CYLINDRICALLY SYMMETRIC SPACETIMES WITH PURE RADIATION

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To My Father

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ABSTRACT

CYLINDRICALLY SYMMETRIC SPACETIMES WITH PURE RADIATION

In this thesis we have investigated some solutions of Einstein's field equations having cylindrical symmetry. The corresponding energy momentum tensor of most of these solutions has pure radiation equation of state. First, the static Levi-Civita solution has been generalized to a Kasner type time dependent solution. Using this solution, we have presented a time dependent Vaidya type solution representing pure and gravitational radiation emitted from a nonstatic cylindrical source. As an application, we have analyzed a radiating nonstatic cosmic string like object. Next, we have presented cylindrically symmetric, static solutions of the Einstein field equations around a line singularity such that the energy momentum tensor corresponds to infinitely thin photonic shells composed of counter propagating pure radiation in certain directions. Positivity of the energy density of the thin shell and the line singularity is discussed. Among these solutions, a particular solution corresponding to a photonic shell whose interior and exterior is flat is interesting since the cylinder becomes a plane for an outside observer. We have also investigated the generalization of these solutions to multiple thin shells and found that line singularities including cosmic strings may be screened by photonic shells until they all appear as a planar wall. Lastly we have investigated solutions corresponding to circulating or counter circulating pure radiation around the axis. The first solution we have studied was an approximate thin shell solution corresponding to counter rotating photons with small contribution from an anisotropic fluid. Next, we study a cylindrical circulating beam of light. The gravitational field of a counter rotating pure radiation field is presented as a last solution for this thesis. These solutions can smoothly match to the corresponding vacuum solutions from either interior or exterior.

ÖZET

SİLİNDİRİKSEL SİMETRİK SAF RADYASYONLU UZAYZAMANLAR

Bu tezde Einstein alan denklemlerinin silindiriksel simetriye sahip bazı çözümleri araştırıldı. Bu çözümlerin çoğunun enerji momentum tensörü, saf radyasyonun durum denklemine sahiptir. İlk olarak statik Levi-Civita çözümü zamana bağlı Kasner tipi bir çözüme genelleştirildi. Bu çözümü kullanarak, statik olmayan silindiriksel bir kaynaktan yayımlanan saf ve kütleçekimsel radyasyona karşılık gelen, zamana bağlı Vaidya tipi bir çözümü sunduk. Bir uygulama olarak, statik olmayan ışıyan kozmik sicim benzeri bir nesneye karşılık gelen bir çözüm incelendi. Daha sonra, enerji momentum tensörü çeşitli yönlerde karşıt hareket eden saf radyasyondan oluşan sonsuz ince bir fotonik kabuğa karşılık gelen, Einstein denklemlerinin bir çizgisel tekillik etrafında, silindiriksel simetrik statik bir çözümünü sunduk. Çizgisel tekilliğin ve ince kabuğun enerji yoğunluğunun pozitifliği tartışıldı. Bu çözümler arasında, dışardaki bir gözlemciye göre silindir düzleme dönüştüğü için, içi ve dışı düz olan ışıksal bir kabuğa karşılık gelen özel çözüm ilginçtir. Bu çözümlerin çoklu kabuk çözümlerine genelleştirilmesini de araştırdık ve kozmik sicimler de dahil olmak üzere çizgisel tekilliklerin fotonik kabuklar tarafından, hepsi birden bir düzlemsel duvar olarak gözükene kadar gizlenebilineceğini bulduk. Son olarak, eksen etrafında dönen ya da karşılıklı dönen saf radyasyona karşılık gelen çözümleri araştırdık. Çalıştığımız ilk çözüm, karşılıklı dönen fotonlar ile birlikte az miktarda eşyönsüz bir akışkandan oluşan yaklaşık bir kalın kabuk cözümüydü. Daha sonra, dönen bir silindiriksel ışık demeti incelendi. Bu tezin son uygulaması olarak, karşılıklı dönen bir saf radyasyon alanının yerçekimsel alanı sunuldu. Bu çözümler, içerden ya da dışarıdan uygun boşluk çözümlerine düzgünce eşlenebilmektedir.

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LIST OF SYMBOLS/ABBREVIATIONS

c	Velocity of light
$C_{\mu\nu\lambda\kappa}$	Weyl tensor
$\mathbf{d}$	Exterior derivative
$\mathbf{D}$	Covariant exterior derivative
$\mathbf{e}^{\mu}$	Orthonormal basis one forms
$F_{\mu\nu}$	Faraday tensor
$g_{\mu\nu}$	Spacetime metric
G	Newton gravitational constant
$G_{\mu\nu}$	Einstein tensor
$\imath_{e_a}$	Interior product operator
$\mathcal{K}$	Kretchmann scalar
K_{ab}	Extrinsic curvature tensor
n_{μ}	Unit normal vector
R	Ricci scalar
$R_{\mu\nu}$	Ricci tensor
$R_{\mu\nu\lambda\kappa}$	Riemann tensor
$T_{\mu\nu}$	Energy momentum tensor
α	Angular deficit parameter
δ	Codifferential
∇	Covariant derivative operator
δ^{μ}_{ν}	Kronecker delta
$\eta_{\mu\nu}$	Minkowski metric
Φ_{ab}	Components of the Ricci tensor in a null frame
κ	Gravitational coupling constant
σ	Levi-Civita constant
σ	Shear
θ	Expansion
ω	Twist

${\omega^{\mu}}_{\nu}$	Connection one forms
${\Omega^{\mu}}_{\nu}$	Curvature two forms
Ψ_{ab}	Components of Weyl tensor in a null frame

ADM	Arnowitt, Deser and Misner
CTC	Closed timelike curve
LC	Levi-Civita
LCK	Levi-Civita-Kasner
NP	Newman-Penrose

1. INTRODUCTION

Exact solutions of the Einstein field equations has played an important role on understanding and analyzing the gravitational behaviour of various distributions of matter. Especially the vacuum solutions have great importance in general relativity because they can carry the imprints of the matter distributions they surround. Thus, we can learn much from analyzing the properties of such solutions. For example, the static spherically symmetric vacuum solution, the Schwarzschild solution [1], is extensively used to explain phenomena where the Newtonian theory is not sufficient. The weak field limit of this solution was used to understand and test Einstein's theory of general relativity whereas strong field limits lead to new problems like gravitational collapse and black holes. Such solutions will help us to understand the nature of the universe since we might need a quantum gravity theory for full resolution of such problems. However, without having the "internal" solutions which can generate these vacuum solutions, the justification that such a matter distribution generates the vacuum solution we have obtained is at least incomplete. For example, the Schwarzschild solution has simply one parameter, M, which is related to the mass of the source. This can be seen from the asymptotic behaviour of this metric. This fact is strengthened by the discovery of spherically symmetric regular internal solutions [2]-[4] since the mass function $m(r)$ of the internal Schwarzschild solution becomes the mass parameter of the vacuum solution at the boundary of the source. On the boundary these metrics can smoothly match. Thus, for spherically symmetric case, the parameter of the vacuum solution is directly related with the mass-energy density of the source. Also, this vacuum solution is unique due to the Birkhoff theorem, which simply says that, the spherically symmetric vacuum solution must be static, and can be represented by the Schwarzschild solution.

When we turn our attention to less symmetrical spacetimes, the case is more complicated than the spherical one. Let us consider cylindrical symmetric solutions since we will focus on such solutions in this thesis. First of all, we can have time dependent vacuum solutions since we do not have a timelike Killing vector in cylindrical symmetry in general. Secondly, the time independent ones can be either static or stationary. Thus before analyzing a vacuum solution, we should first determine its symmetry properties. For example, the *static* cylindrically symmetric vacuum solution- the Levi-Civita metric - has two constant parameters [5]. The interpretation of such parameters has been a controversy and led W. B. Bonnor to say "*the Levi-Civita spacetime continues to puzzle relativists*" [6]. Also, whether or not this vacuum solution describes a cylindrically symmetric space-time for the full range of the parameter related with the energy density of source has been one of the research subjects for this metric.

In this thesis, we will mainly focus on cylindrically symmetric solutions of general relativity. In studying these solutions our emphasis is on the nonvacuum solutions. The vacuum cases were found in the early years of general relativity. We will consider sources whose energy-momentum tensor has a specific form called pure radiation or null dust corresponding to an incoherent radiation directed along a certain direction [7]-[9]. We will call these solutions as "*photonic*" since the energy-momentum tensor of photons in geometric optics limit has the form of pure radiation. The motivation to concentrate on such solutions might have several reasons:

- It is well known that the static cylindrically symmetric vacuum Levi-Civita solution admits helical or circular null geodesics [10];
- The photons can be trapped in the gravitational field of the Levi-Civita spacetime for certain ranges of its parameters [11].

It might be interesting to study the possibility of having cylindrically symmetric *photonic* matter distributions generated by such trapped photons following circular or helical geodesics. One of our main goals in this thesis is to construct cylindrical infinitely thin or thick photonic matter distributions composed of massless particles.

Our analysis is not limited to the shells having above properties. In the first chapters we will discuss a nonstatic generalization of the Levi-Civita solution with Kasner type time dependence and relate it with the solutions in existing literature. We will also study Kasner type generalized cylindrically symmetric analogue of the Vaidya metric [12], describing a null radiation from a spherically symmetric mass distribution.

Cosmic strings are one dimensional topological defects which might arise during the phase transitions in the early universe [13]. Their possible gravitational, astrophysical and cosmological implications has been raised a lot of interest. Since a straight cosmic string can be described by particular values of the parameters of cylindrically symmetric vacuum Levi-Civita metric and also superconducting cosmic strings can be described asymptotically by this metric, we investigate the relation of cosmic strings with the solutions through the thesis.

The cylindrically symmetric solutions have great importance in general relativity. Cylindrical symmetric solutions require an infinite mass distribution in a certain direction. Although one might conclude that such solutions are not so physical, they are simple enough to find and analyze. These solutions are more complex than spherical symmetric solutions and have some properties which the spherical symmetric solutions do not have. The most important consequence is the existence of cylindrically symmetric gravitational wave solutions. Such solutions are also important for the analysis of nonspherical gravitational collapse and the hoop conjecture, quantum gravity, numerical gravity and computational relativity. Thus, it is always worth to study cylindrically symmetric solutions of general relativity.

This thesis is organized as follows. We first present the energy momentum tensor corresponding to null radiation and we review the general properties of a cylindrically symmetric spacetime. Then we turn our attention to the Levi-Civita solution and discuss some of its physical properties. We also consider a nonstatic generalization of this metric and also its Vaidya type radiating generalization. The following two sections are devoted to investigations of thin shells composed of massless particles and multiple shell solutions. An approximate thick shell solution composed of counter rotating photons is given in Section 9. In the following two sections we review the solutions corresponding to light beams moving or counter moving in certain direction and extend these solutions to the rotating cases.

2. ENERGY MOMENTUM TENSOR OF PURE RADIATION

In this thesis we will investigate the solutions of the Einstein field equations

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \tag{2.1}$$

where $T_{\mu\nu}$ is the energy momentum tensor of the fields in spacetime and the Einstein tensor $G_{\mu\nu}$ is given by:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \tag{2.2}$$

where $R_{\mu\nu}$ is the Ricci tensor, R is Ricci curvature scalar and $g_{\mu\nu}$ is the spacetime metric. The gravitational coupling constant is $\kappa = \frac{8\pi G}{c^4}$ where G is the Newton gravitational constant and c is the velocity of light. Through this thesis we will use the units where $G = c = 1$ and the signature $(-, +, +, +)$.

The pure radiation fields will be considered as a main source distribution in this thesis. Pure radiation or null dust fields are fields of radiation composed of massless particles having same propagation direction which are considered as the incoherent superposition of waves with random phases and different polarizations. Such radiation can originate from different types of fields such as null electromagnetic fields [7], neutrino fields [14], massless scalar fields or from the high-frequency limit of gravitational waves [15].

The energy momentum tensor of pure radiation in geometric optics limit is given by

$$T_{\mu\nu} = \rho\, k_\mu k_\nu, \tag{2.3}$$

where k_μ is a null vector representing the lightlike wordline of the null dust and satisfying

$$k^\mu k_\mu = 0, \tag{2.4}$$

and $\rho > 0$ is the energy density of the radiation. A congruence of affinely parameterized null geodesics is characterized by three scalars called twist (rotation) ω, expansion θ, and shear σ. They are given by

$$\omega = \left(\frac{1}{2}\nabla_{[\alpha}k_{\beta]}\nabla^\alpha k^\beta\right)^{1/2}, \quad \theta = \frac{1}{2}\nabla_\alpha k^\alpha, \quad |\sigma| = \left(\frac{1}{2}\nabla_{(\alpha}k_{\beta)}\nabla^\alpha k^\beta - \theta^2\right)^{1/2}, \tag{2.5}$$

where ∇ denotes covariant derivation and pharantheses (brackets) denote symmetrization (antisymmetrization).

This form of the energy momentum tensor is a result of an averaging process [7] and the Maxwell or Weyl equations need not be satisfied. However, it can be derived by a variational procedure [8]. Let us re-derive the main formulas for a pure radiation field by considering an electromagnetic radiation field in the next section.

2.1. A Derivation of Electromagnetic Radiation in Geometric Optics Limit

We consider four dimensional pseudo-Riemannian manifold M equipped with a metric g with Lorentzian signature diag$(-1, 1, 1, 1)$. The spacetime metric $g(= ds^2) = g_{\mu\nu}dx^\mu dx^\nu$. We can also consider an orthonormal frame where the metric takes the form $g = \eta_{ab}\,\mathbf{e}^a\,\mathbf{e}^b$. Here $\mathbf{e}^a$'s orthonormal basis one forms (co-frame forms) dual to the orthonormal basis vectors e_a and $\imath_{e_a}\mathbf{e}^b = \delta^b_a$ where $\imath$ is interior product operator. We use bold letters to denote differential forms. The volume form is defined as $*1 = \mathbf{e}^a \wedge \mathbf{e}^b \wedge \mathbf{e}^c \wedge \mathbf{e}^d = \epsilon$, where the Hodge star $*$ is given by $*w = \imath_w \epsilon$. The following relations are useful:

$$\mathbf{e}_j \wedge *\mathbf{e}^{j1} \wedge \mathbf{e}^{j2} \wedge \ldots \wedge \mathbf{e}^{jp} = (-1)^{p+1} * \imath_{\mathbf{e}_j}(\mathbf{e}^{j1} \wedge \mathbf{e}^{j2} \wedge \ldots \wedge \mathbf{e}^{jp}), \tag{2.6}$$

$$\imath_{\mathbf{e}_j} * \mathbf{e}^{j1} \wedge \mathbf{e}^{j2} \wedge \ldots \wedge \mathbf{e}^{jp} = *\mathbf{e}^{j1} \wedge \mathbf{e}^{j2} \wedge \ldots \wedge \mathbf{e}^{jp} \wedge \mathbf{e}^{j}. \tag{2.7}$$

The codifferential is defined as

$$\delta = (-1)^{m(p+1)+1} * \mathbf{d}*, \tag{2.8}$$

where $\mathbf{d}$ is the exterior derivative. The exterior covariant derivative is denoted by $\mathbf{D}$ and

$$\mathbf{D}. = \mathbf{d}. + \omega \wedge ., \quad D_{e_\alpha}. = \imath_{e_\alpha}\mathbf{D}., \tag{2.9}$$

where ω is connection one form. We follow the notation and convention of [16].

