tag{4.1.9}$$

$$\dot{w} = 0.0000000506060029783\rho^{16}w^5 - 0.00000149326594996\rho^{12}w^4 + 0.0000519977279321\rho^8 w^3 - 0.00267877419007\rho^4 w^2 - w \tag{4.1.10}$$

In the following figure we compared numerical solution of the nf equations having inital points $\rho_0 = 10^{-8}, \varphi_0 = 10^{-1}, w_0 = 10^{-8}$ (transformation of the same initial point) with the numeric solution of original equation having corresponding initial points at orders 6, 11, 16, 21.

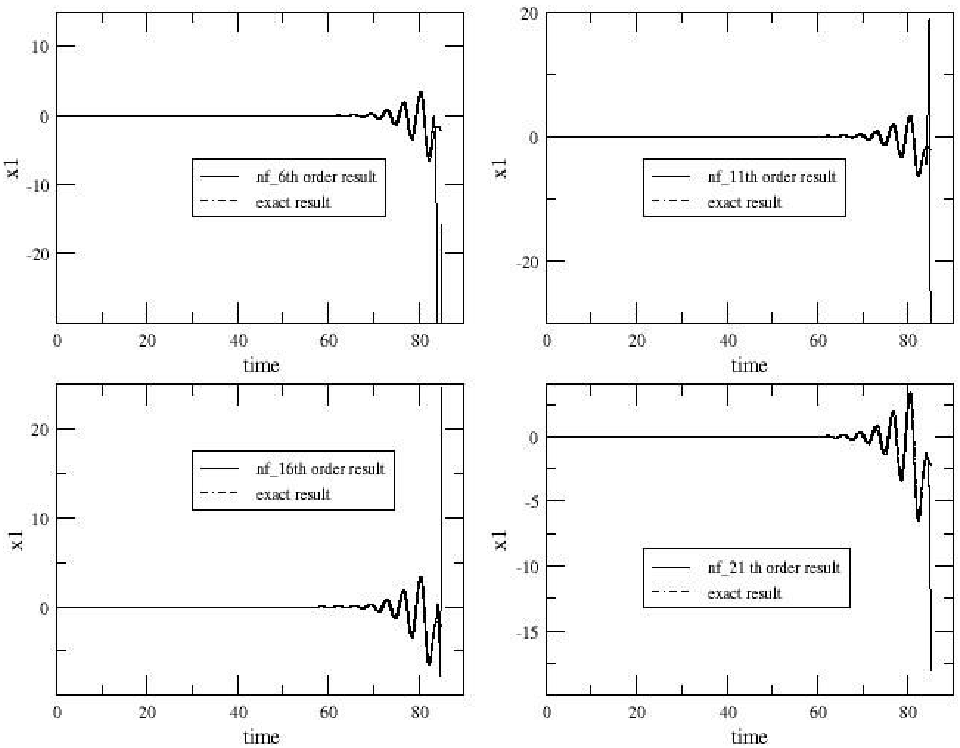

Figure 4.1.2.Comparison of nf results and exact results of the Sprott Q system at 6^{th}, 11^{th},16^{th},21^{th} orders

Now let's zoom in this figures order by order. It is seen in the following figures that as the order of nf expansion increases, the nf solution represents the original system for a longer time interval.

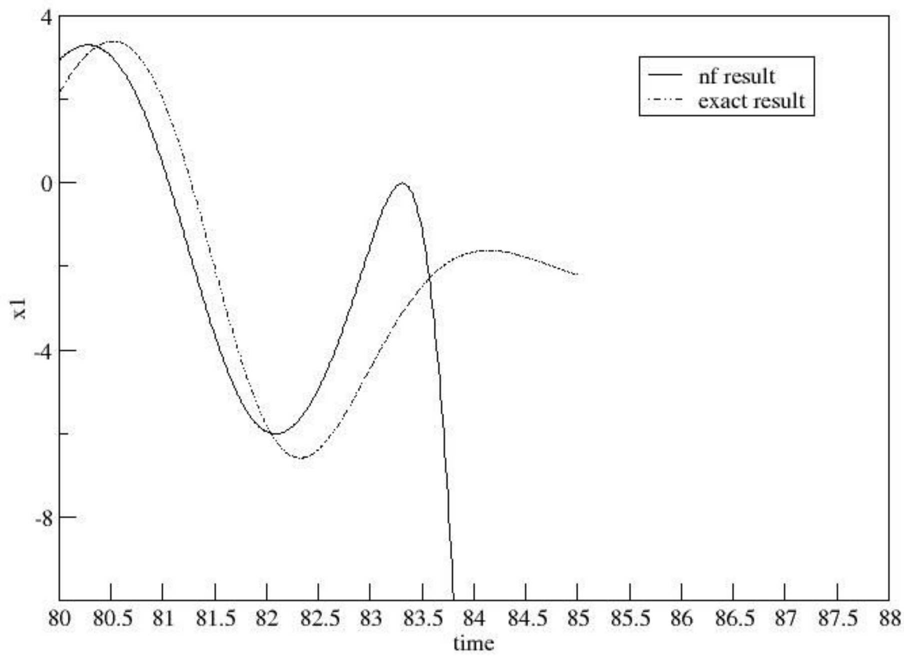

Figure 4.1.3 Nf results and exact results of the Sprott Q system at 6^{th} order

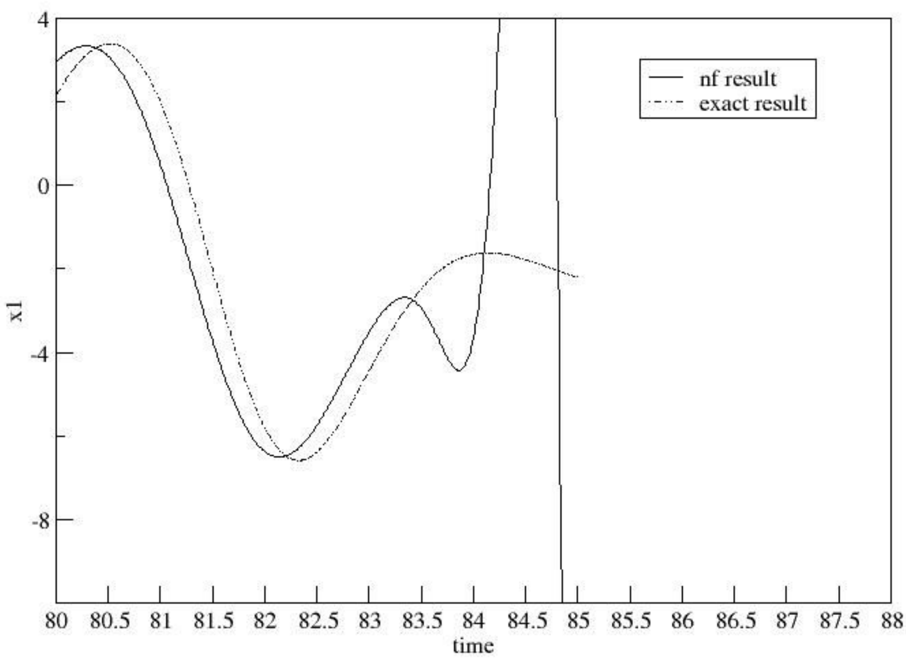

Figure 4.1.4 Nf results and exact results of the Sprott Q system at 11^{th} order

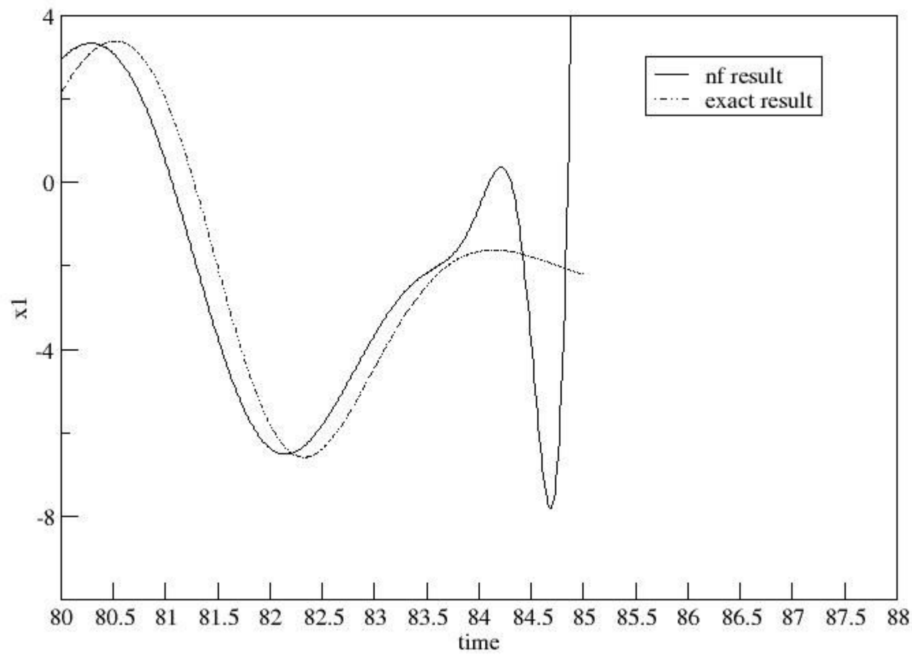

Figure 4.1.5 Nf results and exact results of the Sprott Q system at 16^{th} order