Let us consider Maxwell's equations in geometric optics limit. In this limit the wavelength λ of the electromagnetic waves should be much smaller than a typical length L where the amplitude, the polarization and the wavelength of the waves vary and also it should be much smaller than mean radius R of curvature of the spacetime. If this conditions hold, we can treat them as locally plane waves. We introduce a one form

$$\mathbf{A} = \mathrm{Re}(\mathbf{a}'\, e^{i\theta'}), \tag{2.10}$$

$$\mathbf{a}' = a'_\alpha \mathbf{dx}^\alpha, \tag{2.11}$$

where $\mathbf{A}$ is the electromagnetic potential one form. The one form $\mathbf{a}'$ corresponds to amplitude and $a'_\alpha(x)$ is independent of the wavelength and slowly changing function of spacetime position x in the first approximation. The scalar phase function $\theta'(x)$ is changing rapidly. The real scalar amplitude A is given by

$$A = \sqrt{A_\alpha \bar{A}^\alpha} = \sqrt{a'_\alpha \bar{a}'^\alpha}, \tag{2.12}$$

where $\bar{A}_\alpha$ is complex conjugate of A_α. We expand the amplitude $\mathbf{a}' = \mathbf{a} + \epsilon\mathbf{b} + \epsilon^2\mathbf{c} + ...$

and take the phase $\theta' = \theta/\epsilon$ where $\epsilon = \lambda/\mathrm{Min}(L, R)$. We first demand the potential one form to satisfy the Lorentz gauge condition:

$$\delta \mathbf{A} = 0, \tag{2.13}$$

where δ is the codifferential operator. We introduce one forms $\mathbf{k} = \mathbf{d}\theta$ and $\mathbf{f} = \mathbf{a}/A$. Here $\mathbf{k}$ is dual to the wave vector $k = k^\alpha \partial_\alpha$, and $\mathbf{f}$ is dual to the complex unit polarization vector $f = f^\alpha \partial_\alpha$. We find for the terms of order $O(\epsilon^{-1})$ and $O(1)$

$$O(\epsilon^{-1}): \quad k^\alpha a_\alpha = k^\alpha f_\alpha = 0, \quad (\mathbf{k} \wedge *\mathbf{f} = 0), \tag{2.14}$$

$$O(1): \quad k^\alpha b_\alpha = i\partial^\alpha a_\alpha, \quad (\mathbf{k} \wedge *\mathbf{b} = id * \mathbf{a}). \tag{2.15}$$

The first term shows that the polarization vector is perpendicular to the direction of propagation. Second term says that the geometric optics limit is not valid for the terms higher than $O(\epsilon^{-1})$. We now consider the source free Maxwell equations $d\mathbf{F} = 0$ and $\delta\mathbf{F} = 0$ where $\mathbf{F} = d\mathbf{A}$ is Faraday two form related with field strength $\mathbf{F} = 1/2 F_{\alpha\beta} \mathbf{dx}^\alpha \wedge \mathbf{dx}^\beta$. The first equation is satisfied identically whereas the second one, namely the source-free wave equation $\delta\mathbf{F} = 0$ yields the relations:

$$O(\epsilon^{-2}): \quad k_\alpha k^\alpha = 0, \quad (\mathbf{k} \wedge *\mathbf{k} = 0), \tag{2.16}$$

$$O(\epsilon^{-1}): \; k^\beta \partial_\beta a_\alpha = -\frac{1}{2}\partial_\beta k^\beta a_\alpha, \quad \left((\imath_k D)\mathbf{a} = D_k \mathbf{a} = -\frac{1}{2}(Dk)\mathbf{a}\right), \tag{2.17}$$

where we have also used relations (2.14) and (2.15) when deriving these. We can replace partial derivatives with covariant derivatives acting on scalars in these equations. The first equation shows that the wave vector k is null. Second relation gives the propagation equation for vector amplitude a. Let us take the covariant derivative of (2.16):

$$D_\beta(k_\alpha k^\alpha) = 2k^\alpha D_\beta k_\alpha = 0. \tag{2.18}$$

Since $k_\alpha = \partial_\alpha \theta = D_\alpha \theta$ and $D^2\theta = 0$ we can exchange the indices $D_\beta k_\alpha = D_\beta D_\alpha \theta =$

$D_\alpha D_\beta \theta = D_\alpha k_\beta$ which gives propagation equation for wave vector:

$$k^\alpha D_\alpha k_\beta = 0, \quad (D_k \mathbf{k} = 0). \tag{2.19}$$

This equation shows that the paths of light rays are null geodesics.

Now multiplying (2.17) with $\bar{a}^\alpha$, multiplying the complex conjugate of (2.17) with a^α and adding them we obtain

$$k^\beta D_\beta (A^2) = -D_\beta k^\beta A^2, \tag{2.20}$$

where $A^2 = \bar{a}^\alpha a_\alpha$. This equation gives the propagation law for scalar amplitude A:

$$k^\beta D_\beta A = -\frac{1}{2} D_\beta k^\beta A. \tag{2.21}$$

Now replacing $a_\alpha = A f_\alpha$ in (2.17) and using (2.21) we obtain the propagation law for polarization vector f_α:

$$k^\beta D_\beta f_\alpha = 0, \quad (D_k \mathbf{f} = 0). \tag{2.22}$$

Hence the polarization vector f is perpendicular to the light rays and is parallel-propagated along them.

The equation (2.20) can be written as:

$$D_\alpha (A^2 k^\alpha) = 0. \tag{2.23}$$

which shows that $A^2 k^\alpha$ is a covariantly conserved current. This equation can be considered as the law of conservation of photon number. Actually it is an adiabatic invariant where it varies very slowly for $R >> \lambda$ and not conserved in general. The electromagnetic field tensor is given by

$$F = \mathrm{Re}(iAe^{i\theta}\mathbf{k} \wedge \mathbf{f}), \tag{2.24}$$

representing the electromagnetic field type of a null field satisfying

$$F_{\alpha\beta}k^{\beta} = *F_{\alpha\beta}k^{\beta} = 0, \quad \mathbf{F} \wedge \mathbf{F} = \mathbf{F} \wedge *\mathbf{F} = 0, \tag{2.25}$$

where $*F$ is dual to F.

The equations (2.19) and (2.23) imply the form of the energy momentum tensor of null electromagnetic radiation in geometric optics limit as:

$$T_{\alpha\beta} = A^2 k_{\alpha} k_{\beta}. \tag{2.26}$$

The conservation of energy momentum tensor $D_{\alpha}T^{\alpha}_{\ \beta} = 0$ follows from (2.14) and (2.23). This tensor has the same form with (2.3) where $\rho = A^2$. We see that null radiation discussed in the previous section obeys all the properties of null Maxwell field in geometric optics limit except the polarization properties. However, from a solution of pure radiation we can construct a polarization vector f_{α} such that equations (2.14) and (2.22) are satisfied. Hence a solution having pure radiation energy momentum tensor can be originated from an electromagnetic field configuration as well as other fields in geometric optics limit.

We have followed the lines of the book of Misner-Thorne-Wheeler [7] when deriving these formulas and for a more detailed discussion on this topic we refer to this book.

3. CYLINDRICALLY SYMMETRIC SPACETIMES

Cylindrically symmetric spacetime is characterized by two spacelike Killing vectors which generate an Abelian group G_2 [17]. The first one is the generator of the axial symmetry, which of course has closed orbits. At least part of its axis of symmetry belongs to the spacetime unless we consider some exterior solution. The other Killing vector represents translations along the axis. Let us call these Killing vectors as $\eta = \partial_\phi$ and $\xi = \partial_z$. Then general cylindrically symmetric spacetime metric can be written as:

$$ds^2 = e^{-2U}\left(\gamma_{mn}dx^m dx^n + W^2 d\phi^2\right) + e^{2U}\left(dz + Fd\phi\right)^2, \tag{3.1}$$

where the metric is independent of z and ϕ. The 2-metric γ_{mn} can always be chosen as [17], [18]:

$$\gamma_{mn}dx^m dx^n = e^{2K}\left(dr^2 - dt^2\right), \tag{3.2}$$

where the metric functions K, U, W and F are the functions of the radial and the timelike coordinates r and t in general. The global interpretation of this metric to be cylindrically symmetric also requires the existence of an axis where $\eta_\mu \eta^\mu = e^{-2U}W^2 + F^2 e^{2U}$ vanishes and metric is regular at the axis. If one also takes into account the global topological properties of the metric, such as the existence of regular axis and the identification of periodic angular coordinate having the property $\phi + 2\pi \to \phi$, then the allowed coordinate transformations are too restricted and most of the solutions are excluded. However if we consider only local properties of cylindrically symmetry we can have more room for solutions, such as we can have solutions without an axis or solutions where the metric is not regular on the axis. We may consider such solutions as outer metrics of a cylindrical source, whenever we can replace the singular axis by an regular interior solution. Hence, in this thesis in general, we will not consider the global properties of the cylindrical symmetry and only consider its local properties.

If we have a third (timelike) Killing vector ∂_t, then we have a stationary cylindrical spacetime. For this case at least one of the three Killing vectors is hypersurface orthogonal. Then for a stationary cylindrically symmetric spacetime we can consider two different forms of metrics as

$$ds^2 = e^{2(K-U)}(-dt^2 + dr^2) + e^{2U}(dz + Fd\phi)^2 + e^{-2U}W^2 d\phi^2, \tag{3.3}$$

or

$$ds^2 = e^{2(K-U)}(dz^2 + dr^2) - e^{2U}(dt - Fd\phi)^2 + e^{-2U}W^2 d\phi^2, \tag{3.4}$$

where the metric functions are only the functions of r. These two metrics are not equivalent but one can obtain the one from the other by a complex transformation $t \to iz$, $z \to it$, $F \to iF$.

For static cylindrically symmetric spacetimes, we can take $A = 0$ and the metric becomes diagonal. In this thesis we will mainly consider diagonal cases. Note that we can also have diagonal nonstatic (but nonstationary) metrics. Thus, we generally consider the metric

$$ds^2 = e^{2(K-U)}(-dt^2 + dr^2) + e^{2U}dz^2 + e^{-2U}W^2 d\phi^2. \tag{3.5}$$

This form of the metric is also called the canonical form or the Weyl form or a metric in Levi-Civita gauge. This metric can also be written in the form

$$ds^2 = -e^{2U}dt^2 + e^{2(K-U)}(dr^2 + dz^2) + e^{2U}W^2 d\phi^2. \tag{3.6}$$

which can be derived from (3.5) either by a transformation $t \to iz$, $z \to it$ or by a redefinition of r. We can also write cylindrically symmetric metric in the Kasner gauge as:

$$ds^2 = -A^2(r,t)dt^2 + dr^2 + B^2(r,t)dz^2 + C^2(r,t)d\phi^2. \tag{3.7}$$

We will frequently use any of these form of the metrics whenewer appropriate. In most of the calculations we will consider only the static cases where the metric functions are only the functions of the radial coordinate r.

3.1. Vacuum Solutions and Einstein-Rosen Waves

In order to derive the vacuum solutions, we need the curvature components of the above metrics. Here we consider $F = 0$ case. Considering an orthonormal basis $\mathbf{e}^0 = e^{K-U}\mathbf{dt}$, $\mathbf{e}^1 = e^{K-U}\mathbf{dr}$, $\mathbf{e}^2 = e^{U}\mathbf{dz}$ and $\mathbf{e}^3 = e^{-U}W\mathbf{d}\phi$, one finds the Ricci tensor and Ricci scalar of the metric (3.5) as:

$$
\begin{aligned}
R_{00} &= e^{2(U-K)}\left[(\dot{K}+\dot{U})\frac{\dot{W}}{W} - \frac{\ddot{W}}{W} + (\acute{K}-\acute{U})\frac{\acute{W}}{W} - (2\dot{U}^2 + \ddot{K} - \ddot{U} - K'' + U'')\right], && (3.8)\\
R_{11} &= e^{2(U-K)}\left[(\acute{K}+\acute{U})\frac{\acute{W}}{W} - \frac{W''}{W} + (\dot{K}-\dot{U})\frac{\dot{W}}{W} - (2U'^2 + K'' - U'' - \ddot{K} + \ddot{U})\right], && (3.9)\\
R_{22} &= \frac{e^{2(U-K)}}{W}\left[\dot{U}\dot{W} - \acute{U}\acute{W} + W(\ddot{U} - U'')\right], && (3.10)\\
R_{33} &= \frac{e^{2(U-K)}}{W}\left[-\dot{U}\dot{W} + \acute{U}\acute{W} + \ddot{W} - W'' + W(U'' - \ddot{U})\right], && (3.11)\\
R_{01} &= \frac{e^{2(U-K)}}{W}\left[-\dot{W}\acute{K} - \dot{K}\acute{W} + 2W\dot{U}\acute{U} + \dot{W}'\right], && (3.12)\\
R &= \frac{2e^{2(U-K)}}{W}\left[\ddot{W} - W'' + \acute{U}\acute{W} - \dot{U}\dot{W} + W(\dot{U}^2 - \acute{U}^2 + \ddot{K} - K'' - \ddot{U} + U'')\right]. && (3.13)
\end{aligned}
$$

Here overdot and prime denote partial derivatives with respect to t and r. Let us solve these for $R_{\mu\nu} = 0$. From adding Equations $(3.11) and (3.10)$ we get a wave equation for W:

$$\ddot{W} - W'' = 0, \tag{3.14}$$

which has a general solution of the form $W = f(t-r) + g(t+r)$. Constant W gives flat spacetime. We can take W as r if the gradient of W is spacelike, t if it is timelike, or null. For $W = r$, (3.10) gives a cylindrical wave equation for U:

$$U'' + \frac{U'}{r} - \ddot{U} = 0. \tag{3.15}$$

The remaining field equations give a line integral for K:

$$K = \int \left[r(U'^2 + \dot{U}^2)dr + 2rU'\dot{U}dt \right]. \tag{3.16}$$

These solutions first introduced by Beck [19], are first interpreted by Einstein and Rosen as cylindrical gravitational waves [20]. They are characterized by the existence of two hypersurface orthogonal (which means $F = 0$) spacelike Killing vectors ∂_z, ∂_ϕ. Since cylindrically symmetric sources cannot be localized on a certain direction, these Einstein-Rosen waves cannot describe the exterior regions of bounded radiating sources. However, these solutions and their generalizations have served as a good model for different issues such as the energy loss due to gravitational waves, the asymptotic structure of radiative spacetimes, cosmic censorship, and quantum gravity, among others.

3.1.1. Stationary Vacuum Solutions

The general cylindrically symmetric stationary vacuum solution was found by Lewis [21]. For the form of the metric (3.4) it reads [17]:

$$ds^2 = f^{-1}\left(e^{2K}(dr^2 + dz^2) + r^2 d\phi^2\right) - f(dt + Ad\phi)^2, \tag{3.17}$$

$$f = r\left(ar^n + br^{-n}\right), \quad n^2 ab = -C^2, \tag{3.18}$$

$$A = \frac{C}{nb}\frac{r^n}{ar^n + br^{-n}} + B, \quad f^{-1}e^2k = r^{(n^2-1)/2}. \tag{3.19}$$

This metric has two subclasses. If the all metric coefficients are real then it is called as Weyl class. If n is imaginary (the other constants can be real or imaginary but metric must be real) then it is called as Lewis class. The Lewis metric with Weyl class are locally static and locally equivalent to the Levi-Civita metric [22],[23] although they have different global behaviour.

4. REVIEW OF STATIC VACUUM SOLUTION: THE LEVI-CIVITA SPACETIME

4.1. The Levi-Civita Spacetime in Different Forms

The static cylindrically symmetric vacuum solution, the Levi-Civita metric [5], is one of the earlier solutions of general relativity (1919). Unlike the spherical vacuum solution, this solution is not the unique vacuum solution in cylindrical symmetry since there are stationary vacuum solutions (Lewis metrics [21]) or nonstatic ones. However, it is the only static diagonal vacuum solution. Here we will derive it in different forms.

For the static case where the metric functions K, U, W are the functions of single coordinate r, the R_{01} term vanishes identically and the other components of the Ricci tensor become:

$$R_{00} = \frac{e^{2(U-K)}}{W}\left[(K'-U')W' + W(K''-U'')\right], \tag{4.1}$$

$$R_{11} = \frac{e^{2(U-K)}}{W}\Big[(K'+U')W' - W'' + W(-2U'^2 - K'' + U'')\Big], \tag{4.2}$$

$$R_{22} = \frac{e^{2(U-K)}}{W}\left[-U'W' - U''W\right], \tag{4.3}$$

$$R_{33} = \frac{e^{2(U-K)}}{W}\left[U'W' - W'' + U''W\right], \tag{4.4}$$

$$R = \frac{2e^{2(U-K)}}{W}\Big[-W'' + U'W' + W(-U'^2 - K'' + U'')\Big]. \tag{4.5}$$

$R_{22} + R_{33} = 0$ immediately gives $W'' = 0$ which means that $W \sim r$. Choosing $W = r$ and replacing back to R_{22} equation we get $U = k \ln r + \ln U_0$ where k, U_0 are constants. From $R_{00} + R_{11} = 0$ we find $K = k^2 \ln r + \ln K_0$ where K_0 is a constant. Replacing these into the metric and redefining the coordinates t, r, z in order to get rid of the unrelated constants, the static vacuum solution results to the metric of the form:

$$ds^2 = r^{2(k^2-k)}(-dt^2 + dr^2) + r^{2k}dz^2 + \alpha^2 r^{2(1-k)}d\phi^2. \tag{4.6}$$

Here we cannot set the constant α to unity by a coordinate redefinition if we consider the coordinate ϕ as an angular coordinate with the range $0 < \phi \leq 2\pi$, since any redefinition will change this range. Thus, it is obvious that the Levi-Civita metric has two constant parameters k and α.

Actually, the Levi-Civita metric is written by different authors in very different forms. The most familiar form can be derived starting from the metric (3.6) and it is of the form:

$$ds^2 = -\rho^{4\sigma}dt^2 + \rho^{4\sigma(2\sigma-1)}(d\rho^2 + dz^2) + Q^2\rho^{2(1-2\sigma)}d\phi^2. \tag{4.7}$$

Here ρ is again a radial coordinate, t is the time, z is the axial and ϕ is the angular coordinate with ranges $-\infty < t, z < \infty$, $0 \leq \rho < \infty$, $0 < \phi \leq 2\pi$ and Q is a new constant constructed from α and other terms coming from the transformation given below. This metric can be derived from the metric (4.6) either by a substitution $t \to iz$, $z \to it$, $k \to 2\sigma$, $r \to \rho$, or by considering the coordinate transformations below. We first make a coordinate transformation with $\rho = r^K/K$ with $K = k^2 - k + 1$ which brings the metric into the form

$$ds^2 = -(K\rho)^{2k(k-1)/K}dt^2 + d\rho^2 + (K\rho)^{2k/K}dz^2 + (K\rho)^{2(1-k)/K}\alpha^2 d\phi^2. \tag{4.8}$$

Introducing a new parameter $s = k/2(k-1)$ and making another coordinate transformation $\rho = (Nr)^{1/N}$ with $N = 4s^2 - 2s + 1$ we arrive to the original Levi-Civita solution (4.7).

The vacuum solution can also be written in Kasner form which will be useful for our later discussions. To derive it we will consider the metric in Kasner gauge (3.7). It can be written in an orthonormal basis $ds^2 = \eta_{ab}\mathbf{e}^a\mathbf{e}^b$ with $\mathbf{e}^0 = A\mathbf{dt}$, $\mathbf{e}^1 = \mathbf{dr}$, $\mathbf{e}^2 = B\mathbf{dz}$, $\mathbf{e}^3 = C\mathbf{d}\phi$ and η_{ab} is the Minkowski metric. The nonzero components of the

Einstein tensor in this frame is given as:

$$G_{00} = -(\frac{B_{rr}}{B} + \frac{C_{rr}}{C} + \frac{B_r C_r}{BC}), \tag{4.9}$$

$$G_{11} = \frac{A_r B_r}{AB} + \frac{A_r C_r}{AC} + \frac{B_r C_r}{BC}, \tag{4.10}$$

$$G_{22} = \frac{A_{rr}}{A} + \frac{C_{rr}}{C} + \frac{A_r C_r}{AC}, \tag{4.11}$$

$$G_{33} = \frac{A_{rr}}{A} + \frac{B_{rr}}{B} + \frac{A_r B_r}{AB}. \tag{4.12}$$

Let us solve these for $G_{ab} = 0$. We first define $U(r) = ABC$. From the combinations $G_{00} + G_{11} + G_{22} + G_{33}$, $-G_{00} + G_{11} - G_{22} + G_{33}$, $-G_{00} + G_{11} + G_{22} - G_{33}$, one obtains

$$\left(\frac{A'}{A}U\right)' = 0, \ \left(\frac{B'}{B}U\right)' = 0, \ \left(\frac{C'}{C}U\right)' = 0. \tag{4.13}$$

Combining these we get

$$U'' = 0. \tag{4.14}$$

This gives $U = U_0 r$, where U_0 is a constant. Using this, we can integrate the equations (4.13) as:

$$A = A_0 r^a, \ B = B_0 r^b, \ C = C_0 r^c, A_0 B_0 C_0 = U_0. \tag{4.15}$$

From the definition of U and from the remaining field equation (4.10) we finally obtain the so called Kasner conditions:

$$a + b + c = 1, \quad a^2 + b^2 + c^2 = 1. \tag{4.16}$$

These gives the cylindrically symmetric, static Levi-Civita metric in the Kasner form:

$$ds^2 = -r^{2a}dt^2 + dr^2 + r^{2b}dz^2 + C_0^2 r^{2c} d\phi^2. \tag{4.17}$$

Here by rescaling the coordinates t, z we have set the constants of integration A_0, B_0

to unity. In this solution we have four constants a, b, c, C_0 and two relations for a, b, c, thus the solution has only two independent constants. Now let us discuss the relation between the conventional form (4.7) of the Levi-Civita metric and its Kasner form (4.17). Transforming the radius ρ into a proper radius r by defining

$$dr = \rho^{2\sigma(2\sigma-1)} d\rho, \tag{4.18}$$

puts the Levi-Civita metric (4.7) in its gaussian normal form [31]:

$$ds^2 = -R^{4\sigma/N} dt^2 + dr^2 + R^{4\sigma(2\sigma-1)/N} dz^2 + Q^2 R^{2(1-2\sigma)/N} d\phi^2, \tag{4.19}$$

where

$$\rho = R^{1/N}, \quad R = Nr, \quad N = 4\sigma^2 - 2\sigma + 1. \tag{4.20}$$

Actually, one can easily realize that, by considering the definitions

$$a = \frac{2\sigma}{N}, \quad b = \frac{-2\sigma(1-2\sigma)}{N}, \quad c = \frac{1-2\sigma}{N}, \tag{4.21}$$

and rescaling the coordinates t, z, this metric can be written as vacuum Levi-Civita metric in Kasner form (4.17). The dependence of a, b and c on σ is shown in Figure 4.1. Since there are so called Kasner conditions between the constants a, b, c, only one

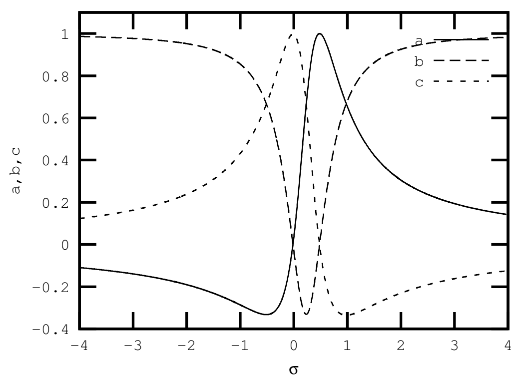

Figure 4.1. The dependence of a, b, c on σ

of them is actually free. Choosing the free parameter as b, a and c are given by

$$a, c = \frac{(1-b) \pm \sqrt{(1-b)(1+3b)}}{2}, \tag{4.22}$$

where $-\frac{1}{3} \leq b \leq 1$. For every value of b, a and c can take two values. This is shown in Figure 4.2. In general, the choice of b as free parameter is arbitrary, so a, b, c can take values in between ($-\frac{1}{3} \leq a, b, c \leq 1$) but only one of them can be negative at the same time. Note that, the Riemann tensor vanishes only for the case where one of the parameters is equal to one and the others are equal to zero.

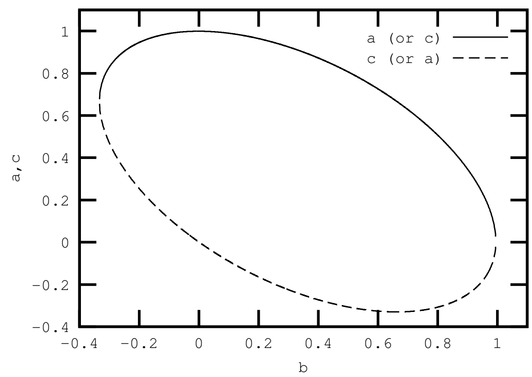

Figure 4.2. The parameters a and c with b as a free parameter

4.2. The Physical Meanings of the Parameters of the Levi-Civita Metric

As we have stressed several times, the vacuum Levi-Civita metric has two independent constant parameters, which cannot be removed by a coordinate transformation. Since we have considered several different forms for this vacuum solution, we have used different letters for these parameters. For the metric (4.6) we have k, α, for (4.7) we have σ, Q and for (4.17) we have the one of the constants a, b, c and C_0. Among these, the constants α (or Q or C_0) is the conicity parameter and related with the global topology of the spacetime. For example when $\sigma = 0$ in (4.7), the metric reduces to the famous straight cosmic string metric [13],

$$ds^2 = -dt^2 + d\rho^2 + dz^2 + Q^2\rho^2 d\phi^2, \tag{4.23}$$

which describes a locally flat metric but its global properties are not trivial. It has a conical singularity at the symmetry axis. If we consider a coordinate transformation $\phi' = Q\phi$ then the metric becomes flat metric in cylindrical coordinates

$$ds^2 = -dt^2 + d\rho^2 + dz^2 + \rho^2 d\phi'^2. \tag{4.24}$$

However, the range of the angular coordinate is $0 < \phi' \leq 2\pi Q$, which means that we have an angular deficit. When we travel one tour around any $z =$ const, $r =$ const. hypersurface, we spend an angle $2\pi Q$. Thus, unless these conicity constants equivalent to unity, these vacuum solutions contain an angular deficit parameter. This property of cosmic strings has observational implications on Astrophysics [13]. Especially, it may lead to double images of distant objects if there is such a string between the object and the observer.

The other parameter k (or σ or one of a, b, c) is the puzzling one. It is obviously related with the mass energy density of the source generating such spacetime. However, the relation has some confusing properties. Let us consider the metrics (4.7) and (4.17) for the moment. The Newtonian limit of the metric in Kasner form (4.17) can be found considering $a = q \ll 1$. Using the constraints (4.16) on b & c with:

$$b + c = 1 - q, \quad b^2 + c^2 = 1 - q, \tag{4.25}$$

we see that there are two limits for b, c in which one of them goes to zero and the other to one. To identify the angular coordinate correctly we have to choose $b \to 0$ and $c \to 1$. Then

$$\begin{aligned} r^a &= 1 + q\ln(r) + O(k^2), && (4.26)\\ r^b &= 1 - q\ln(r) + O(q^2), && (4.27)\\ r^c &= r + O(q^2). && (4.28) \end{aligned}$$

Hence one obtains

$$ds^2 = -\left(1 + q\ln(r)\right)^2 dt^2 + dr^2 + (1 - q\ln(r))^2 dz^2 + \alpha^2 r^2 d\phi^2. \tag{4.29}$$

Calculating the Einstein tensor for this metric verifies that it is zero up to order q^2. Thus one can identify the Newtonian potential per unit mass as $V = q\ln r$. For positive q this corresponds to an attractive force field while for negative q this corresponds to a repulsive force field $\overrightarrow{g}$,

$$\overrightarrow{g} = -\overrightarrow{\nabla} V = -q\frac{\overrightarrow{r}}{r^2}. \tag{4.30}$$

Thus, when a (or σ) is positive we have a source with positive energy density and it applies an attractive force on a test particle whereas when it is negative we have a source with negative energy density and it applies a repulsive force. Also, the strength of this force is proportional to a (σ) in the Newtonian limit. However, these properties dramatically change when we go beyond the Newtonian limit.

The existence of spacetime singularities of a metric can be identified if one considers the scalars constructed from the curvature components of that metric: such as Ricci scalar, the contraction of Ricci or Riemann tensors with themselves, etc. For a vacuum spacetime the first two of these identically vanish, thus we have to calculate the Kretchmann scalar for the metric (4.7),

$$\mathcal{K} = R_{abcd}R^{abcd} = \frac{64\sigma^2(1-2\sigma^2)(1-2\sigma+4\sigma^2)}{r^{4+8\sigma(2\sigma-1)}}. \tag{4.31}$$

It is obvious that we have a spacetime singularity at $r = 0$ unless we have $\sigma = 0, 1/2, \pm\infty$. The strength of the singularity increases when σ increases form 0 to 1/4, and gets a maximum at 1/4, then it starts to decrease by increasing σ and it vanishes for $\sigma = 1/2$. Then it again increases by increasing σ. Thus we have intervals of σ where the strenght of the singularity decreases by increasing mass parameter, σ. This is rather confusing, and makes it difficult to relate σ as proportional to mass density of the source at least for some interval of σ, although there is a relation between them. This confusing behaviour may be due to the fact that the Kretchmann scalar is not a good measure of the gravitational force caused by the source and we have to analyze the behaviour of test particles in this spacetime. The radial acceleration of a test particle initially at rest in the coordinate system of (4.7) is given by (here dot represents the derivative with respect to proper time):

$$\ddot{\rho} = -\frac{2\sigma}{\rho^{1+4\sigma(2\sigma-1)}}, \tag{4.32}$$

and for small σ we have

$$\ddot{\rho} = -\frac{2\sigma}{\rho}, \tag{4.33}$$

which has the same form as the Newtonian expression for the acceleration of a test particle which has a distance r from a line mass of mass per unit length 2σ and consistent with (4.30). Thus, the interpretation that the Levi-Civita spacetime represents the gravitational field of an infinitely long static cylindrical source is clear for small σ where the Newtonian approximation is valid. Also, there are cylindrical sources which could be smoothly matched to the Levi-Civita spacetime for different ranges of σ. Static dust cylinders composed of equal amounts of dust rotating in opposite directions around the axis with zero net angular momentum and matching to the Levi-Civita metric have been found for the range $0 \leq \sigma < 1/4$ [24], [10]. Static perfect fluid cylinders are also considered and a perfect fluid smoothly matching to the exterior with the above range is found in [25], whereas this range is extended to $0 \leq \sigma < 1/2$ by perfect fluid sources of [26], [27], [28]. Thus, at least for this range, the interpretation that the Levi-Civita spacetime represents the exterior field of a static cylindrical source is acceptable. If we analyze the radial acceleration of test particles at rest for the metric in Gaussian normal form, we get

$$\ddot{r} = -\frac{2\sigma}{(4\sigma^2 - 2\sigma + 1)\, r}. \tag{4.34}$$

The behaviour of the radial acceleration for a test particle initially at a given radius for changing σ is given in Figure (4.3). For positive σ the source on the axis applies an attractive force to the particle whereas for negative σ it applies a repulsive force. Thus, for positive σ the source has positive energy density and for negative σ it has negative energy density. In the interval $0 \leq \sigma \leq 1/2$ it increases smoothly and gets its maximum value for $\sigma = 1/2$ where the spacetime is actually flat! When σ increases further, the force decreases and goes to zero when σ goes to infinity. For negative σ there is a similar behaviour. The repulsive force increases when σ goes from 0 to $-1/2$ and becomes maximum. Then it starts to decrease when σ goes to $-\infty$.