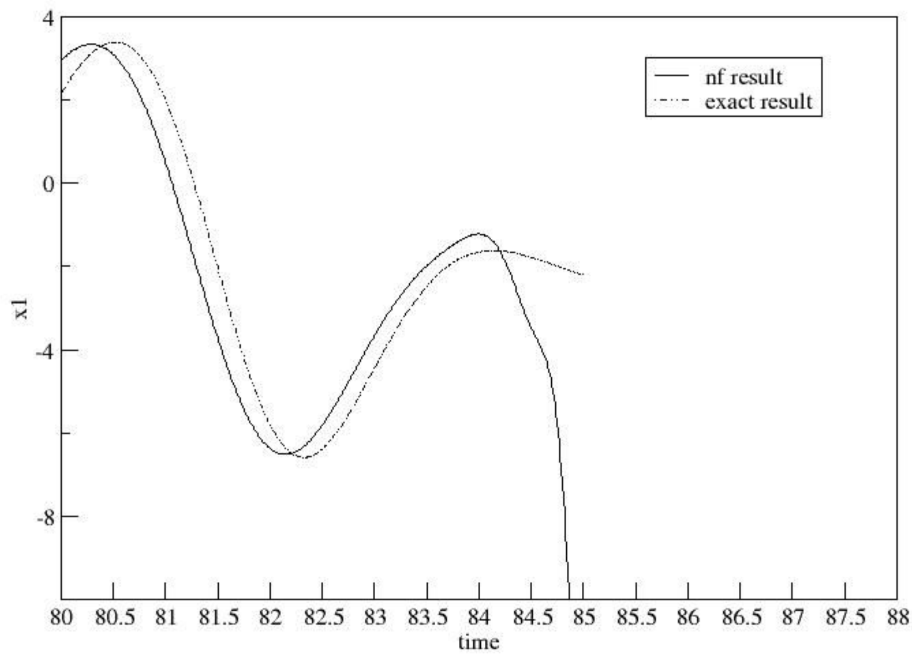

Figure 4.1.6 Nf results and exact results of the Sprott Q system at 21^{th} order

4.2. The Sprott F System

The Sprott F system is represented by the following set of equations;

$$\frac{dx_1}{dt} = x_2 + x_3 \tag{4.2.1}$$

$$\frac{dx_2}{dt} = -x_1 + 0.5x_2 \tag{4.2.2}$$

$$\frac{dx_3}{dt} = x_1^2 - x_3 \tag{4.2.3}$$

For the initial points $x_1 = -0.1588366999 \times 10^{-7}$, $x_2 = -0.4998334242 \times 10^{-8}$, $x_3 = -0.1000000006 \times 10^{-7}$, the Sprott F system is shown in th following figure for the first 1000 seconds.

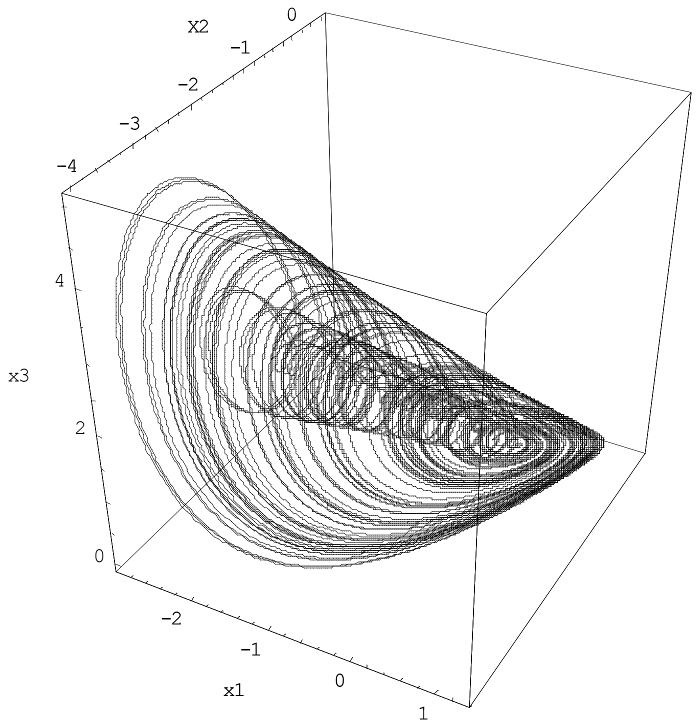

Figure 4.2.1. The Sprott F system in a phase space

The standard (complex diagonal) form of the Sprott F system is given as

$$
\begin{aligned}
\frac{dz}{dt} &= 0.144ir^2 + 0.154919333848irz + 0.309838667697ir\bar{z} \\
&+ 0.025iz^2 + 0.2iz\bar{z} + 0.15i\bar{z}^2 - 0.185903200618r^2 - 0.36rz - 0.24r\bar{z} \\
&- 0.161374306092z^2 - 0.258198889747z\bar{z} - 0.0645497224368\bar{z}^2 \\
&+ 0.968245836552iz + 0.25z
\end{aligned}
\tag{4.2.4}
$$

$$
\begin{aligned}
\frac{dr}{dt} &= 0.15irz - 0.15ir\bar{z} + 0.121030729569iz^2 \\
&- 0.121030729569i\bar{z}^2 + 0.36r^2 + 0.580947501931rz \\
&+ 0.580947501931r\bar{z} - r + 0.21875z^2 + 0.5z\bar{z} + 0.21875\bar{z}^2
\end{aligned}
\tag{4.2.5}
$$

Then the 21th order nf expansion is,

$$
\begin{aligned}
\frac{du}{dt} &= 795.186235293iu^9\bar{u}^8w^4 + 38.9400262058iu^7\bar{u}^6w^3 \\
&+ 2.2198465555iu^5\bar{u}^4w^2 + 0.176734525397iu^3\bar{u}^2w \\
&+ 0.968245836552iu - 1687.81442023u^9\bar{u}^8w^4 \\
&- 86.6172031936u^7\bar{u}^6w^3 - 5.34829808898u^5\bar{u}^4w^2 \\
&- 0.515898748371u^3\bar{u}^2w + 0.25u
\end{aligned}
\tag{4.2.6}
$$

$$
\begin{aligned}
\frac{dw}{dt} &= 6215.20912547u^8\bar{u}^8w^5 + 305.565695119u^6\bar{u}^6w^4 \\
&+ 17.8056492347u^4\bar{u}^4w^3 + 1.54769624511u^2\bar{u}^2w^2 - w
\end{aligned}
\tag{4.2.7}
$$

Then we apply the transformation $u = \rho e^{-i\varphi}$ to the nf expansion and obtain

$$
\begin{aligned}
\dot{\rho} &= -1687.81442023\rho^{17}w^4 - 86.6172031936\rho^{13}w^3 \\
&- 5.34829808898\rho^9w^2 - 0.515898748371\rho^5w + 0.25\rho
\end{aligned}
\tag{4.2.8}
$$

$$
\begin{aligned}
\dot{\varphi} &= -795.186235293\rho^{16}w^4 - 38.9400262058\rho^{12}w^3 \\
&- 2.2198465555\rho^8w^2 - 0.176734525397\rho^4w - 0.968245836552
\end{aligned}
\tag{4.2.9}
$$

$$\dot{w} = 6215.20912547\rho^{16}w^{5} + 305.565695119\rho^{12}w^{4} + 17.8056492347\rho^{8}w^{3} + 1.54769624511\rho^{4}w^{2} - w \quad (4.2.10)$$

In the following figure we compared the numerical solution of nf equations with the inital point $\rho_0 = 10^{-8}, \varphi_0 = 10^{-1}, w_0 = 10^{-8}$ with the numerical solution of the original equation having the corresponding transformed initial point at orders 6, 11, 16, 21.

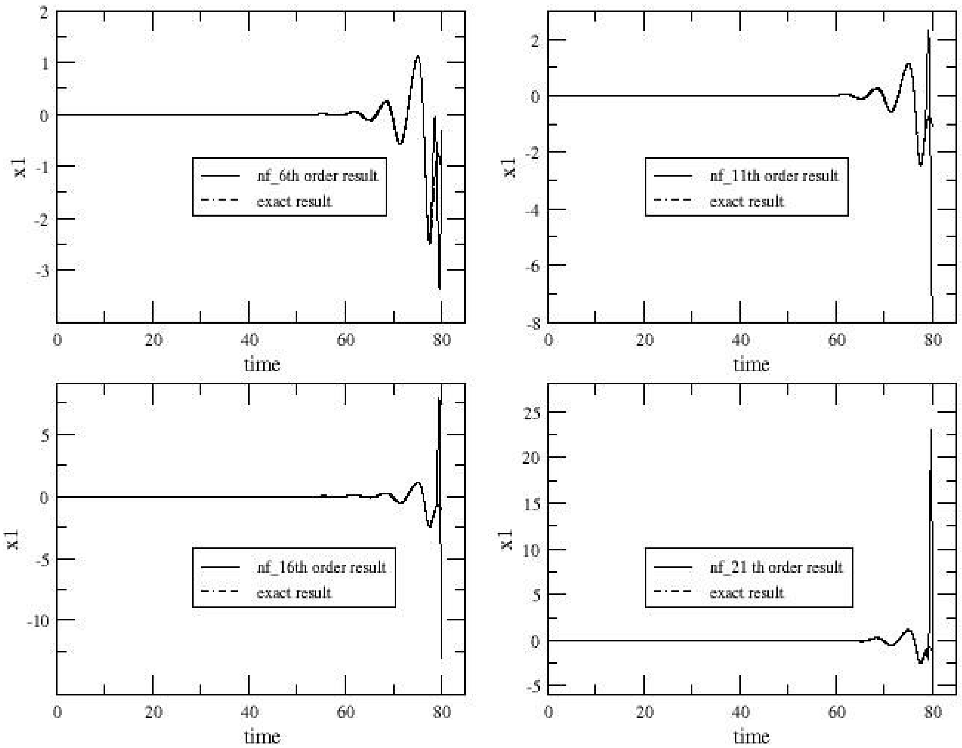

Figure 4.2.2.Comparison of nf results and exact results of the Sprott F system at 6th, 11th,16th,21th orders

Similar to the Sprott Q system, let's zoom in this figures order by order. It is seen in the following figures that as the order of nf expansion increases, the nf solution represents the original system for a longer time interval.