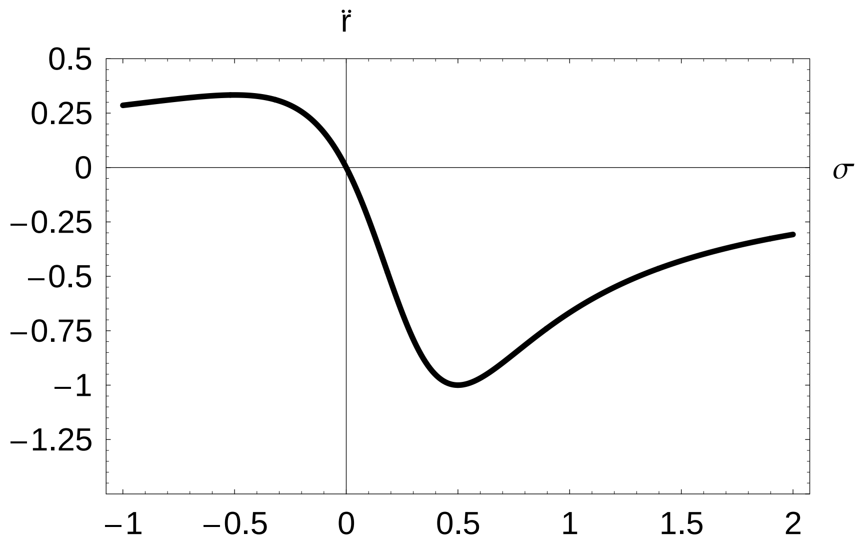

Figure 4.3. The radial acceleration of a test particle in Levi-Civita spacetime

The $\sigma = 1/2$ case needs further discussion. The Levi-Civita metric has three Killing vectors, ∂_t, ∂_z, ∂_ϕ. For $\sigma = \pm 1/2$ we have another Killing vector $\phi\partial_z - z\partial_\phi$. For $\sigma = 1/2$ the metric (4.7) becomes:

$$ds^2 = -\rho^2 dt^2 + d\rho^2 + dz^2 + Q^2 d\phi^2. \tag{4.35}$$

This metric is known as the Rindler metric [29] when the coordinates are Cartesian coordinates. This spacetime is flat, has uniform gravitational field similar to a massive plane in Newtonian theory. The test particles are accelerated in this field. This extreme case happens when the gravitational force on a test particle, hence the mass density of the source, becomes maximum. Thus, these properties suggest that the Levi-Civita spacetime can describe an infinite plane (a cylinder with infinite radius?) for $\sigma = 1/2$. This fact been suggested by different authors. For $\sigma = -1/2$ the metric (4.7) becomes

$$ds^2 = -\rho^{-2}dt^2 + \rho^4(d\rho^2 + dz^2 + Q^2 d\phi^2). \tag{4.36}$$

This metric is the Taub's plane symmetric metric [30] written in different coordinates. This metric is not flat, has a negative energy density and represents a plane symmetric spacetime. Hence, there is an agreement that Levi-Civita metric can describe planes rather than cylinders for its certain values ($\sigma = \pm 1/2$). Actually, Philbin [27] concluded that for $|\sigma| < 1/2$ Levi-Civita metric describes a spacetime generated by a cylindrical source and presented a perfect fluid source smoothly matching to the Levi-Civita spacetime. He also suggested that for $|\sigma| \geq 1/2$ it represents the exterior field of an infinite plane mass homogenous spacetime and he presented a perfect fluid thick wall with positive mass density and smoothly matching to the exterior Levi-Civita spacetime for $\sigma \geq 1/2$; and a perfect fluid thick wall with negative energy density and again smoothly matching to the exterior Levi-Civita metric with $\sigma \leq -1/2$. This suggestion, however is not conclusive and there is another idea for the interpretation of the metric for $\sigma > 1/2$.

Herrera et. al. [31] considered the vacuum solution without identifying the coordinates as angular or axial at the beginning, since the solutions of the Einstein equations are local and one cannot say anything about the topology of the spacetime unless one considers global properties such as the existence of regular axis, or asymptotic flatness, etc. Thus, they present the vacuum Levi-Civita solution as:

$$ds^2 = -\rho^{4\sigma}dt^2 + \rho^{4\sigma(2\sigma-1)}(d\rho^2 + P^2dp^2) + Q^2\rho^{2(1-2\sigma)}dq^2, \tag{4.37}$$

where t and ρ are the usual time and the radial coordinates and σ, P and Q are constants. The behaviour of the metric components determines the nature of the coordinates p and q. One of the coordinates p and q has to be interpreted as the axis of symmetry and the other as the angle about this axis. If we fix the angular coordinate to a finite range then one of the constants P and Q can be transformed away by a scale transformation depending on the behaviour of the coordinates p and q. This leaves the metric with only two independent parameters with the interpretations given above. The idea of Herrera et. al. is that the Levi-Civita metric describes the cylindrically symmetric vacuum spacetime for the range $0 \leq \sigma < \infty$ provided that at $\sigma = \frac{1}{2}$ the coordinates p and q interchange their roles, that is, for $0 \leq \sigma < \frac{1}{2}$, q is the axial (z) coordinate and p is the angular (ϕ) coordinate, and for $\sigma > \frac{1}{2}$, p becomes the angular coordinate and q becomes the z coordinate. For $\sigma = \frac{1}{2}$, neither p nor q is entitled to be an angular coordinate, and the three coordinates (r, p, q) are better visualized as Cartesian coordinates x, y and z [32]. The $\sigma < 0$ case is isomorphic to the $\sigma > 0$ case. They also presented a thin shell source satisfying some physical requirements with this changing of the role of the coordinates.

Let us discuss this interpretation for the Kasner form (4.17) of the metric which manifests itself into the relations between σ and a, b, c. When $0 < \sigma < \frac{1}{2}$, $b < 0, c > 0$ and when $\sigma > \frac{1}{2}$, $b > 0, c < 0$. Since for σ positive one always has $a \geq 0$, (Figure 4.1) one concludes that for the Kasner form of the metric (4.17), for positive σ, the coordinate corresponding to the parameter whose value is negative has the role of z coordinate and whose value is positive has the role of angular coordinate of cylindrical coordinates. To avoid confusion, for the rest of the thesis, either we should consider only $0 \leq \sigma \leq 1/2$ for which b is always negative or we should consider $0 \leq \sigma < \infty$ for which b and c change sign at $\sigma = 1/2$ (Figure 4.1) if we follow the interpretation of Herrera et al. For negative σ, same change of the roles of the coordinates may occur at $\sigma = -1/2$, since for small σ, $r^{2c} \approx r^2$ and for large σ, $r^{2b} \approx r^2$.

In summary, for the range $|\sigma| \leq 1/2$ the generally accepted interpretation of the Levi-Civita spacetime is the following: For $0 < \sigma < 1/2$, the Levi-Civita spacetime describes an exterior region of a static cylindrically symmetric source (an infinite line mass- line singularity), for $\sigma = 0, \alpha \neq 1$ it is locally flat cosmic string spacetime, for $\sigma = 0, \alpha = 1$ it is Minkowski spacetime and for $\sigma = 1/2$ a planar source (flat spacetime in accelerated coordinates). For negative values of the Levi-Civita spacetime, the only change in the interpretation is the negative energy density of the source and for $\sigma = -1/2$ spacetime is again planar but not flat. However for the $|\sigma| > 1/2$ as we have discussed there is no generally accepted interpretation and one must be careful to use the term cylindrically symmetric for these values of σ for the Levi-Civita metric.

4.3. Circular Geodesics of Levi-Civita Spacetime

Let us first study the equations of a test particle following a circular geodesics in the spacetime of (4.7). A more detailed discussion can be found in [10]. Let us denote the angular velocity of a particle moving along a geodesics as $\omega = d\phi/dt$ and its tangential velocity as $W^\mu = (0, 0, 0, W^\phi)$ with $W^\phi = \omega/\sqrt{-g_{tt}}$, then we have (here t is the time coordinate, and dot represents derivative with respect to proper time τ):

$$\left(\frac{ds}{dt}\right)^2 = -\rho^{4\sigma} + Q^2\rho^{2(1-2\sigma)}\left(\frac{d\phi}{dt}\right)^2, \tag{4.38}$$

$$\omega^2 = \left(\frac{\dot{\phi}}{\dot{t}}\right)^2 = \frac{2\sigma}{Q^2(1-2\sigma)}\rho^{2(4s-1)}, \tag{4.39}$$

$$\ddot{\tau} = 0, \quad \dot{r} = 0, \quad \dot{z} = 0. \tag{4.40}$$

Then,

$$W^2 = \frac{2\sigma}{1-2\sigma}. \tag{4.41}$$

Replacing this into (4.38) we get:

$$\left(\frac{ds}{dt}\right)^2 = \left(W^2 - 1\right)\rho^{4\sigma}, \tag{4.42}$$

which shows that the circular geodesics are timelike for $W < 1$ ($0 < \sigma < 1/4$), spacelike for $W > 1$ ($\sigma > 1/4$) and null for $W = 1$ ($\sigma = 1/4$). Hence, a particle in the Levi-Civita spacetime can follow circular geodesics for $\sigma < 1/4$ and even light can follow such geodesics for $\sigma = 1/2$. For the Kasner form of the metric (4.17), the circular goedesics are timelike for $a > c$, spacelike for $a > c$ and lightlike for $a = c = 2/3, b = -1/3$ since $W^2 = a/c$ for this form of the metric. Up to this point we have not considered a possible change of the role of the coordinates ϕ and z for $\sigma = 1/2$. If it happens for

$\sigma > 1/2$ as we have discussed in the previous subsection, then for $1/2 < \sigma < 1$ the circular geodesics are spacelike, for $\sigma > 1$ they are timelike and for $\sigma = 1$ it is lightlike, otherwise all geodesics are spacelike for $\sigma > 1/4$.

The Levi-Civita metric also admits helical null geodesics for a certain range of σ with the angular velocity ω and the pitch velocity v (the constant velocity along z direction of the particles following a helical path). For the Kasner form of the metric (4.17), they are given by

$$\omega^2 = (\frac{a-b}{c-b}) \frac{1}{\alpha^2 r_0^2} (\frac{r}{r_0})^{2(a-c)}, \tag{4.43}$$

$$v^2 = (\frac{c-a}{c-b}) (\frac{r}{r_0})^{2(a-b)}. \tag{4.44}$$

For $b < 0$ ($\sigma < 1/2$), since $|c| \geq |b|$, there exist helical null geodesics only for $c \geq a$ ($0 < \sigma \leq 1/4$) and for $a = c = 2/3$ ($\sigma = 1/4$) they become circular. For $c < 0$, $b > 0$ ($\sigma > 1/2$), there is no such geodesics if no change of the role of the coordinates happens. However if there exists such change, then for the helical null geodesics, which are circular for $a = b = 2/3$ ($\sigma = 1$), exist for $b \geq a$ ($\sigma \geq 1$) (Figure 4.1).

The existence of the circular and helical geodesics which light can follow is one of our motivations to look for solutions of the Einstein equations describing an infinitely thin or thick shell composed of counter circulating or following a helical path with massless particles. Also, as we have said, in the Newtonian limit, we have a force field $\overrightarrow{g} = -\overrightarrow{\nabla} V = -k\frac{\overrightarrow{r}}{r^2}$. A nonrelativistic particle in a circular orbit in this force field has centripetal acceleration,

$$\frac{k}{r} = \frac{v^2}{r}, \tag{4.45}$$

so that the particle moves with constant velocity independent of the radius. It is plausible that this generalizes to light which always moves with constant velocity.

4.4. Trapping of Photons by a Line Singularity

Let us investigate whether outgoing light rays are trapped in the gravitational field of a line singularity or can escape to radial infinity. Actually the answer is already known [11]. For the metric (4.7), by simple calculations the null geodesics for $z =$ const. plane has:

$$\begin{aligned} \dot{t} &= p\,\rho^{-4\sigma}, && (4.46)\\ \dot{\phi} &= q\,Q^{-2}\,\rho^{-2(1-2\sigma)}, && (4.47) \end{aligned}$$

where p, q are constants. Thus, we have

$$\frac{dt}{d\phi} = \left(\frac{p}{q}\right) Q^2 \rho^{2(1-4\sigma)}. \tag{4.48}$$

Putting this into the metric (4.7) for $z =$ const. and remembering that for light $ds^2 = 0$, we get

$$\left(\frac{d\rho}{d\phi}\right)^2 = Q^2 \rho^{2-8\sigma^2} \left[\frac{p^2}{q^2}\,Q^2\,\rho^{2-8\sigma} - 1\right]. \tag{4.49}$$

In order the light rays to escape to infinity without being trapped, we need $(p\,Q/\,q)^2 > \rho^{-2+8\sigma}$. If at some stage $d\rho/d\phi = 0$, then the photons turn back and may be trapped in the gravitational field of the line source generating the Levi-Civita spacetime. For $\sigma > 1/4$, the photons are always trapped since $\lim_{\rho\to\infty} \rho^{8\sigma-2} = \infty$ unless $q = 0$ which corresponds to purely radial motion $\dot{\phi} = 0$. For $\sigma < 1/4$ if photons emitted from a surface in any direction, then it can escape to infinity since $8\sigma - 2$ is negative and $(p\,Q/\,q)^2$ is always bigger than $\rho^{2-8\sigma}$ if it is bigger for a $\rho = \rho_0$.

In summary, the photons emitted from the surface of a cylindrically symmetric source generating Levi-Civita spacetime are trapped due to the gravitational field of this spacetime when the parameter related with the mass density of the source exceeds certain value ($\sigma > 1/4$). Together with the existence of helical and circular null geodesics in this spacetime, this fact motivates us to study the possibility that such trapped photons can produce some photonic structures such as thin or thick shells composed of photons surrounding such line sources? Or even are there exist such shells with locally flat interiors and generating exterior metric? However before trying to answer these questions we first investigate time dependent vacuum solutions and their generalizations to the spacetimes which describe emitting gravitational and null radiation from cylindrical sources.

5. KASNER GENERALIZATION OF LEVI-CIVITA SPACETIME

As we have discussed in the introduction, one of the most important differences of the spherically and the cylindrically symmetric vacuum solutions of general relativity is that, according to the Birkhoff theorem, there is a timelike Killing vector in the spherically symmetric vacuum solution. Thus, it can be said that the spherically symmetric vacuum is necessarily static. Also there is no degrees of freedom giving rise to gravitational waves in spherically symmetric vacuum. However, the situation drastically changes when we consider the cylindrically symmetric systems since there is no analogue of Birkhoff's theorem in cylindrical symmetry. Cylindrically symmetric spacetimes also admit gravitational wave degrees of freedom and during the gravitational collapse of a cylindrically symmetric system, gravitational waves can be produced and the exterior region of a collapsing cylindrical body is not static [33]-[37]. This fact has important consequences in the studies of gravitational waves, cosmological models, quantum gravity and numerical relativity.

We have rewieved the static and nonstatic vacuum solutions in the previous chapters. For further discussion we refer to the book of Stephani et. al. [17] and references therein. Furthermore, the time dependent cylindrically symmetric nonvacuum solutions of Einstein equations were studied for different cylindrical systems. An expanding cylindrical radiation filled universe [38], a radiation universe with heat and null radiation flow [39], nonstatic cosmic strings with a time dependent vacuum exterior [40]-[44], nonstatic global strings [45] are some examples of such solutions. Some of these solutions have an interesting property that their exterior vacuum solutions correspond to some particular values of the parameters of the Levi-Civita metric having also a Kasner type time dependence. This fact motivates us to study the Kasner generalization of the Levi-Civita solution with the full range of its parameters [46]. Thus in this chapter, we will study the properties of cylindrically symmetric time dependent vacuum solutions in Kasner form. This solution can be considered as a Kasner generalization of the Levi-Civita solution since for every constant time slice it reduces to the Levi-Civita solution. These kind of generalized Kasner solutions having more than one variable are known and studied by different authors [47]-[49]. This solution is also equivalent to the Einstein-Rosen soliton wave solutions [50]-[52] by a coordinate transformation. Although this solution is well known, we will establish a direct relation between the parameters of the Levi-Civita solution with the parameters of its Kasner generalization. We will also perform a detailed comparison of the Levi-Civita solution and its nonstatic Kasner generalization by studying their singularity behaviour, geodesic structure and radial acceleration of test particles in these spacetimes.