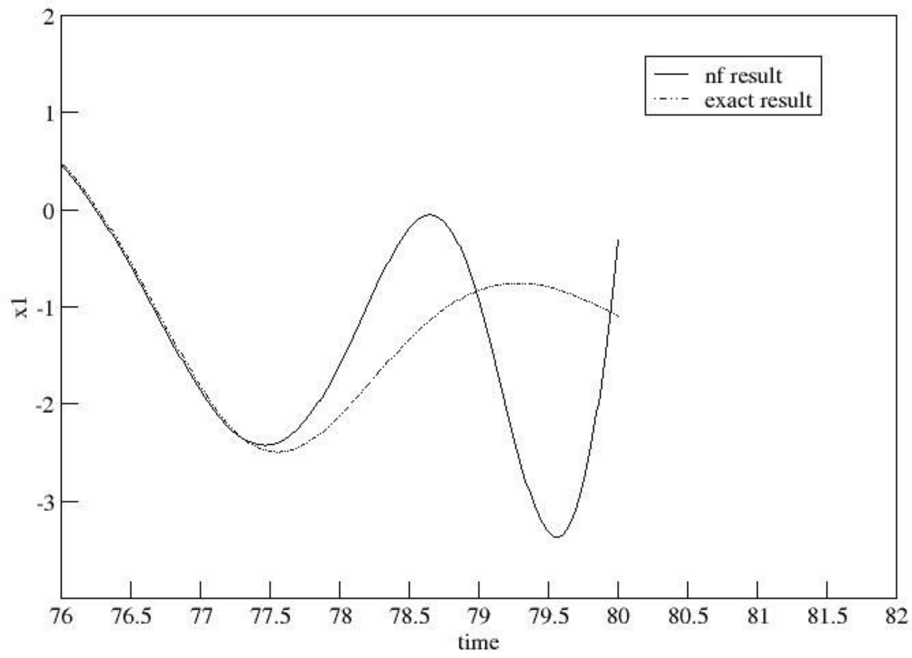

Figure 4.2.3 Nf results and exact results of the Sprott F system at 6^{th} order

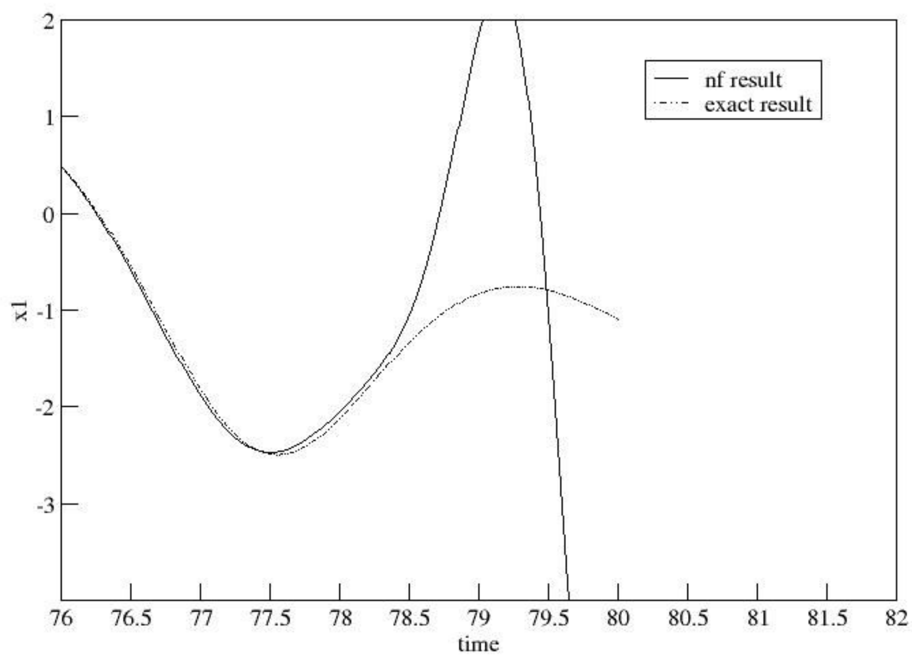

Figure 4.2.4 Nf results and exact results of the Sprott F system at 11^{th} order

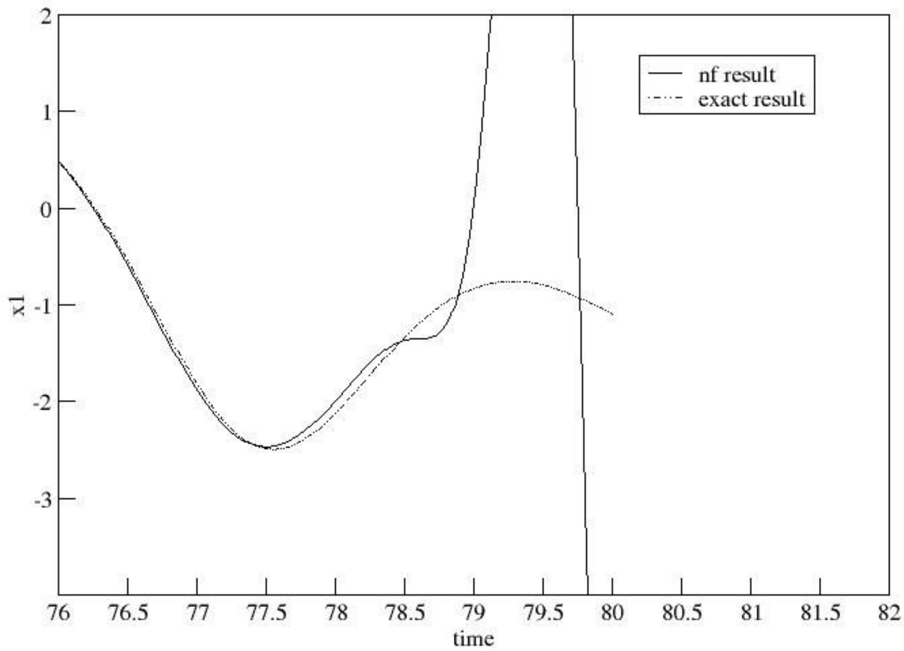

Figure 4.2.5 Nf results and exact results of the Sprott F system at 16^{th} order

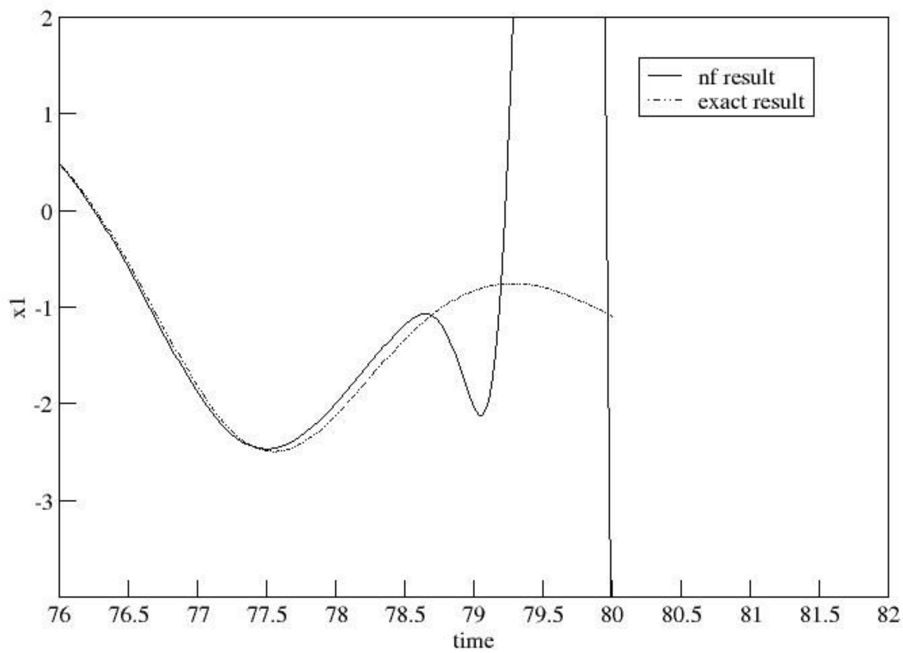

Figure 4.2.6 Nf results and exact results of the Sprott F system at 21^{th} order

4.3. The Sprott H System

The Sprott H system is represented by the following set of equations;

$$\frac{dx_1}{dt} = -x_2 + x_3^2 \tag{4.3.1}$$

$$\frac{dx_2}{dt} = x_1 + 0.5x_2 \tag{4.3.2}$$

$$\frac{dx_3}{dt} = x_1 - x_3 \tag{4.3.3}$$

For the initial point $x_1 = 0.8386168602 \times 10^{-8}$, $x_2 = -0.150750883 \times 10^{-7}$, $x_3 = 0.9001665996 \times 10^{-8}$, the Sprott H system is shown in the following figure for the first 1000 seconds.