5.1. Levi-Civita-Kasner Solution

Let us recall the nonstatic and nonstationary cylindrically symmetric metric in the form:

$$ds^2 = e^{2(K-U)}(-dt^2 + dr^2) + e^{2U}dz^2 + e^{-2U}W^2 d\phi^2, \tag{5.1}$$

and let us consider the following ansatz for the functions of the metric:

$$W = \alpha(c_1 r + c_2)(c_3 t + c_4), \tag{5.2}$$

$$U = k\ln(c_1 r + c_2) + q\ln(c_3 t + c_4), \tag{5.3}$$

$$K = k^2\ln(c_1 r + c_2) + q^2\ln(c_3 t + c_4), \tag{5.4}$$

where $k,\ q,\ \alpha$ and c_i's are constants. Here, when $c_3 = 0,\ c_4 \neq 0,\ c_1 \neq 0$ we get the Levi-Civita solution of the form:

$$ds^2 = r^{2(k^2-k)}(-dt^2 + dr^2) + r^{2k}dz^2 + \alpha^2 r^{2(1-k)}d\phi^2, \tag{5.5}$$

where we have rescaled the coordinates r, t and z. As we have demonstrated in the Section (4.1), one can recover the conventional form of the Levi-Civita solution (4.7) by applying the coordinate transformations:

$$R = \frac{r^{\kappa}}{\kappa}, \; R = \frac{\rho^{S}}{S}, \; \kappa = k^2 - k + 1, \; S = 4\sigma^2 - 2\sigma + 1, \; \sigma = \frac{k}{2(k-1)}. \tag{5.6}$$

When we choose $c_1 = 0$, $c_2 \neq 0$, $c_3 \neq 0$ the solution reduces to well known vacuum Kasner solution [53]:

$$ds^2 = t^{2(q^2-q)}(-dt^2 + dr^2) + t^{2q} dz^2 + t^{2(1-q)} d\phi^2, \tag{5.7}$$

where q is a real constant. For this case the coordinates can be thought of as the Cartesian coordinates. The coordinate transformation $t' = (Q\,t)Q^{-1}$ puts the Kasner solution in its familiar form as [53]:

$$ds^2 = -dt^2 + t^{2a} dr^2 + t^{2b} dz^2 + t^{2c} d\phi^2, \tag{5.8}$$

where we have again rescaled the metric, removed prime for clarity and $a = (q^2 - q)Q^{-1}$, $b = q\,Q^{-1}$, $c = (1-q)Q^{-1}$ with $Q = q^2 - q + 1$. Kasner solution corresponds to an anisotropic homogenous cosmology. Here the constants a, b, c satisfy the Kasner constraints:

$$a + b + c = 1 = a^2 + b^2 + c^2. \tag{5.9}$$

Also, for $c_1 = c_3 = 0$ and others nonvanishing we get flat Minkowski spacetime. Notice that c_1 and c_2 cannot vanish simultaneously in (5.2). Same is true also for c_3 and c_4.

If one calculates the Ricci tensor of the metric (5.2), the only nonvanishing term is:

$$R_{01} = -c_1 c_3(-1 + k^2 - 2kq + q^2)(c_1 r + c_2)^{(-1+2k-2k^2)}(c_3 t + c_4)^{(-1+2q-2q^2)}. \tag{5.10}$$

Here we see that when c_1 or c_3 vanish we have a vacuum solution as it should be. Assuming they do not vanish, equating (5.10) to zero we get $q = k \pm 1$ which results (Hereafter we choose $c_2 = c_4 = 0$ and we absorb c_1 and c_3 in the coordinates r, t, z by redefining them):

$$\begin{aligned} ds^2 &= r^{2(k^2-k)}\, t^{2((k+\epsilon)^2-k-\epsilon)}(-dt^2 + dr^2) \\ &\quad + r^{2k}\, t^{2(k+\epsilon)} dz^2 + \ P^2\, r^{2(1-k)}\, t^{2(1-k-\epsilon)} d\phi^2, \end{aligned} \tag{5.11}$$

with $\epsilon = \pm 1$. Thus, we have obtained the desired Kasner generalization of the Levi-Civita solution, where we can call it as Levi-Civita-Kasner space-time (LCK). Let us express it with the conventional Levi-Civita parameters since we want to compare it with the static solution. The transformations:

$$R = r^{\kappa}\, \kappa^{-1},\ \tau = Q^{-1} t^{Q},\ k = 2\sigma/(2\sigma - 1),\ Q = (k+\epsilon)^2 - (k+\epsilon) + 1, \tag{5.12}$$

lead to the following metric:

$$ds^2 = -R^{2D} d\tau^2 + \tau^{2A} dR^2 + R^{2E}\tau^{2B} dz^2 + \alpha^2\, R^{2F}\tau^{2C} d\phi^2, \tag{5.13}$$

where we have again rescaled the coordinates τ, R, z, absorbed all constant into α and

$$H = \epsilon(4\sigma^2 - 1) + (1 - 2\sigma)^2, \quad A = \frac{2\sigma + H}{S + H}, \tag{5.14}$$

$$B = \frac{(2\sigma - 1)(2\sigma + \epsilon(2\sigma - 1))}{S + H}, \quad C = \frac{(1 - 2\sigma)(1 + \epsilon(2\sigma - 1))}{S + H}, \tag{5.15}$$

$$D = \frac{2\sigma}{S}, \quad E = \frac{2\sigma(2\sigma - 1)}{S}, \quad F = \frac{1 - 2\sigma}{S}. \tag{5.16}$$

For any value of σ we have in general two different solutions depending on $\epsilon = \pm 1$. These solutions are in the form of the generalized Kasner spacetimes [47]-[49] and the metric functions A, B, C and E, F, G satisfy the Kasner constraints:

$$A + B + C = A^2 + B^2 + C^2 = 1, \quad D + E + F = D^2 + E^2 + F^2 = 1, \tag{5.17}$$

separately. The Levi-Civita-Kasner solution (5.11) is also equivalent to Einstein-Rosen soliton wave solutions [50]-[52] by a transformation:

$$r = \sqrt{T-\sqrt{T^2-\varrho^2}}, \quad t=\sqrt{T+\sqrt{T^2-\varrho^2}}, \tag{5.18}$$

which puts the metric functions into the form:

$$W = rt=\varrho, \tag{5.19}$$

$$U = k\ln\varrho+\frac{\epsilon}{2}\ln\left(T+\sqrt{T^2-\varrho^2}\right), \tag{5.20}$$

$$K = k^2\ln\varrho+\left(\epsilon k+\frac{1}{2}\right)\ln\left(T+\sqrt{T^2-\varrho^2}\right)-\frac{1}{2}\ln(\left(2\sqrt{T^2-\varrho^2}\right). \tag{5.21}$$

This transformation is valid only for $t^2>r^2$. For $r^2>t^2$ we need the following transformation:

$$r = \sqrt{\varrho+\sqrt{\varrho^2-T^2}}, \quad t=\sqrt{\varrho-\sqrt{\varrho^2-T^2}}, \tag{5.22}$$

which gives:

$$W = rt=T, \tag{5.23}$$

$$U = k\ln T+\frac{\epsilon}{2}\ln\left(\varrho-\sqrt{\varrho^2-T^2}\right), \tag{5.24}$$

$$K = k^2\ln T-\left(\epsilon k+\frac{1}{2}\right)\ln\left(\varrho-\sqrt{\varrho^2-T^2}\right)-\frac{1}{2}\ln\left(2\sqrt{\varrho^2-T^2}\right). \tag{5.25}$$

Here the first metric is not valid for $\varrho>T$ and the other is not valid for $\varrho<T$. Then we need to extend one to join with the other. After achieving this, the resulting spacetime is the solution we consider in this article. Hence, the metric we discuss covers both regions.

5.2. Newmann-Penrose Spin and Weyl Coefficients

5.2.1. Newmann-Penrose Formalism

The Newmann-Penrose formalism [54] requires a complex null tetrad. Let us first introduce an orthonormal tetrad as follows:

$$ds^2 = -\mathbf{e}^0 \otimes \mathbf{e}^0 + \mathbf{e}^1 \otimes \mathbf{e}^1 + \mathbf{e}^2 \otimes \mathbf{e}^2 + \mathbf{e}^3 \otimes \mathbf{e}^3, \tag{5.26}$$

where $\mathbf{e}^0$ is timelike. Now, let us introduce null tetrads as follows:

$$\sqrt{2}\,\mathbf{l} = \mathbf{e}^0 + \mathbf{e}^1, \sqrt{2}\,\mathbf{n} = \mathbf{e}^0 - \mathbf{e}^1,\ \sqrt{2}\,\mathbf{m} = \mathbf{e}^2 + i\mathbf{e}^3,\ \sqrt{2}\,\bar{\mathbf{m}} = \mathbf{e}^2 - i\mathbf{e}^3. \tag{5.27}$$

Here we have two real null vectors $\mathbf{l} = l_a \mathbf{dx}^a$ and $\mathbf{n} = n_a \mathbf{dx}^a$ satisfying $l^a n_a = 1$, and two complex conjugate vectors $\mathbf{m} = m_a \mathbf{dx}^a$ and $\bar{\mathbf{m}} = \bar{m}_a \mathbf{dx}^a$ satisfying $m^a m_a = -1$. All other scalar products vanish. The spacetime metric becomes:

$$ds^2 = \mathbf{l} \otimes \mathbf{n} - \mathbf{m} \otimes \bar{\mathbf{m}}. \tag{5.28}$$

For a suitably chosen tetrad, the Newmann-Penrose formalism yields some physical properties of the metric since some curvature coefficients such as spin (connection) coefficients, some scalars constructed from Ricci and Weyl tensors has direct physical meaning [55], [56]. There are 12 spin coefficients:

$$-\kappa = l_{a;b} m^a l^b, \quad -\rho = l_{a;b} m^a \bar{m}^b, \quad -\sigma = l_{a;b} m^a m^b, \quad -\tau = l_{a;b} m^a n^b, \tag{5.29}$$

$$\nu = n_{a;b} \bar{m}^a n^b, \quad \mu = n_{a;b} m^a \bar{m}^b, \quad \lambda = n_{a;b} \bar{m}^a \bar{m}^b, \quad \pi = n_{a;b} \bar{m}^a l^b, \tag{5.30}$$

$$-\epsilon = \frac{1}{2}(l_{a;b} n^a l^b - m_{a;b} \bar{m}^a l^b), \quad -\beta = \frac{1}{2}(l_{a;b} n^a m^b - m_{a;b} \bar{m}^a m^b), \tag{5.31}$$

$$\gamma = \frac{1}{2}(n_{a;b} l^a n^b - \bar{m}^a n^b), \quad \alpha = \frac{1}{2}(n_{a;b} l^a \bar{m}^b - \bar{m}_{a;b} m^a \bar{m}^b). \tag{5.32}$$

These spin coefficients have some physical meanings. For example, expansion (or divergence) θ, twist (or rotation) ω, and shear σ (given above) of the null vector $\mathbf{l}$ can be found from these spin coefficients. The expansion and twist is given by:

$$\rho = -(\theta + i\omega). \tag{5.33}$$

Also vanishing of κ shows that $\mathbf{l}$ is geodesic. From these complex spin coefficients one can express all connection coefficients.

The curvature components in this complex null tetrad can be expressed in terms of traceless Ricci tensor $S_{ab} = R_{ab} - \frac{1}{4} g_{ab} R$, and Weyl tensor C_{abcd}. The Ricci scalars Φ_{ab} and Weyl scalars Ψ_a are given as:

$$\Phi_{00} = \frac{1}{2} S_{ab} l^a l^b, \quad \Phi_{01} = \frac{1}{2} S_{ab} l^a m^b, \quad \Phi_{02} = \frac{1}{2} S_{ab} m^a m^b, \tag{5.34}$$

$$\Phi_{11} = \frac{1}{4} S_{ab}(l^a n^b + m^a \bar{m}^b), \quad \Phi_{12} = \frac{1}{2} S_{ab} n^a m^b, \quad \Phi_{22} = \frac{1}{2} S_{ab} n^a n^b, \tag{5.35}$$

$$\Psi_0 = C_{abcd} l^a m^b l^c m^d, \quad \Psi_1 = C_{abcd} l^a n^b l^c m^d, \quad \Psi_2 = -C_{abcd} l^a m^b n^c \bar{m}^d, \tag{5.36}$$

$$\Psi_3 = C_{abcd} n^a l^b n^c \bar{m}^d, \quad \Psi_4 = C_{abcd} n^a \bar{m}^b n^c \bar{m}^d. \tag{5.37}$$

Szekeres interpreted the Weyl scalars as follows: Ψ_4 represents a transverse wave and Ψ_3 represents a longitudinal wave component in the $\mathbf{l}$ direction, Ψ_2 term represents a Coulomb component whereas Ψ_0 and Ψ_1 terms represent transverse and longitudinal wave components in the $\mathbf{n}$ direction.

The Petrov classification is the classification of the vacuum gravitational fields according to the algebraic properties of the curvature (or equivalently Weyl) tensor. There are different methods for determining the Petrov types of given spacetime and refer to [17] for the details. By adapting the null tetrad for the Weyl tensor under consideration, one can set $\Psi_0 = 0$, which has a corresponding null vector l^a. By rotating the null frame ($l \to l'$), this equation yields an equation of fourth order for Ψ'_a's having four roots. According to multiplicity of these roots one can divide spacetimes into the following Petrov types. Type I: four distinct roots (Ψ_0 vanishing). Type II: one double root and two simple roots (Ψ_0, Ψ_1 vanishing). Type D: two double roots (Ψ_0, Ψ_1 vanishing). Type III : one triple root and one simple root (Ψ_0, Ψ_1, Ψ_2 vanishing).

Type N: one four-fold root (Ψ_0, Ψ_1, Ψ_2, Ψ_3 vanishing). Type O: the Weyl tensor vanishes identically (all Ψ_a 's are vanishing).

5.2.2. Spin and Weyl Scalars of LCK Spacetime

The static Levi-Civita metric is Petrov type I in general except it is flat for $\sigma = 0, 1/2, \infty$ and it is Petrov type D for $\sigma = -1/2, 1/4, 1$ (see [10]). Let us compare with LCK spacetime.