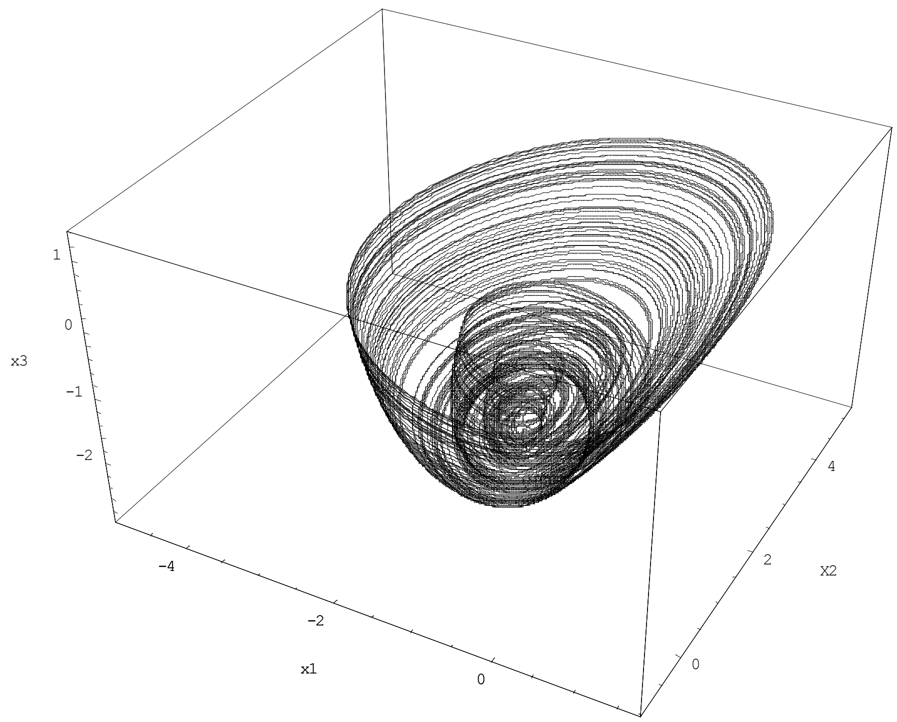

Figure 4.3.1. The Sprott H system in a phase space

The standard (complex diagonal) form of the Sprott H system is given as

$$\frac{dz}{dt} = 0.6ir^2 - 0.258198889747irz + 0.258198889747ir\bar{z}$$
$$- 0.15iz^2 + 0.3iz\bar{z} - 0.15i\bar{z}^2 + 0.258198889747r^2 + 0.6rz - 0.6r\bar{z} \quad (4.3.4)$$
$$- 0.0645497224368z^2 + 0.129099444874z\bar{z} - 0.0645497224368\bar{z}^2$$
$$+ 0.968245836552iz + 0.25z$$

$$\frac{dr}{dt} = 0.6irz - 0.6ir\bar{z} - 0.6r^2 - r + 0.15z^2 - 0.3z\bar{z} + 0.15\bar{z}^2 \quad (4.3.5)$$

Then 21$^{\text{th}}$ order nf expansion is,

$$\frac{du}{dt} = -792.20844668iu^9\bar{u}^8w^4 - 180.277899101iu^7\bar{u}^6w^3$$
$$+ 6.16624043194iu^5\bar{u}^4w^2 - 0.294557542328iu^3\bar{u}^2w$$
$$+ 0.968245836552iu - 3589.23388444u^9\bar{u}^8w^4 \quad (4.3.6)$$
$$+ 401.005570341u^7\bar{u}^6w^3 - 14.8563835805u^5\bar{u}^4w^2$$
$$+ 0.859831247285u^3\bar{u}^2w + 0.25u$$

$$\frac{dw}{dt} = 75839.1745963u^8\bar{u}^8w^5 - 1414.65599592u^6\bar{u}^6w^4$$
$$+ 49.460136763u^4\bar{u}^4w^3 - 2.57949374186u^2\bar{u}^2w^2 - w \quad (4.3.7)$$

Then we apply the transformation $u = \rho e^{-i\varphi}$ to the nf expansion and obtain

$$\dot{\rho} = -3589.23388444\rho^{17}w^4 + 401.005570341\rho^{13}w^3$$
$$-14.8563835805\rho^9w^2 + 0.859831247285\rho^5w + 0.25\rho \quad (4.3.8)$$

$$\dot{\varphi} = 792.20844668\rho^{16}w^4 + 180.277899101\rho^{12}w^3$$
$$- 6.16624043194\rho^8w^2 + 0.294557542328\rho^4w - 0.968245836552 \quad (4.3.9)$$

$$\dot{w} = 75839.1745963u^8\bar{u}^8w^5 - 1414.65599592u^6\bar{u}^6w^4$$
$$+ 49.460136763u^4\bar{u}^4w^3 - 2.57949374186u^2\bar{u}^2w^2 - w \quad (4.3.10)$$

In the following figure we compare the numerical solution of the nf equations with the inital point $\rho_0 = 10^{-8}, \varphi_0 = 10^{-1}, w_0 = 10^{-8}$ with the numerical solution of the original equation with the corresponding transformed initial point at orders 6, 11, 16, 21.

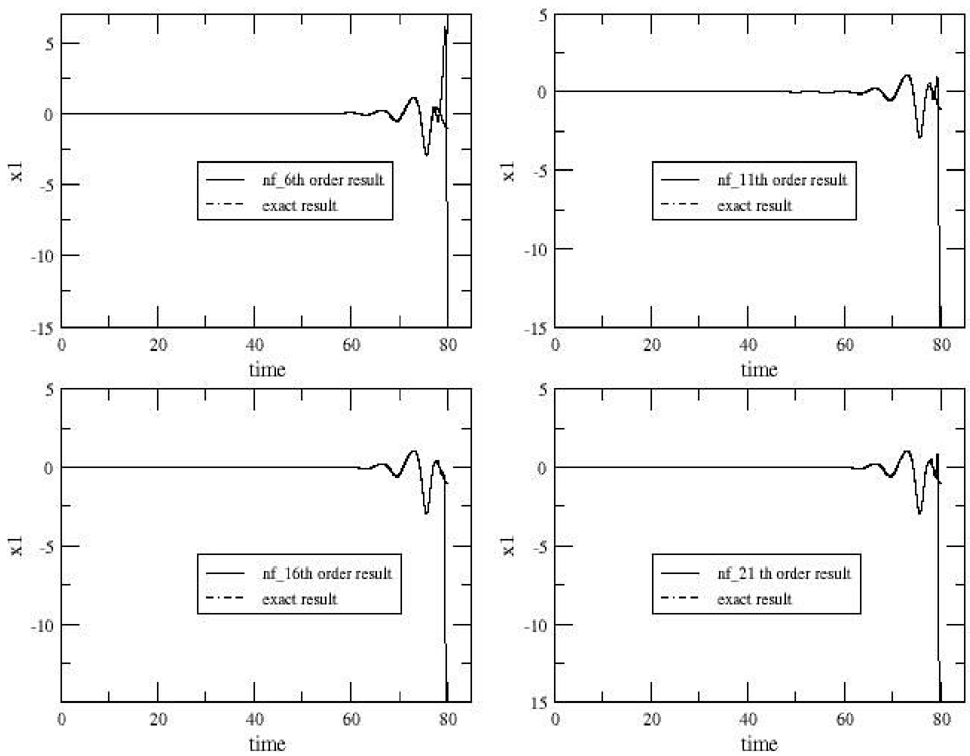

Figure 4.3.2.Comparison of nf results and exact results of the Sprott H system at $6^{th}, 11^{th}, 16^{th}, 21^{th}$ orders

Similar to other systems, let's zoom in these figures order by order. Here it is also seen that as we expand the nf equations to higher orders, the nf solution represents the original system for a longer time interval.

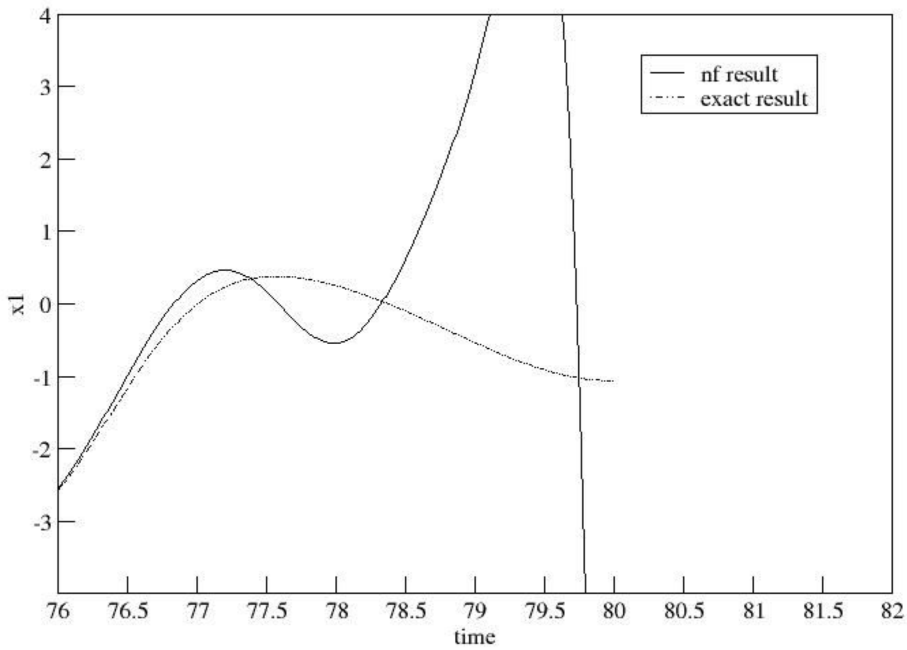

Figure 4.3.3 Nf results and exact results of the Sprott H system at 6^{th} order

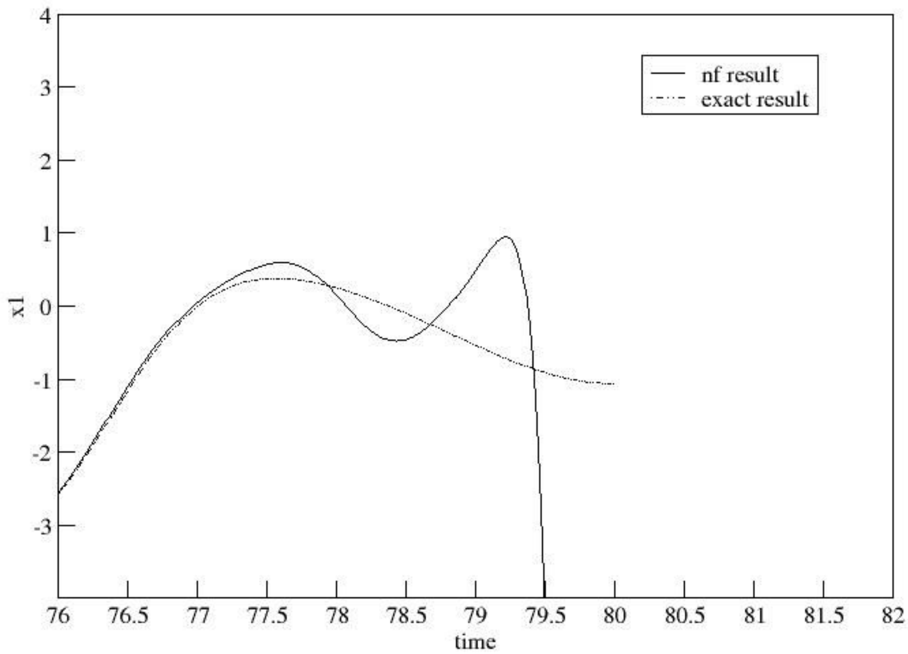

Figure 4.3.4 Nf results and exact results of the Sprott H system at 11^{th} order

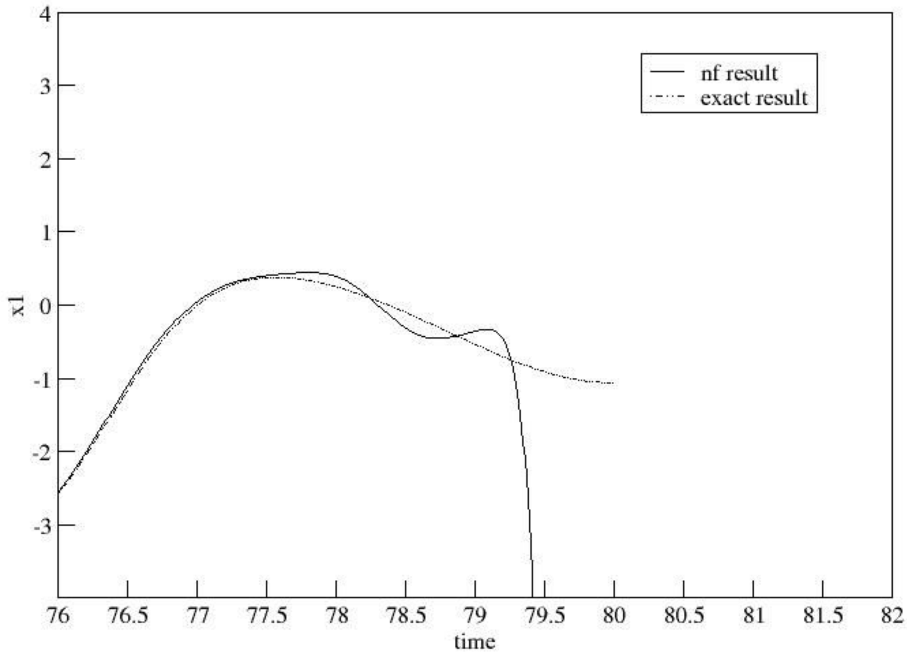

Figure 4.3.5 Nf results and exact results of the Sprott H system at 16^{th} order

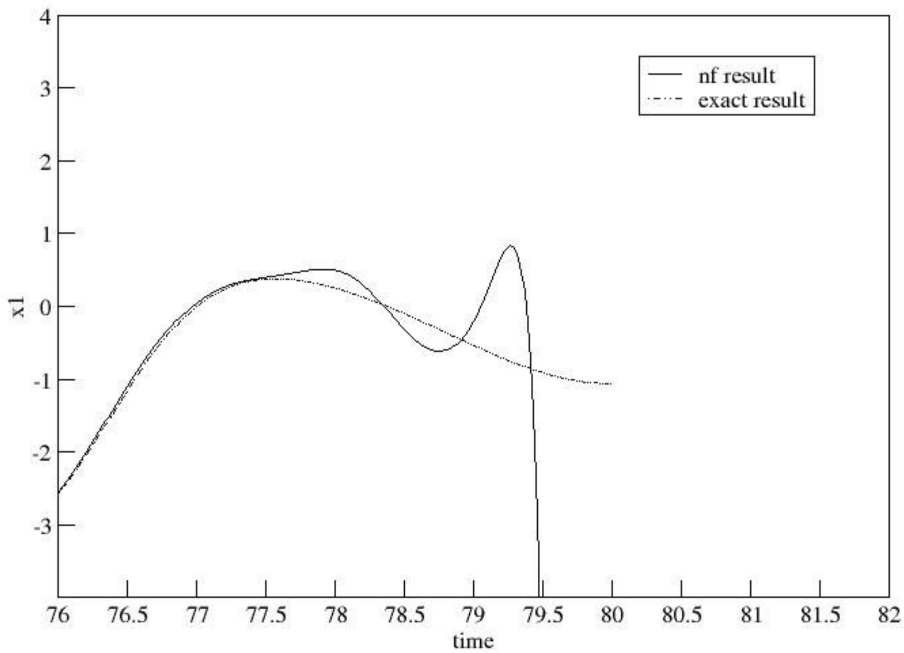

Figure 4.3.6 Nf results and exact results of the Sprott H system at 21^{th} order

Although in all cases the time interval for which the normal form expansion is valid increases with the order, the gain in the time interval is minimal beyond the first order resonance term.

These three systems represents dynamical system for about 80 time unit where a similiar expansion for the Sprott C and E systems represent the systems for 1800 and 1000 time units respectively. Behavior of the nf equations determine this time interval while converting the normal form coordinates (ρ, φ, w) to the original phase space coordinates, (x_1 ,x_2, x_3). Because Sprott Q, F and H do not have purely imaginary eigenvalues of their linearized parts, ρ and φ vary much more rapidly compared to Sprott C and E. Therefore the magnutitude of the perturbative part of our transformations reach the magnutitude of linearized part in a shorter time. For Sprott Q, F and H this time is about 70 time units. Numeric comparisons show that the time for which the normal form expansion represents the system varies between two special times. The first one is the time needed for the magnitude of the perturbative part to reach the magnitude of the linearized part. The second time is the time needed for the magnitude of perturbative part to reach two times the magnitude of the linearized part.

5. SIGNIFICANCE OF AN APPROXIMATELY CONSERVED QUANTITY

When a variable or a combination of variables has a behavior which is polynomial bounded, it is impossible to observe an exponential separation or convergence of nearby trajectories. Hence neither exponential growth or shrinking can occur, this can be taken as an approximate indication that there is a zero Liapunov exponent [7]. Through numerical simulation, all 19 of the Sprott systems are shown to have a zero Lyapunov exponent. It is also remarkable that their linearized eigenvalue spectrum contains a complex conjugate pair. It is attractive to relate such a pair to a zero Lyapunov exponent [15,16].