Here, using a Newmann-Penrose tetrad, we will present the nonvanishing spin and Weyl scalars of this spacetime. The canonical form of the metric (5.11) is more appropriate for our purposes. With the help of *grtensor* [57], we will calculate NP spin, Ricci and and Weyl scalars. We chose the Newmann-Penrose tetrad as follows:

$$ds^2 = \mathbf{l} \otimes \mathbf{n} - \mathbf{m} \otimes \bar{\mathbf{m}}, \tag{5.38}$$

$$\sqrt{2}\,\mathbf{l} = \mathbf{e}^0 + \mathbf{e}^1, \sqrt{2}\,\mathbf{n} = \mathbf{e}^0 - \mathbf{e}^1,\ \sqrt{2}\,\mathbf{m} = \mathbf{e}^2 + i\mathbf{e}^3, \tag{5.39}$$

$$\mathbf{e}^0 = r^{k^2-k} t^{(k+\epsilon)^2-k-\epsilon} dt,\ \mathbf{e}^1 = r^{k^2-k} t^{(k+\epsilon)^2-k-\epsilon} dr, \tag{5.40}$$

$$\mathbf{e}^2 = r^k t^{k+\epsilon} dz,\ \mathbf{e}^3 = \alpha r^{1-k} t^{1-k-\epsilon} d\phi. \tag{5.41}$$

For $\epsilon = 1$ the nonvanishing components of spin coefficients and Weyl scalars are:

$$\sigma = -\frac{(1+2k)\,r + (1-2k)\,t}{2\sqrt{2} r^{k^2-k+1} t^{k^2+k+1}}, \quad \lambda = \frac{(1+2k)\,r - (1-2k)\,t}{2\sqrt{2}\, r^{k^2-k+1}\, t^{k^2+k+1}}, \tag{5.42}$$

$$\rho = \frac{t-r}{2\sqrt{2}\, r^{k^2-k+1}\, t^{k^2+k+1}}, \quad \mu = \frac{t+r}{2\sqrt{2}\, r^{k^2-k+1}\, t^{k^2+k+1}}, \tag{5.43}$$

$$\epsilon = \frac{k((1+k)\,r + (1-k)\,t)}{2\sqrt{2}\, r^{k^2-k+1}\, t^{k^2+k+1}}, \quad \gamma = \frac{k((1-k)\,t - (1+k)\,r)}{2\sqrt{2}\, r^{k^2-k+1}\, t^{k^2+k+1}}, \tag{5.44}$$

$$\kappa = \nu = \tau = \pi = \alpha = \beta = 0, \tag{5.45}$$

$$\Psi_0 = \frac{k\,((1+k)(1+2k)\,r^2 + 4(1-k^2)\,r\,t + (1-k)(1-2k)\,t^2)}{2\, r^{2k^2-2k+2}\, t^{2k^2+2k+2}}, \tag{5.46}$$

$$\Psi_2 = \frac{k((1+k)\,r^2 + (1-k)\,t^2)}{2\, r^{2k^2-2k+2}\, t^{2k^2+2k+2}}, \tag{5.47}$$

$$\Psi_4 = \frac{k\,((1+k)(1+2k)\,r^2 - 4(1-k^2)\,r\,t + (1-k)(1-2k)\,t^2)}{2\, r^{2k^2-2k+2}\, t^{2k^2+2k+2}}. \tag{5.48}$$

This shows us that the LCK spacetime with $\epsilon = 1$ is again Petrov type I in general except it is flat for $k = 0 (\sigma = 0)$ and $k \to \infty$ $(\sigma = 1/2)$ and Petrov type D for $k = -1, 1$ $(\sigma = -1/4, \infty)$. Also, since $\kappa = 0$, $\mathbf{l}$ is geodesics but it is not affinely parameterized since $\epsilon \neq 0$ except $k = 0 (\sigma = 0)$. It also has expansion $(-\rho \neq 0)$ and shear $(|\sigma| \neq 0)$ but it is not twisting.

For $\epsilon = -1$ we have

$$\sigma = \frac{(3-2k)\,r + (2k-1)\,t}{2\sqrt{2}\,r^{k^2-k+1}\,t^{k^2-3k+3}}, \quad \lambda = \frac{(2k-3)\,r + (2k-1)\,t}{2\sqrt{2}\,r^{k^2-k+1}\,t^{k^2-3k+3}}, \tag{5.49}$$

$$\rho = \frac{t-r}{2\sqrt{2}\,r^{k^2-k+1}\,t^{k^2-3k+3}}, \quad \mu = \frac{t+r}{2\sqrt{2}\,r^{k^2-k+1}\,t^{k^2-3k+3}}, \tag{5.50}$$

$$\epsilon = \frac{(k-1)((k-2)r - k\,t)}{2\sqrt{2}\,r^{k^2-k+1}\,t^{k^2-3k+3}}, \quad \gamma = \frac{(k-1)((2-k)r - k\,t)}{2\sqrt{2}\,r^{k^2-k+1}\,t^{k^2-3k+3}}, \tag{5.51}$$

$$\kappa = \nu = \tau = \pi = \alpha = \beta = 0, \tag{5.52}$$

$$\Psi_0 = \frac{(k-1)((6-7k+2k^2)\,r^2 - 4k(k-2)\,r\,t + k(2k-1)\,t^2)}{2\,r^{2k^2-2k+2}\,t^{2k^2-6k+6}}, \tag{5.53}$$

$$\Psi_2 = \frac{(k-1)((k-2)\,r^2 - kt)}{2\,r^{2k^2-2k+2}\,t^{2k^2-6k+6}}, \tag{5.54}$$

$$\Psi_4 = \frac{(k-1)((6-7k+2k^2)\,r^2 + 4k(k-2)\,r\,t + k(2k-1)\,t^2)}{2\,r^{2k^2-2k+2}\,t^{2k^2-6k+6}}. \tag{5.55}$$

Thus, Levi-Civita-Kasner with $\epsilon = -1$ is also Petrov type I in general except $k = 1$ $(\sigma \to \infty)$ and $k \to \infty (\sigma = 1/2)$ where it is flat and Petrov type D for $k = 0, 2$ $(\sigma = 0, 1)$. Again, the vector $\mathbf{l}$ is geodesic but not affinely parameterized except for $k = 1$. Also, it has nonvanishing expansion and shear but it is not twisting. Thus, the Levi-Civita and Levi-Civita-Kasner solutions have common Petrov types in general, but they differ for some particular values of the parameter σ.

5.3. Singularity Behaviour

The Kretschmann scalar $\mathcal{K} = R_{abcd}R^{abcd}$ of the metric (5.13) are

$$\mathcal{K} = 64\,\sigma^2\,(1-2\sigma)^2 \Bigg(\frac{(1-4\sigma)^2 r^{-8\sigma/(1-2\sigma+4\sigma^2)}}{(1-6\sigma+12\sigma^2)^3\,t^4} + \frac{t^{8\sigma(1-4\sigma)/(1-6\sigma+12\sigma^2)}}{(1-2\sigma+4\sigma^2)^3\,r^4}$$

$$
- \frac{2\,(1-4\sigma)^2\, r^{-4\sigma/(1-2\sigma+4\sigma^2)}\, t^{4\sigma(1-4\sigma)/(1-6\sigma+12\sigma^2)}}{(1-2\sigma+4\sigma^2)^2(1-6\sigma+12\sigma^2)^2\, r^2\, t^2}\Bigg), \ (\epsilon=1), \tag{5.56}
$$

$$
\mathcal{K} = 64\,(1-2\sigma)^2\Bigg(\frac{(\sigma-1)^2\; r^{-8\sigma/(1-2\sigma+4\sigma^2)}}{(3-6\sigma+4\sigma^2)^3 t^4} + \frac{\sigma^2\, t^{4(1-2\sigma)^2/(3-6\sigma+4\sigma^2)}}{(1-2\sigma+4\sigma^2)\, r^4\, t^4}
$$

$$
- \frac{8\sigma^2(\sigma-1)^2 r^{4\sigma/(1-2\sigma+4\sigma^2)} t^{2(1-2\sigma)^2/(3-6\sigma+4\sigma^2)}}{(3-6\sigma+4\sigma^2)(1-2\sigma+4\sigma^2) r^2 t^4}\Bigg), \ (\epsilon=-1). \tag{5.57}
$$

We see that, this spacetime has singularities as in the static case. It is well known that the static Levi-Civita spacetime is singular at $r=0$, except for $\sigma=0$, $\sigma=\pm 1/2$ and $\sigma\to\infty$. For these values of σ, the solution is regular and flat. If one compares the static solution with the nonstatic solutions, one realizes that there are similarities and differences. For $\epsilon=1$ only when $\sigma=0$ or $\sigma=1/2$, the solution is locally flat. For other cases, there are singularities. For $\sigma=1/4$ we have a singularity at $r=0$ whereas for $\sigma\to\infty$ we have singularity at $t=0$. For all other values of σ we have singularities at both $r=0$ and $t=0$. When $\epsilon=-1$ the situation is also different. For this case when $\sigma=0$ the solution is not locally flat but contains a singularity at $t=0$. There are locally flat solutions when $\sigma=1/2$ or $\sigma\to\infty$. For $\sigma=1$ we have a singularity at $r=0$. For other values of σ we have both line and big-bang singularities.

As we have mentioned, for $\sigma=0$ and $\sigma=1/2$ static Levi-Civita solution (4.7) is flat. Corresponding solutions for $\epsilon=\pm 1$ are:

$$
ds^2 = -d\tau^2 + dr^2 + \tau^2 dz^2 + \alpha^2 r^2 d\phi^2, \quad (\sigma=0,\ \epsilon=1), \tag{5.58}
$$

$$
ds^2 = -d\tau^2 + \tau^{4/3} dR^2 + \tau^{-2/3} dz^2 + \alpha^2 R^2 \tau^{4/3} d\phi^2, \quad (\sigma=0,\ \epsilon=-1), \tag{5.59}
$$

$$
ds^2 = -R^2 d\tau^2 + \tau^2 dR^2 + dz^2 + \alpha^2 d\phi^2, \quad (\sigma=1/2,\ \epsilon=\pm 1). \tag{5.60}
$$

Here the first and third metrics are flat whereas the second one is curved. The first and third metrics can be put into standard Minkowski form with a suitable coordinate transformation. The first solution is presented in [40] as a possible nonstatic exterior solution corresponding to a nonstatic string. Also, the second solution (5.59) was introduced in [39],[42]-[45] as an exterior vacuum solution to their interior nonstatic stringlike cylindrical source. It is not singular at $r = 0$ but has a big bang singularity at $t = 0$ since its Kretschmann scalar is $\mathcal{K} \sim t^{-4}$.

5.4. Radial Acceleration of Test Particles

The radial acceleration of a free test particle at rest in the coordinate system of (5.13) is given by:

$$\frac{d^2R}{d\tau^2} = -\frac{D}{RT^{2A}}. \tag{5.61}$$

The radial acceleration in the static Levi-Civita spacetime can be found by taking $A = 0$. As we have discussed in the previous chapter, for the Levi-Civita spacetime, when the parameter σ is positive, the axis is attractive and when σ is negative, the axis is repulsive. For a particle in a constant radius, the magnitude of the acceleration is increasing with increasing σ when $0 < \sigma < 1/2$, and decreases with increasing σ when $\sigma > 1/2$. For $\sigma = 0$, no radial force is exerted on a particle at rest. For nonzero σ, when the radial distance increases, radial acceleration decreases.

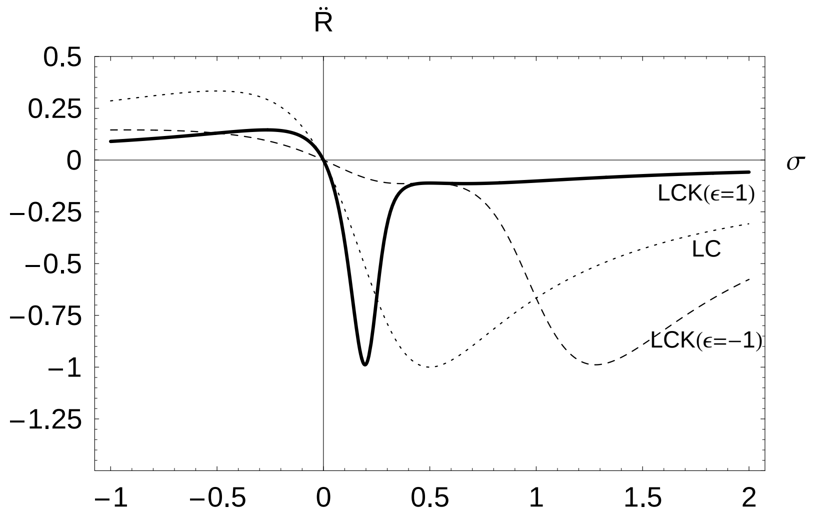

Figure 5.1. The radial acceleration of a particle at $r = 1$, $t = 3$ for the static Levi-Civita and Levi-Civita-Kasner spacetimes with $\epsilon = \pm 1$

As in the Levi-Civita spacetime, for the Levi-Civita-Kasner spacetimes (5.13) when σ positive the axis is attractive and when σ is negative the axis is repulsive. And also when $\sigma = 0$, no radial acceleration is felt by a particle at rest. However, since the solution is time dependent, the behaviour of acceleration is changing with time. A typical behavior for $t = 3$ can be seen in Figure 5.1. Here, for $\epsilon = 1$ the magnitude of acceleration increases with increasing σ up to $\sigma \sim 0.17$, then it starts to decrease sharply up to $\sigma \sim 0.3$, and then it decreases monotonically with increasing σ. For $\epsilon = -1$ situation is different. It increases monotonically up to $\sigma = 1/2$ then increases more sharply up to $\sigma = 3/2$ then starts to decrease with increasing σ. When the time evolves, the radial acceleration is getting stronger for certain ranges of σ and out of this range, particle feels very tiny force. For $\epsilon = 1$ this range is in between 0 and 0.2. For $\epsilon = -1$ the situation is reverse. For small σ particle feels very small force. The region where acceleration is very strong is near $\sigma \sim 1$. For other values of σ a test particle feels very tiny radial force on it.

5.5. Circular Geodesics

Here we study the equations of a test particle following a circular geodesics in the spacetime (5.13) and compare with the circular geodesics in the Levi-Civita spacetime [10] where we have reviewed in the previous chapter.

Let us again denote the angular velocity of a particle moving along a geodesics as $\omega = d\phi/d\tau$ and its tangential velocity as $W^{\mu} = (0, 0, 0, W^{\phi})$ with $W^{\phi} = \omega/\sqrt{-g_{tt}}$, then we have (here τ is the time coordinate, not the proper time and dot represents derivation with respect to a proper time parameter η):

$$\left(\frac{ds}{d\tau}\right)^2 = -R^{2D} + R^{2F}\tau^{2C}\left(\frac{d\phi}{d\tau}\right)^2, \tag{5.62}$$

$$\omega^2 = \left(\frac{\dot{\phi}}{\dot{\tau}}\right)^2 = \frac{D}{\alpha^2 F}R^{2(D-F)}\tau^{-2C}, \tag{5.63}$$

$$\ddot{\tau} = -C\alpha^2 R^{2(F-D)}\tau^{2C-1}\dot{\phi}^2, \tag{5.64}$$

$$\dot{r} = 0, \quad \dot{z} = 0. \tag{5.65}$$

Then,

$$W^2 = \frac{D}{F}. \tag{5.66}$$

Replacing this into the first and the third equations, we get

$$\left(\frac{ds}{d\tau}\right)^2 = \left(W^2 - 1\right) R^{2D}, \tag{5.67}$$

$$\dot{\tau} = \frac{d\tau}{d\eta} = \frac{\tau_0}{\tau^{C\sqrt{W}}}. \tag{5.68}$$

Thus, the circular geodesics are timelike for $W < 1$ ($\sigma < 1/4$), spacelike for $W > 1$ ($\sigma > 1/4$) and null for $W = 1$ ($\sigma = 1/4$). We have the same conditions with the static Levi-Civita spacetime. Thus the time dependence does not affect the character of the circular geodesics. Also, as in the static case, for a given σ the tangential velocity of a particle is constant. The only difference between Levi-Civita and Levi-Civita-Kasner spacetimes is that ∂_τ is not a Killing vector for Levi-Civita-Kasner spacetimes in general. This does not affect the dependence of the character of the circular geodesics to the parameter σ, although they have different gravitational fields, as the previous section suggests.

In summary we have investigated some physical properties of the nonstatic vacuum solutions in cylindrical coordinates with Kasner type time dependence. They can describe the exterior regions of nonstatic line sources and nonstatic straight strings [40]-[45] having nonvanishing gravitational potential. For each constant time slice they reduce to the Levi-Civita metric. For each Levi-Civita parameter, σ, there are in general two corresponding nonstatic vacuum solutions of this form depending on $\epsilon = \pm 1$. We have studied some physical properties of this space-time and compared with the static Levi-Civita spacetime. This metric is in the form of generalized Kasner solutions studied before [47]-[49]. Also, by a coordinate transformation, it reduces to Einstein-Rosen soliton waves [17],[50]-[52]. We have discovered some differences and similarities between Levi-Civita and Levi-Civita-Kasner spacetimes. Now, we show that a small deviation in the Levi-Civita-Kasner metric leads to a nonempty cosmological solution.