Let us recall that the normal form equations can be written as

$$\frac{du_i}{dt} = \lambda_i u_i + \sum_{n=1}^{N} \varepsilon^{n-1} f^n(u_1,\ldots u_n) \tag{5.1}$$

where the special form $f_n^i(u_1,\ldots u_n) = \sum a_{i,jkm}^n u_1^j u_2^k u_3^m$ is used. The normal form near identity transformation can be written as $x_i = u_i + \sum_n \varepsilon^{n-1} h_n^i(u_1,\ldots u_n)$. Again, the special form $h_n^i(u_1,\ldots u_n) = \sum \beta_{i,jkm}^n u_1^j u_2^k u_3^m$ had previously been used. Instead of using this special basis, let us write the condition for the elimination of the order n terms as a Lie partial differential equation, where y={y_1,...y_n}

$$L(h_n) = \sum_\beta \lambda_\beta y_\beta \frac{\partial h_n(y)}{\partial u_\beta} - \lambda_a h_n(y) = f_n(y)\,. \tag{5.2}$$

Substituting the form $h_n(y) = y_1^j y_2^k y_3^m$ would give back the resonance relation. Instead we look at the characteristics of this system of partial differential equations that span its solution space .

$$\frac{dy_1}{\lambda_1 y_1} = \frac{dy_2}{\lambda_2 y_2} = \ldots = \frac{dy_m}{\lambda_m y_m} = \ldots = \frac{dh_n}{f_n(y) + \lambda_n h_n(y)} \quad . \tag{5.3}$$

For a zero Lyapunov exponent λ_m, we must have dy_n =0, hence y_n =constant must be a combination of variables that are conserved. Since an exactly conserved quantity is not easy to find, we can try to detect the existance of a combination which is an approximately conserved quantity [7]. To do this let's take the lowest order components of NF equations,

$$\dot{\rho} = a\rho \tag{5.4}$$

$$\dot{\varphi} = -b \tag{5.5}$$

we obtain the following equation by combining them,

$$\frac{d\rho}{d\varphi} = -\frac{a}{b}\rho + \ldots \tag{5.6}$$

then integrate this equation to get the approximately conserved quantity

$$\ln \rho + \frac{a}{b}\varphi = const. \tag{5.7}$$

Now we should investigate the behavior of this combination in the phase space. Thus (ρ, φ, ω) coordinates should be transformed into (x_1, x_2, x_3) coordinates. It is done by numerically for the system of Sprott C [7]. For this calculation one should reverse the multivariable series which are used at NIT transformations,

$$z = u + \sum_{n=2}^{k} T_n(u, \bar{u}, w) \tag{5.8}$$

$$\bar{z} = \bar{u} + \sum_{n=2}^{k} \overline{T}_n(u, \bar{u}, w) \tag{5.9}$$

$$r = w + \sum_{n=2}^{k} S_n(u, \bar{u}, w) \tag{5.10}$$

According to the inverse function theorem, [17] in order to be able to invert the normal form transformations, the transformations should be one to one and continuous for the domain of function. During our repsentation time the NF equations represent attractors which never pass through the same point. Hence in the specific time interval for all set of $(u, \bar{u}, w)$ which are in the domain of the $z, \bar{z}, r$ functions we have different $z, \bar{z}, r$ values. This means that in representation time NIT transformations are one-to-one for our chaotic systems. Additionally, the Jacobian has 1's on the diagonals and hence it is invertible at the origin. Since the near identity transformation is analytic in a ball containing the origin, the Taylor expansion is available up to the first zero of the Jacobian determinant. The inverse will then exist in a ball (not necessarily of the same radius) around the origin.

Then we define the inverses of these functions as following,

$$u = z + \sum_{n=2}^{k} G_n(z, \bar{z}, r) \tag{5.11}$$

$$\bar{u} = \bar{z} + \sum_{n=2}^{k} \overline{G_n}(z, \bar{z}, r) \tag{5.12}$$

$$w = r + \sum_{n=2}^{k} H_n(z, \bar{z}, r) \tag{5.13}$$

By these calculations we obtained ρ and φ as a function of x_1, x_2, x_3 . To see the accuracy of the reversion functions we compared the two sets of ρ, φ values, one obtained from integration of NF, another obtained from integration of $\rho(x_1, x_2, x_3)$ and $\varphi(x_1, x_2, x_3)$. We also calculated percentage errors of $\rho(x_1, x_2, x_3)$ and $\varphi(x_1, x_2, x_3)$ for about 8000 steps corresponding to about 80 time units.

5.1. The Sprott Q System

The integration of the lowest order NF gives us,

$$\ln \rho + \frac{5}{9\sqrt{15}} \varphi = const. \tag{5.1.1}$$

Behavior of this function is shown in the following figure.

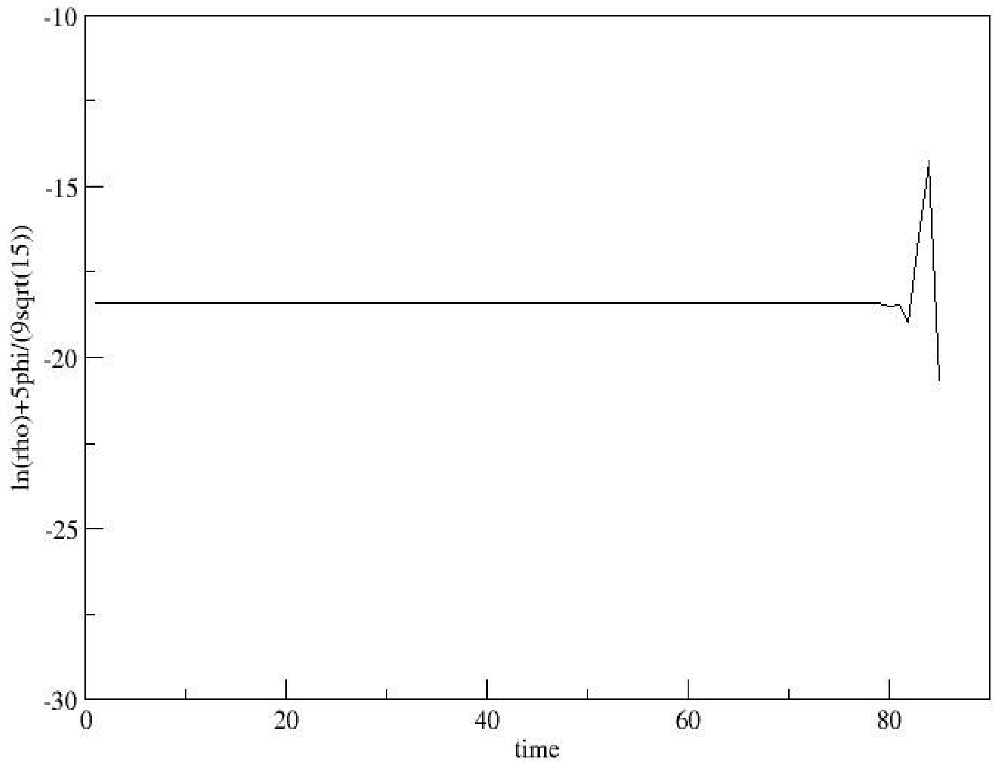

Figure 5.1.1. Trajectory of the aproximately conserved quantity in (x_1, x_2, x_3) space for the Sprott Q system

It is seen that the indicated quantity is approximately conserved in the range that the NF expansion represents the correct solution. Comparison of the NF equations and ρ(x_1, x_2, x_3) and φ(x_1, x_2, x_3) are represented in the Figure 5.1.2. and in the Figure 5.1.3.

Errors are calculated for 8115 steps during 81.15 time units. Average of errors is 0.796 % for ρ(x_1, x_2, x_3). Average of errors is 0.260% for φ(x_1, x_2, x_3).

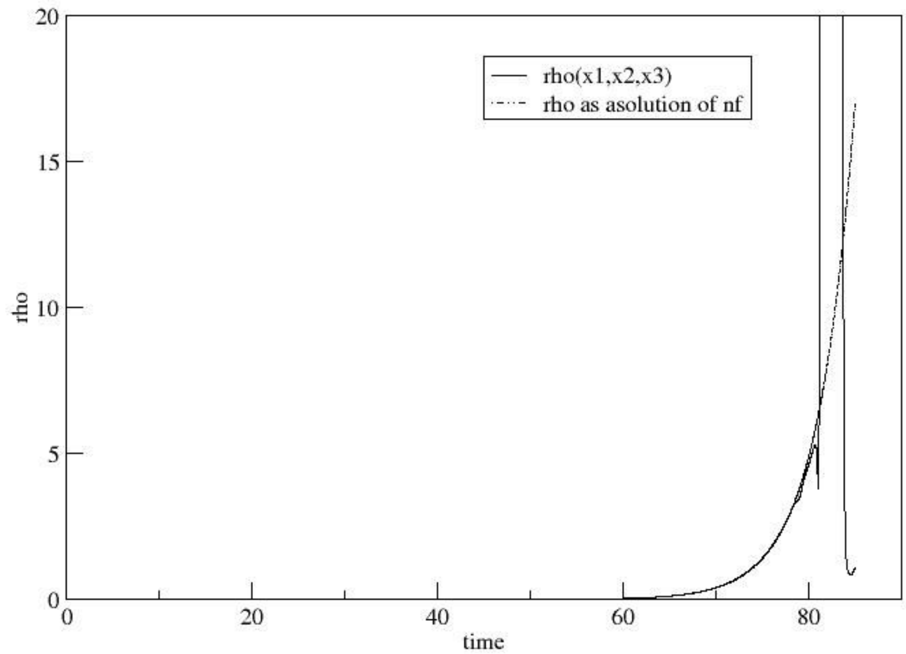

Figure 5.1.2 Solution of $\rho(x_1,x_2,x_3)$ and ρ obtained by nf equations for the Sprott Q system

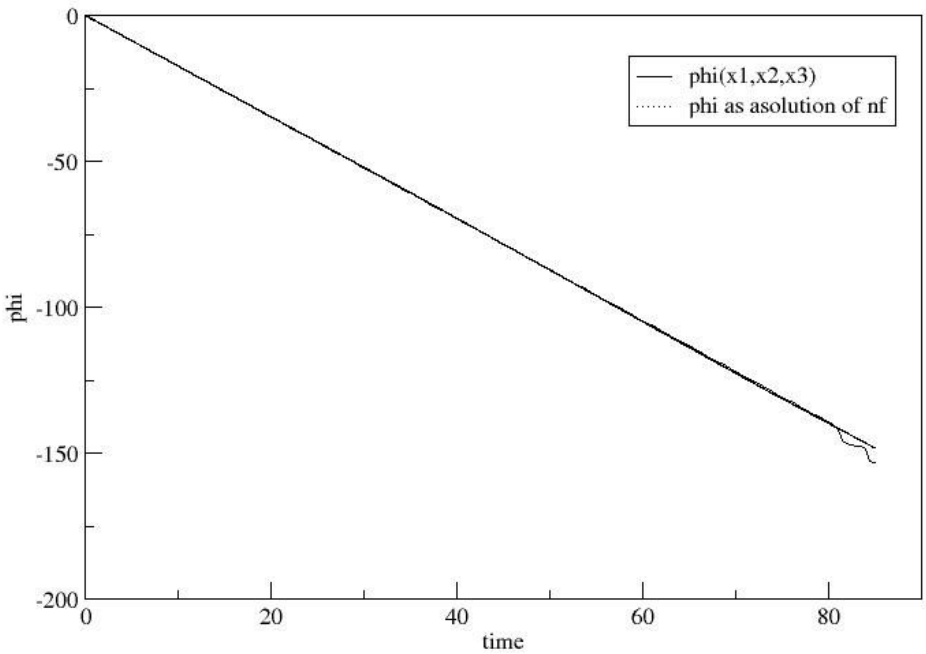

Figure 5.1.3 Solution of $\varphi(x_1,x_2,x_3)$ and φ obtained by nf equations for the Sprott Q system

5.2. The Sprott F System

The integration of the lowest order NF equations gives the following approximately conserved quantity:

$$\ln \rho + \frac{\varphi}{\sqrt{15}} = const. \qquad (5.2.1)$$

The behavior of this function is shown in the following figure.

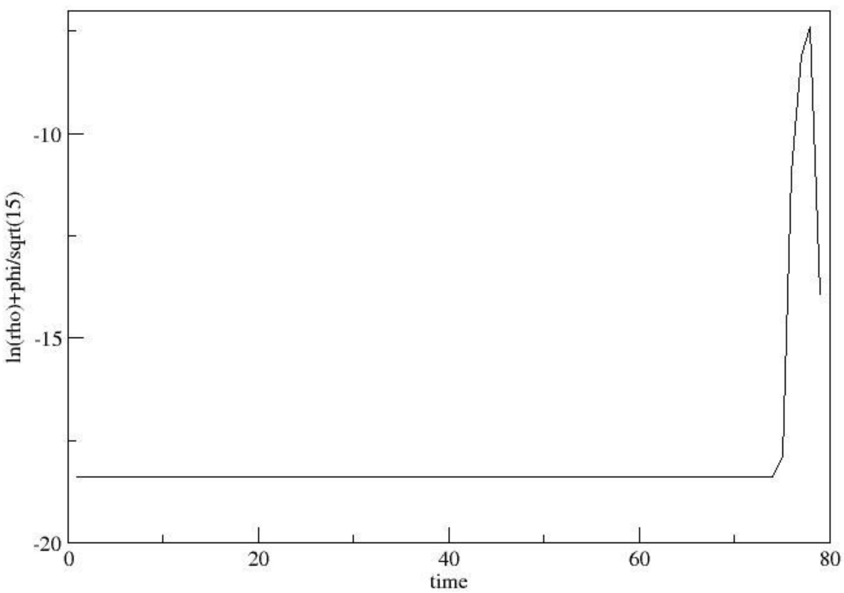

Figure 5.2.1. Trajectory of the aproximately conserved quantity in (x_1, x_2, x_3) space for the Sprott F system

It is seen that the indicated quantity is approximately conserved in the range that the NF expansion represents the correct solution. Comparison of the NF equations and ρ(x_1, x_2, x_3) and φ(x_1, x_2, x_3) are represented in the Figure 5.2.2. and in the Figure 5.2.3.

Errors are calculated for 7400 steps during the first 74 time units. Average of errors is 0.251 % for ρ(x_1, x_2, x_3). Average of errors is 0.299 % for φ(x_1, x_2, x_3).

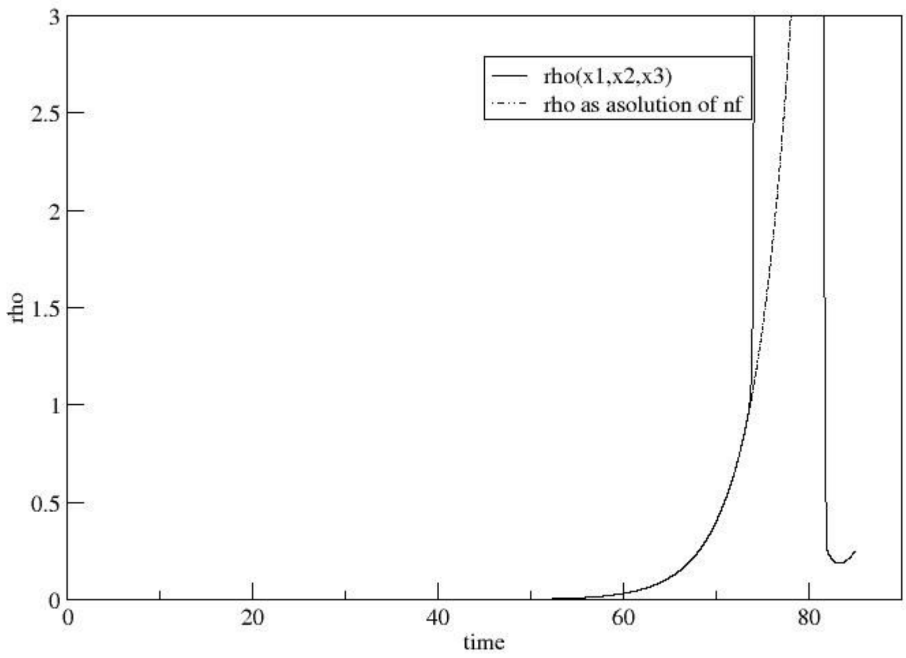

Figure 5.2.2 Solution of $\rho(x_1,x_2,x_3)$ and ρ obtained by nf equations for the Sprott F system

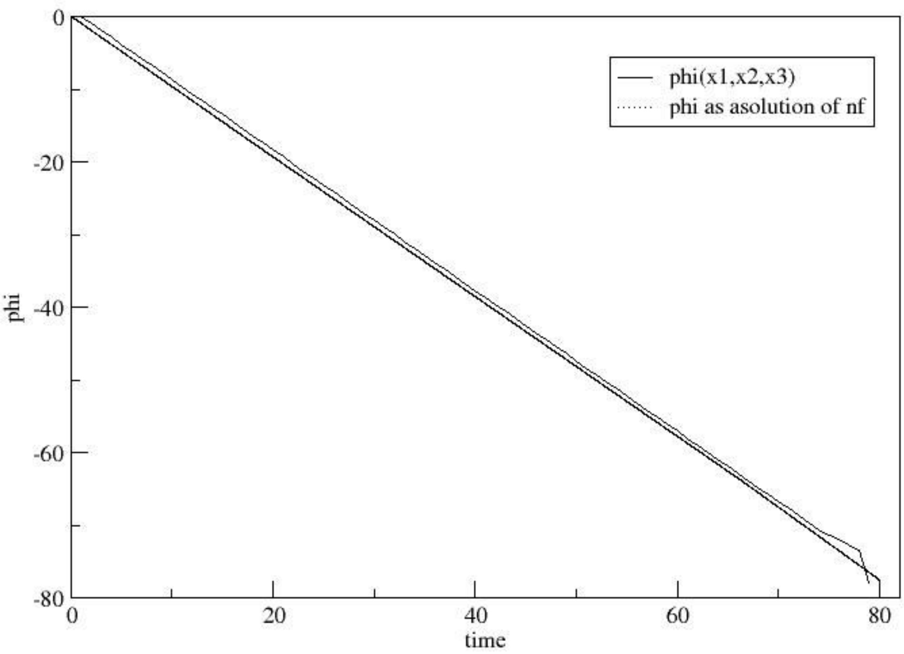

Figure 5.2.3 Solution of $\varphi(x_1,x_2,x_3)$ and φ obtained by nf equations for the Sprott F system

5.3. The Sprott H System

The integration of the lowest order NF equations gives us,

$$\ln \rho + \frac{\varphi}{\sqrt{15}} = const. \tag{5.3.1}$$

The behavior of this function is shown in the following figure.