5.6. A Stiff Fluid of Generalized Kasner Form

Let us consider the following metric:

$$ds^2 = r^{2(k^2-k)}\, t^{2(q^2-q+a)}(-dt^2 + dr^2) + r^{2k}\, t^{2q} dz^2 + \ P^2\, r^{2(1-k)}\, t^{2(1-q)} d\phi^2, \tag{5.69}$$

which deviates from the Levi-Civita-Kasner solution by a parameter a. For $a = -k^2 - q^2 + 2kq + 1$ we have a nonstatic stiff fluid with the equation of state

$$-G^0{}_0 = G^r{}_r = G^z{}_z = G^\phi{}_\phi = a\ r^{2(k-k^2)} t^{2(q-q^2-a-1)}. \tag{5.70}$$

When $a \to 0$ we recover the Levi-civita-Kasner solution. This solution has a similar singularity behavior as the Levi-Civita-Kasner metric has and it is singular in general at $r = 0$ and $t = 0$. For a special values of k and q we can avoid the singularity at the axis. For example for $k = q = 0$ we have $a = 1$ and the metric becomes

$$ds^2 = t^2(-dt^2 + dr^2 + r^2 d\phi^2) + dz^2, \tag{5.71}$$

with

$$-G^0{}_0 = G^r{}_r = G^z{}_z = G^\phi{}_\phi = t^{-4}. \tag{5.72}$$

This metric describes a cosmological solution with an universe filled with an isotropic stiff fluid having the equation of state $\rho = p$. We have a big bang singularity at $t = 0$, since the Kretchman scalar is $K \sim t^{-8}$. Also, since the energy density goes with t^{-4}, it becomes negligible for large t and practically at $(t \to \infty)$ we get vacuum universe.

6. RADIATING LEVI-CIVITA SOLUTION AND ITS KASNER GENERALIZATION

6.1. Review of Radiating Levi-Civita Spacetime

One of the most difficult and important problems in physics is the gravitational collapse of a realistic matter distribution since spacetime singularities may be formed during collapse of matter and extremely high spacetime curvature and energy density may be realized around the singularities starting from regular initial conditions. In the case of collapsing of a spherically symmetric body we are not faced with naked singularities since the singularity, formed when a body collapses, is hidden inside a horizon. Also when a spherically symmetric body collapses, it does not produce gravitational radiation. This may be a property of spherical symmetry since, according to Birkhoff theorem, outside of any spherically symmetric system is necessarily static.

However, a spherically symmetric body can emit other forms of radiation. A solution with null fluid emitted radially from a static spherically symmetric source is given by the Vaidya solution[12]:

$$ds^2 = -\left(1 - \frac{2m(u)}{r}\right) du^2 + 2du\,dr + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{6.1}$$

Here $m(u) = m(t - r)$. When we take the mass parameter $m(u) = m_0 = const.$, the solution reduces to the most general spherically symmetric vacuum solution, the Schwarzschild solution. This solution describes an incoherent radiation originated from a spherically symmetric object due to some physical process and directed to a certain direction in geometrical optics limit. If we consider k_μ as a null vector satisfying $k_\mu k^\mu = 0$ and representing the direction of propogation and η is energy density of the pure radiation, then energy-momentum tensor of Vaidya solution has the form:

$$T_{\mu\nu} = \eta\, k_\mu k_\nu. \tag{6.2}$$

When we consider the cases with lesser symmetry, we see that if the body is not sufficiently compact in every direction, then according to hoop conjecture [33], naked singularities may be formed. In order to better understand this problem, the collapse of nonspherical or cylindrical matter distributions is studied analytically or numerically with several configurations [58]-[62]. Actually, when studying cylindrical collapse, one should consider the possibility that, the outside metric could be a nonstatic metric in general since there is no analogue of the Birkhoff theorem in cylindrical symmetry. Thus, since there could be gravitational waves outside of a collapsing cylinder, it could be a good approximation to take the outside metric as a radiative metric [63] with gravitational and null radiation emitted radially from the body, as several authors [64]-[70] have considered in the framework of the thin shell formalism. In those works the exterior region is in general represented by the cylindrical analogue of the Vaidya solution, the radiating Levi-Civita solution [71]:

$$ds^2 = e^{2F(t-r)} r^{2(k^2-k)}(-dt^2 + dr^2) + r^{2k} dz^2 + r^{2(1-k)} \alpha^2 d\phi^2, \tag{6.3}$$

which describes cylindrical null and gravitational radiation in radial direction. In most of the works, the $k = 0$ case is considered which corresponds to a cylindrical null and gravitational radiation in otherwise flat Minkowski spacetime (or a cosmic string can be present at the axis if we have also an angular deficit parameter in the metric (6.3) [72]). When the radiation is switched of, this solution reduces to the Levi-Civita solution which corresponds to the static cylindrically symmetric vacuum solution representing the exterior field of a static cylindrical source. It has two essential parameters [6], the first one, k, is related with the energy density and the curvature of the source, whereas the second one, α, is the angular deficit parameter. However, unlike the spherical symmetry, there is no time-like Killing vector in cylindrical symmetry and the cylindrical vacuum may not be static. Motivated by these considerations, in this section a cylindrical radiative metric is presented which corresponds to Kasner generalization of the Levi-Civita solution which we have discussed in the previous section when the radiation is switched off.

It is well known that for any Einstein-Rosen wave ($K = K_0$, $U = U_0$, $W =$

W_0) solving vacuum Einstein equations for the metric (5.1), there is a corresponding radiative solution ($K = F(r-t) + K_0,\ U = U_0,\ W = W_0$) satisfying (6.2) [17],[73],[74]. Thus we can immediately write the radiating version of the Levi-Civita metric (6.3) if we consider the Einstein-Rosen form of the static vacuum Levi-Civita metric (4.6) for the functions K_0, U_0, W_0.

The energy density of the null dust is

$$\eta = -\frac{\dot{F}}{r}. \tag{6.4}$$

To have a positive energy density, here we need $\dot{F} < 0$. The energy momentum tensor satisfies the energy conservation equation $\nabla^{\nu} T_{\mu\nu} = 0$.

When $k = 0$ our solution reduces to an exterior of a radiating cosmic string [72]:

$$ds^2 = e^{2F(r-t)}(dr^2 - dt^2) + dz^2 + \alpha^2 r^2 d\phi^2, \tag{6.5}$$

where the constant α is the angular deficit parameter. There is a corresponding interior solution with $g_{\phi\phi} = \beta^2(r)$ where $\beta(r)$ is a smooth function.

The Krestchmann scalar $\mathcal{K} = R_{abcd}R^{abcd}$ of the metric (6.3) is

$$\mathcal{K} = \left(\mathcal{K}_0 + \frac{8k(k-1)(1-2k)^2 r\dot{F})}{r^{4k^2-4k+4}}\right) e^{-4F}, \tag{6.6}$$

where $\mathcal{K}_0$ is the Kretschmann scalar of the static Levi-Civita solution with

$$\mathcal{K}_0 = \frac{16k^2(k-1)^2(k^2-k+1)}{r^{4k^2-4k+4}}. \tag{6.7}$$

This shows that the singularity behaviour is the same with the Levi-Civita solution. Here at $k = 0$ and $k = 1$ the solution is regular and corresponds to a locally flat spacetime. For other values of k, the axis is not regular and there is a line singularity at $r = 0$.

Let us calculate the Ricci and Weyl scalars using the Newmann-Penrose formalism [54] since in this formalism some of the curvature components have direct physical meaning [55],[56]. We choose the NP tetrad as follows:

$$ds^2 = \mathbf{l} \otimes \mathbf{n} - \mathbf{m} \otimes \bar{\mathbf{m}}, \tag{6.8}$$

$$\sqrt{2}\,\mathbf{l} = \mathbf{e}^0 + \mathbf{e}^1, \sqrt{2}\,\mathbf{n} = \mathbf{e}^0 - \mathbf{e}^1,\ \sqrt{2}\,\mathbf{m} = \mathbf{e}^2 + i\mathbf{e}^3, \tag{6.9}$$

$$\mathbf{e}^0 = e^F r^{k^2-k} dt,\ \mathbf{e}^1 = e^F r^{k^2-k} dr,\ \mathbf{e}^2 = r^k dz,\ \mathbf{e}^3 = \alpha r^{1-k} d\phi. \tag{6.10}$$

The nonvanishing Ricci and Weyl scalars are:

$$\Phi_{00} = -\frac{\dot{F}}{e^{2F}\, r^{2k^2-2k+1}}, \tag{6.11}$$

$$\Psi_0 = \frac{(2k-1)\left(k(k-1) - 2r\dot{F}\right)}{2\,e^{2F}\, r^{2k^2-2k+2}}, \tag{6.12}$$

$$\Psi_2 = \frac{k(1-k)}{2\,e^{2F} r^{2k^2-2k+2}}, \tag{6.13}$$

$$\Psi_4 = \frac{k(k-1)(2k-1)}{2\,e^{2F} r^{2k^2-2k+2}}. \tag{6.14}$$

In order to have energy density of null dust positive, Φ_{00} must be positive, thus $\dot{F}$ must be negative. The Weyl tensor is thought of as to represent the purely gravitational effects, such as gravitational radiation, of the spacetime [55],[56]. The Weyl scalar Ψ_0 denotes outgoing gravitational transverse wave component in the direction of $\mathbf{n}$, Ψ_4 gives the incoming transverse gravitational wave component in the direction of $\mathbf{l}$ and Ψ_2 denotes a Coulomb field. When $k = 0, 1$ ($\sigma = 0, \infty$) we have only Ψ_0 nonzero and spacetime is Petrov type N, which is in accordance with the results of a previous work [72], which studied particle and gravitational wave emission from cosmic strings. For $k = 1/2$ ($\sigma = -1/2$), Ψ_0 and Ψ_4 are vanishing and it is Petrov type D. They are nonvanishing for all other values of k and it is Petrov type I. Thus the solution (6.3) has in general an outgoing null dust and both incoming and outgoing gravitational radiation.

The metric (6.3) is Petrov type I in general except for $k = 0, 1/2, 1$. For $k = 0, 1$ it is Petrov type N whereas for $k = 1/2$ it is Petrov type D.

6.2. Kasner Generalization of Radiating Levi-Civita Solution

Using the property of the Einstein-Rosen solutions we can easily construct the Kasner generalization of radiating Levi-Civita solution [46]. We will use Kasner generalization of the Levi-Civita solution discussed in the previous section for the functions K_0, U_0, W_0. Then, we have the metric functions:

$$\begin{aligned}
&K = F(r,t) + k^2 \ln(c_1 r + c_2) + q^2 \ln(c_3 t + c_4),\\
&U = k \ln(c_1 r + c_2) + q \ln(c_3 t + c_4),\\
&W = \alpha(c_1 r + c_2)(c_3 t + c_4),\\
&F(r,t) = F(r - t),\ q = k + \epsilon,\ \epsilon = \pm 1,
\end{aligned} \tag{6.15}$$

which are solutions of (6.2) with the energy density:

$$\eta = \frac{(c_2 c_3 - c_1(c_4 + c_3(t - r)))\,\dot{F}}{(c_1 r + c_2)(c_3 t + c_4)}. \tag{6.16}$$

Notice that both c_1 and c_2 cannot vanish simultaneously. This is also true for c_3 and c_4. When $F =$ constant this solution reduces to Levi-Civita-Kasner solution. Also, when we take $c_3 = 0$ $c_4 \neq 0$ we get the radiating Levi-Civita solution (6.3) whereas for $c_1 = 0$ $c_2 \neq 0$ we get the radiating Kasner solution with the metric:

$$ds^2 = e^{2F(t-r)} t^{2(k^2-k)}(-dt^2 + dr^2) + t^{2k} dz^2 + t^{2(1-k)} d\phi^2. \tag{6.17}$$

For this radiating Kasner solution it is better to think the coordinates as the Cartesian coordinates. This metric describes a pure radiation moving in the r direction in the Kasner spacetime. The energy density is the negative of the Levi-Civita case and $\dot{F}$ must be positive in order to have positive energy density. At $t = 0$ this metric has a Kasner type cosmological singularity except $k = 0$ and $k = 1$.

If the following conditions is satisfied for the Levi-Civita-Kasner solution (6.16), then the energy density of the solution (6.15) is positive:

$$c_2c_3 - c_1c_4 \geq 0,\ c_1c_3(r-t)\dot{F} > 0,\ \dot{F} > 0, \tag{6.18}$$

or

$$c_2c_3 - c_1c_4 \leq 0,\ c_1c_3(r-t)\dot{F} < 0,\ \dot{F} < 0. \tag{6.19}$$

For example the following choice

$$c_2 = c_4 = 0,\ c_1 = c_3 = 1,\ F = -a(t-r)^n,\ n = 1, 2, 3..., \tag{6.20}$$

where $a > 0$ is a constant leads to positive energy solutions when n is even.

For $k = 0$ and $\epsilon = -1$ this solution can represent an exterior solution of a radiating nonstationary cosmic string with the metric:

$$ds^2 = e^{2F}t^4(-dt^2 + dr^2) + t^{-2}dz^2 + \alpha^2 t^4 r^2 d\phi^2, \tag{6.21}$$

since for F = constant this metric represents the exterior field of a nonstationary cosmic string-like object [41, 44].

This spacetime (6.15) contains in general a Kasner type cosmological singularity at $t = 0$ and also it is singular at the axis (We take $c_2 = c_4 = 0$ in (6.15)). The spacetime is not singular for the particular values of the parameters $\epsilon = 1$, $k = 0$ and $\epsilon = -1$, $k = 1$. The cosmological singularity seems to unavoidable but if one is able to find a regular interior radiating solution containing the symmetry axis, then we can avoid having a line singularity at $r = 0$ since our solution could be an exterior solution of a radiating nonstatic cylindrical source. The spacetime is well behaved for $t > 0$ and $r > 0$.

6.3. Some Physical Properties of the Solution

6.3.1. Newmann-Penrose Coefficients

Here we analyze Ricci and Weyl scalars of the metric (6.15) using a null tetrad. For $\epsilon = 1$ we have the spin coefficients:

$$\Phi_{00} = \frac{(t-r)F'}{e^{2F}\, r^{2k^2-2k+1}\, t^{2k^2+2k+1}}, \tag{6.22}$$

$$\Psi_0 = (k((1+k)(1+2k)r^2 - 4(k^2-1)rt + (k-1)(2k-1)t^2) \tag{6.23}$$

$$-\,((1+2k)r + (1-2k)t)2rtF')/(2e^{2F}\, r^{2k^2-2k+2}\, t^{2k^2+2k+2}), \tag{6.24}$$

$$\Psi_2 = \frac{k((1+k)r^2 + (1-k)t^2)}{2e^{2F}\, r^{2k^2-2k+2}\, t^{2k^2+2k+2}}, \tag{6.25}$$

$$\Psi_4 = \frac{k((1+k)(1+2k)r^2 + 4(k^2-1)rt + (k-1)(2k-1)t^2}{2e^{2F}\, r^{2k^2-2k+2}\, t^{2k^2+2k+2}}. \tag{6.26}$$

This shows that only for $\epsilon = 1$ case, for $k = 0$, Ψ_2 and Ψ_4 vanishes and the spacetime is Petrov type N. For other values of k, Ψ_0, Ψ_2 and Ψ_4 is nonvanishing and Petrov type is I. For $\epsilon = -1$ we have:

$$\Phi_{00} = \frac{(t-r)F'}{e^{2F}\, r^{k^2-2k+1}\, t^{2k^2-6k+5}}, \tag{6.27}$$

$$\Psi_0 = ((k-1)((k-2)(2k-3)r^2 - 4k(k-2)rt + k(2k-1)t^2) \tag{6.28}$$

$$-\,((2k-3)r + (1-2k)t)2rtF')/(2\, e^{2F}\, r^{2k^2-2k+2}\, t^{2k^2-6k+6}), \tag{6.29}$$

$$\Psi_2 = \frac{(k-1)(k-2)r^2 - kt^2}{2\, e^{2F}\, r^{2k^2-2k+2}\, t^{2k^2-6k+6}}, \tag{6.30}$$

$$\Psi_4 = \frac{(k-1)(k-2)(2k-3)r^2 + 4k(k-2)rt + k(2k-1)t^2}{2\, e^{2F}\, r^{2k^2-2k+2}\, t^{2k^2-6k+6}}. \tag{6.31}$$

For the $\epsilon = -1$ case, Ψ_0, Ψ_2 and Ψ_4 is nonvanishing and the spacetime is Petrov type I except for $k = 1$ where Ψ_2 and Ψ_4 is vanishing and the spacetime is Petrov type N.