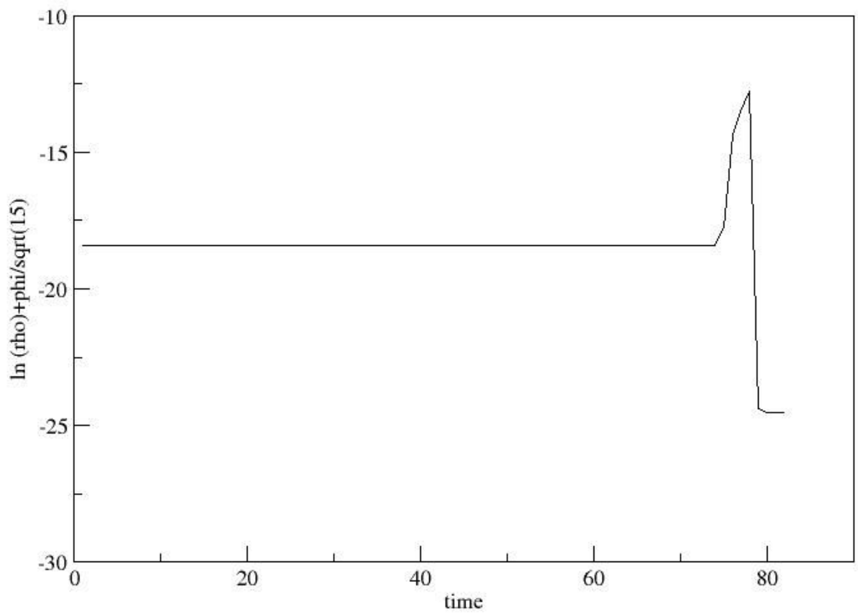

Figure 5.3.1. Trajectory of the aproximately conserved quantity in (x_1, x_2, x_3) space for the Sprott H system

It is seen that the indicated quantity is approximately conserved in the range that the NF expansion represents the correct solution. Comparison of the NF equations and $\rho(x_1, x_2, x_3)$ and $\varphi(x_1, x_2, x_3)$ are represented in the Figure 5.3.2. and in the Figure 5.3.3.

Errors are calculated for 7300 steps during the first 73 time units. Average of errors is 0.289 % for $\rho(x_1, x_2, x_3)$. Average of errors is 0.244 % for $\varphi(x_1, x_2, x_3)$.

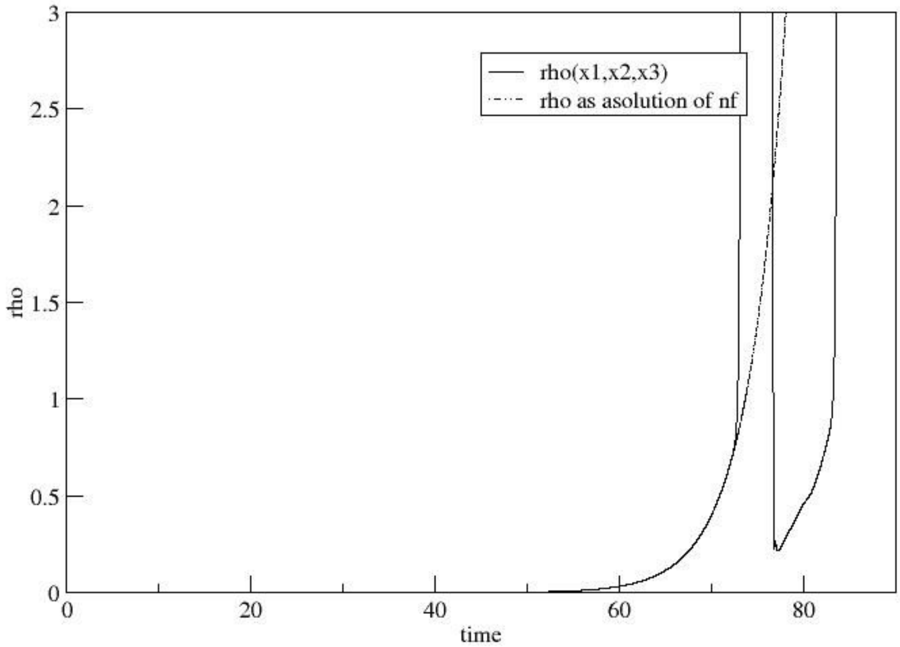

Figure 5.3.2 Solution of $\rho(x_1,x_2,x_3)$ and ρ obtained by nf equations for the Sprott H system

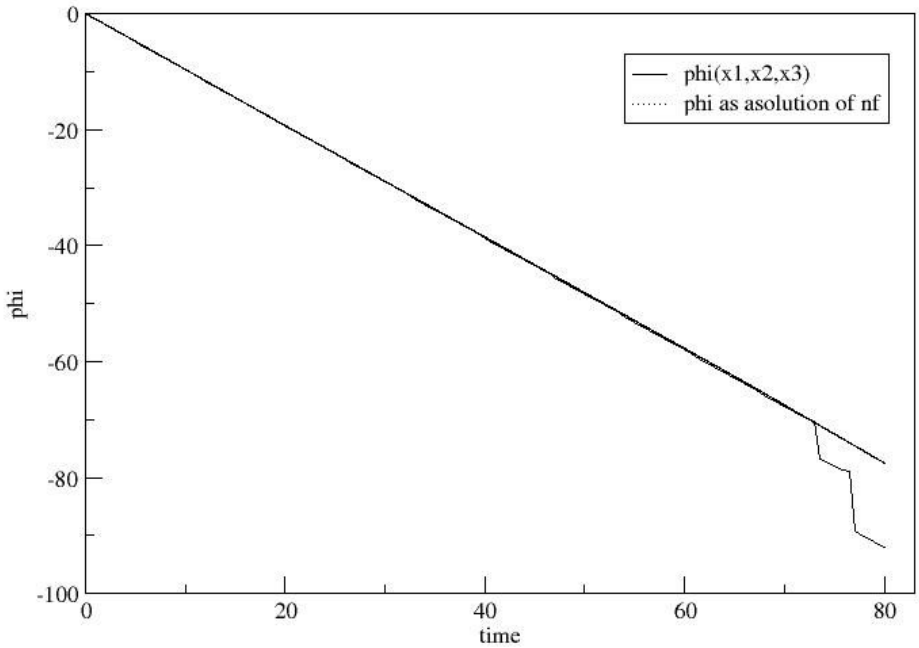

Figure 5.3.3 Solution of $\varphi(x_1,x_2,x_3)$ and φ obtained by nf equations for the Sprott H system

CONCLUSIONS

We have applied the normal form method to three Sprott systems consisting of three dimensional first-order differential equations with one quadratic nonlinear term and five linear terms. It is shown that as we expand the normal form equations to higher orders, we find that the expansion represents the dynamical system for a longer time. However the gain in the time interval is minimal beyond the first order resonance term. Therefore, the lowest order resonant term contains a lot of information about the system.

When a variable or a combination of variables does not have exponential growth or decay in the system phase space, an approximately conserved quantity is assumed to exist in this space. This behavior indicates a zero Liapunov exponent. Using normal form equations we obtained such a combination. For this calculation we inverted the multivariable series which are used at NIT transformations. It is seen that the indicated quantity is approximately conserved quantity in the range that the NF expansion represents the correct solution .

APPENDIX A: COMPUTATIONAL DETAILS

In this Appendix, technical questions concerning the integration of the Sprott systems and their normal forms and the methods used in the inversion of the normal form near identity transformation are presented.

All numerical integrations have been performed by using the subroutines based on an Adams predictor and corrector algorithm given by Shampine et. al [18]. This algorithm is started by a polynomial approximation. The calculation has been repeated by using the ode package available in SuSe Linux [19]. The latter package is based on a switching Runge Kutta 5th order algorithm. Up to the onset of the divergence, the two approaches have given comparable solutions. However, the divergent portions are significantly different. This is due to the fact that the Runge Kutta algorithm is a variable step size algorithm with quality control, so that when the divergent portion is encountered, the step size is continually reduced until the calculation becomes meaningless due to truncation error. On the other hand, the fixed step size Adams mechanism continues the calculation in a meaningless way without attempting to correct itself.

The divergence of the normal form equations can be attributed to the polynomial transformation, since the resonant part can always be reduced to a quadrature.

A second point in the calculation is the necessity of inverting the normal form near identity transformation. In Reference [7], this calculation had been carried out numerically. Here, an analytic approach is used.

If the transformation is

$$x_i = u_i + \sum_n \varepsilon^{n-1} T_n(u_1, \ldots u_n) \tag{A.1}$$

We assume that the inverse transformation is

$$u_i = x_i + \sum_m \varepsilon^{m-1} R_n(x_1,...x_n) \tag{A.2}$$

where $T_n(u_1,...u_n)$ is a given homogeneous polynomial of degree n. For R_n, we assume a generalized homogenous polynomial

$$R_n(x_1...x_n) = \sum_{m_1,m_2...} c(m_1,m_2...m_n)\, x_1^{m_1} x_2^{m_2} ... x_n^{m_n} \quad \text{with} \quad m_1 + m_2 + ...m_n = 1 \tag{A.3}$$

Substituting (A.3) into (A.2) and the resulting equation into (A.1), we get a series of linear equations that can be solved for the coefficients $c(m_1,m_2,...m_n)$. This furnishes the inverted series.

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