6.3.2. Radial Acceleration of Test Particles

The radial acceleration of a test particle initially at rest in a constant radius in the spacetime (6.15) is given by:

$$\ddot{r} = \frac{(k-k^2)r^{-1} - F'}{e^F r^{k^2-k}\, t^{q^2-q}}. \tag{6.32}$$

If we compare (6.32) with the Levi-Civita-Kasner metric, we see that the main difference is the term $\sim F'$ which characterizes the null radiation. When the F' is positive, the axis is more attractive whereas when it is negative, the axis is less attractive. Thus, the presence of the null dust may alter the particle motion.

6.3.3. Circular Geodesics

Let us study the equations of a test particle following a circular geodesics in the spacetime (6.15). Let us denote the angular velocity of a particle moving along a geodesics as ω, then we have:

$$\omega^2 = \frac{(k^2-k)r^{-1} + F'}{(1-k)e^{2F} r^{2k^2-1} t^{2(q^2-1)}}, \tag{6.33}$$

which results

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{k^2 - k + rF'}{1-k} - 1\right) e^{2F}\, r^{2(k^2-2)}\, t^{2(q^2-q)}. \tag{6.34}$$

Thus, the circular geodesics are timelike if the expression inside the parentheses is negative, null if it is zero and spacelike if it is positive. For the Levi-Civita and Levi-Civita-Kasner metrics the ranges of k where the geodesics are timelike, spacelike or null are the same. However, here we have extra terms proportional to $r\,F'$ and they are in general depend on time and the radial coordinate. This might have some consequences on particle motion. For example, when time passes, a particle following a circular geodesics may not continue to its motion since such geodesics become spacelike. Also for a given k, the circular geodesics might be restricted to a certain radius. Hence, the presence of the null radiation clearly affects the dependence of these ranges to the parameter k.

6.4. A Radiating Nonstatic String-like Object

Using the property of the Einstein-Rosen type solutions, we can construct examples of interior solutions having a nonstatic radiating object with a cosmic string like equation of state and generating outer radiating spacetime for particular values of the parameters k and q. The interior and exterior metrics are given by:

$$ds_-^2 = t^4 \left(e^{2F(r-t)}(-dt^2 + dr^2) + A(r)^2\, d\phi^2\right) + t^{-2} dz^2, \tag{6.35}$$

$$ds_+^2 = t^4 \left(e^{2F(r-t)}(-dt^2 + dr^2) + \alpha^2 r^2\, d\phi^2\right) + t^{-2} dz^2, \tag{6.36}$$

with corresponding energy momentum tensors:

$$T_{\mu\nu -} = T^{(R)}_{\mu\nu -} + T^{(S)}_{\mu\nu -}, \quad T_{\mu\nu +} = \eta_+ \mathbf{k}_\mu \mathbf{k}_\nu, \tag{6.37}$$

$$T^{(R)}_{\mu\nu -} = \kappa\, \eta_- \mathbf{k}_\mu \mathbf{k}_\nu,\ k_\mu = (1,1,0,0), \tag{6.38}$$

$$T^{0\ (S)}_{\ 0-} = T^{z\ (S)}_{\ z-} = \kappa\, \mu, \tag{6.39}$$

$$\eta_- = \frac{(t\, A' - A)\, F'}{t\, A}, \quad \mu = \frac{-A''}{A\, e^{2F} t^{-4}}, \quad \eta_+ = \frac{(t-r)\, F'}{t\, r}. \tag{6.40}$$

In these solutions, the interior and exterior metrics can smoothly match if the metrics and their first derivatives are continuous on the boundary of the stringlike object. Since we have chosen same inner and outer coordinates, this can be fulfilled if $A(r_0) = \alpha r_0$ and $A'(r_0) = \alpha$. These are called Lichnerovicz conditions and can be satisfied for the present case easily. For example if we choose $A(r) = \sin(b\, r)$ then the junction conditions yield:

$$\alpha = \sin(b r_0), \quad r_0 = \frac{\tan(b\, r_0)}{b}, \tag{6.41}$$

which can be easily satisfied since we have more parameters than equations.

Here the problem of these solutions is that, unlike μ, it seems that it may be impossible to η_- to be positive for all ranges of r and t. However, we can avoid negative energy density if we limit the ranges of the r and t where $\eta_- > 0$. Then, the solution can represent a radiating nonstationary cosmic string like object emitting null radiation.

7. CYLINDRICALLY SYMMETRIC STATIC PHOTONIC THIN SHELLS

As we have discussed in the previous chapters, the motion of massless test particles in the gravitational field of a cylindrical line source has interesting features. They can follow a circular or helical geodesics. Also due to the strong gravitational effects of such sources, photons not moving radially can be trapped. Although in general extended bodies do not follow geodesics, it might be interesting to analyze whether or not there are some solutions representing some structures composed of massless particles. For example the existence of a source distribution composed of counter propogating massless particles might be interesting. We use the term counter propogating to describe equal amounts of particles moving in opposite directions. This is necessary because we only consider the static Levi-Civita spacetime where the above interesting features hold.

In this chapter we will try to construct static thin shells composed of counter moving massless particles. We will find that there exist such sources satisfying certain energy conditions for different ranges of the exterior and interior metric parameters. Actually, since the static cylindrical thin shells have been studied extensively before [31], [65], [75]-[77] for different motivations by different authors, our solutions [78] reduce to these solutions for some special values of metric parameters. Actually, since previous studies generally considered such shells as sources generating the exterior Levi-Civita metric, in all solutions the interior region of the shell was taken as flat or locally flat. Our main difference is that we will treat the problem as general as possible. This requires that we have to take the interior and exterior regions as curved and characterized by different Levi-Civita metrics. This treatment enable us to interpret the solutions as shells surrounding a cylindrical source at the axis in general. We will also concentrate on the shell itself rather than the parameters of the exterior metric. Our construction can be pictured as follows. We have a line singularity at the symmetry axis, which can always be replaced by a regular interior cylinder solution since there are solutions which can generate and match smoothly to the vacuum Levi-Civita metric. Then there is a thin cyindrical shell composed of counter moving massless particles surrounding this line singularity. The exterior region of the shell is another vacuum Levi-Civita solution with different parameters. Before analyzing these shells we first review the standart thin shell formalism and using this formalism we derive the energy momentum tensor of the shell and the conditions when this shell can be composed of massless particles.

7.1. Darmois-Israel Thin Shell Formalism

Thin shells are highly idealized matter distributions. They are infinitely thin and have a dirac delta support. Thus, they cannot describe real physical sources. However, they can represent some sources when the *thickness* of the source is very small in one direction. Also, it is sometimes much easier to construct such sources than more regular sources, due to nonlinearity of the equations of general relativity. Thus, one can learn much by studying them. Actually, they have played an important role in studies of gravitational collapse, the evolution of bubbles and domain walls in cosmology, wormholes, the brane world models among others.

7.1.1. Gauss-Codazzi Equations

Here we introduce the gravitational quantities where the four dimensional spacetime is splited into three dimensional hypersurface with coordinates x^i, and one dimensional extra coordinate, y. Let us choose the spacetime metric in the Gaussian normal form:

$$ds^2 = \epsilon dy^2 + g_{ij}(y, x^i)\, dx^i dx^j, \tag{7.1}$$

where $\epsilon = \pm 1$ specifies the coordinate y as a space-like or a time-like coordinate and we choose hereafter $\epsilon = 1$. In this setting Riemann tensor has the form:

$$R^i_{\ jkl} \;=\; {}^{(3)}R^i_{\ jkl} + (K^i{}_l K_{kj} - K^i{}_k K_{lj}), \tag{7.2}$$

$$R^{y}{}_{ijk} = {}^{(3)}\nabla_k K_{ij} - {}^{(3)}\nabla_j K_{ik}, \tag{7.3}$$

$$R^{y}{}_{iyj} = -\partial_y K_{ij} - K_{il}K^{l}{}_{j}, \tag{7.4}$$

where K_{ij} is the exterior curvature tensor and ${}^{(3)}$ denotes quantities corresponding to the 3-metric g_{ij}. The first two equations are called as Gauss and Codazzi equations. Using these we can find the Einstein tensor as

$$G_{ij} = {}^{(3)}G_{ij} - \frac{\partial}{\partial y}(K_{ij} - \eta_{ij}K) + K^{l}{}_{i}K_{jl} - KK_{ij} + \frac{1}{2}g_{ij}(K^2 + K^{il}K_{jl}), \tag{7.5}$$

$$G_{yi} = {}^{(3)}\nabla_j K^{j}{}_{i} - {}^{(3)}\nabla_i K, \tag{7.6}$$

$$G^{y}{}_{y} = -{}^{(3)}R + \frac{1}{2}(K^2 - K^{ij}K_{ij}). \tag{7.7}$$

Using these we will now derive the equations of Darmois-Israel junction conditions and thin shell formalism [79],[80].

7.1.2. Junction Conditions

Now let us consider two spacetimes M^+ and M^- with Lorentzian signature $(-1, 1, 1, 1)$. We can define independent coordinate systems on both manifolds x^μ_+, x^μ_- with the metrics $g^+_{\mu\nu}$ and $g^-_{\mu\nu}$. We define two non-null 3-surfaces Σ^+ and Σ^- in M^+ and M^-. They induce metrics on the boundaries with $g_{ij}(\xi^i_\pm)$, where $\xi^i_\pm$ are intrinsic coordinates on the boundaries $\Sigma^\pm$. These surfaces $\Sigma^\pm$ divide the manifolds $M^\pm$ into two parts $M^\pm_1$ and $M^\pm_2$. Our aim is to construct a new manifold $\mathcal{M}$ using distinct parts of these manifolds $M^\pm$ by identifying $\Sigma^+ = \Sigma^- = \Sigma$ which means that the intrinsic coordinates of these surfaces become identical ($\xi^i_+ = \xi^i_- = \xi^i$). There are four possibilities $M^+_1 \cup M^-_1$, $M^+_1 \cup M^-_2$, $M^+_2 \cup M^-_1$, $M^+_2 \cup M^-_2$.

The parametric equations defining $\Sigma^\pm$ is given by

$$f_\pm(x^\mu(\xi^i)) = 0. \tag{7.8}$$

The unit 4-normals of timelike 3-surfaces $\Sigma^{\pm}$ in $M^{\pm}$ are given by

$$n_{\mu\pm} = \varepsilon \left(\left| g_{\pm}^{\nu\lambda} \frac{\partial f_{\pm}}{\partial x_{\pm}^{\nu}} \frac{\partial f_{\pm}}{\partial x_{\pm}^{\lambda}} \right| \right)^{-1/2} \frac{\partial f_{\pm}}{\partial x_{\pm}^{\mu}}, \tag{7.9}$$

where $n_\mu n^\mu = 1$ and $n_{\mu\pm} \neq 0$. Here $\varepsilon = \pm 1$ is an yet unspecified function. The basis vectors tangent to $\Sigma^{\pm}$ can be given as:

$$e_{i\pm}^{\mu} = \frac{\partial x_{\pm}^{\mu}}{\partial \xi_{\pm}^{i}}. \tag{7.10}$$

The induced metric on $\Sigma^{\pm}$ can be calculated as:

$$g_{ij\pm} = e_{i\pm}^{\mu}\, e_{j\pm}^{\nu}\, g_{\mu\nu\pm} = \frac{\partial x_{\pm}^{\mu}}{\partial x_{\pm}^{i}} \frac{\partial x_{\pm}^{\nu}}{\partial x_{\pm}^{j}}\, g_{\mu\nu\pm}. \tag{7.11}$$

The extrinsic curvature or second fundamental form is given by

$$K_{ij\pm} = \frac{\partial x_{\pm}^{\mu}}{\partial x_{\pm}^{i}} \frac{\partial x_{\pm}^{\nu}}{\partial x_{\pm}^{j}} \nabla_{\mu} n_{\nu\pm} \tag{7.12}$$

$$= n_{\lambda\pm} \left(\frac{\partial^2 x_{\pm}^{\lambda}}{\partial \xi_{\pm}^{i} \partial \xi_{\pm}^{j}} + \Gamma_{\mu\nu\pm}^{\lambda} \frac{\partial x_{\pm}^{\mu}}{\partial \xi_{\pm}^{i}} \frac{\partial x_{\pm}^{\nu}}{\partial \xi_{\pm}^{j}} \right). \tag{7.13}$$

Identifying the boundaries $\Sigma^{\pm}$ as a common boundary separating these two space-times $M^{\pm}$, implies that $\xi_{+} = \xi_{-} = \xi$. This leads the first junction condition on Σ :

$$[g_{ij}] = 0. \tag{7.14}$$

where $[X] = X^{+}|_{\Sigma} - X^{-}|_{\Sigma}$. This means that the induced metric from either sides on Σ must be continuous or the first fundamental form $g_{ij} = g_{\mu\nu} - n_\mu n_\nu$ should be continuous on Σ.

Next we assume that the energy momentum tensor $T^{\mu}_{\ \nu}$ of the whole manifold $M = M^{+} \cup M^{-}$ has a delta function contribution on Σ. Then the integral of $T^{\mu}_{\ \nu}$ with respect to proper distance y measured perpendicularly through Σ:

$$S^{\mu}{}_{\nu} = \lim_{\eta \to 0} \left(\int_{-\eta}^{\eta} T^{\mu}{}_{\nu}\, dy \right), \tag{7.15}$$

is non vanishing and represents the surface energy momentum tensor of Σ. Since the induced metric is the same from the both sides of the Σ, if it is regular then the only terms contributing to the integral for $\kappa T^{\mu}{}_{\nu} = G^{\mu}{}_{\nu}$ with $G_{\mu\nu}$ is given in (7.5)-(7.7) are the terms proportional to $\frac{\partial}{\partial y}$. The above integral gives us:

$$\begin{aligned} \kappa S^{y}{}_{y} &= 0, & (7.16)\\ \kappa S^{y}{}_{i} &= 0, & (7.17)\\ \kappa S^{i}{}_{j} &= -\left([K^{i}{}_{j}] - g^{i}{}_{j}[K] \right), & (7.18) \end{aligned}$$

since the extrinsic curvature tensor may have step function discontinuities on the both sides of the Σ. These are the second junction conditions on Σ. When Σ is just a boundary surface with no energy momentum tensor, the second junction conditions becomes

$$[K_{ij}] = 0. \tag{7.19}$$

As we have shown above, if Σ has a delta function support with a matter source on it then the second junction condition gives the surface energy momentum tensor S_{ij} of the shell

$$\kappa S^{i}{}_{j} = -\left([K^{i}{}_{j}] - g^{i}{}_{j}[K] \right). \tag{7.20}$$

This equation is called as Lanczos equation and it defines the dynamics of the shell together with an equation of state of the matter content of the shell. The remaining Gaus-Codazzi equations behave as constraints. The equation (7.6) is called as "ADM" constraint and (7.7) is called as "Hamiltonian" constraint. These equations are valid on both sides of the shell and take their limiting values when one approaches to Σ.

The ADM constraint gives the conservation identity:

$$ {}^{(3)}\nabla_i S^i{}_j = -[T_{\mu\nu} n^\mu e^\nu_j], \tag{7.21} $$

and

$$ {}^{(3)}\nabla_i (\overline{K}^i{}_j - \overline{K}) = \kappa \overline{(T_{\mu\nu} n^\mu e^\nu_i)}, \tag{7.22} $$

where $\overline{X} = (X^+|_\Sigma + X^-|_\Sigma)/2$.

The Hamiltonian constraint along with the Einstein and Lanczos equations yield the evolution identity:

$$ S^{ij}\overline{K}_{ij} = -[T_{\mu\nu} n^\mu n^\nu], \tag{7.23} $$

and

$$ {}^{(3)}R + \left(\overline{K}_{ij}\overline{K}^{ij} - K^2\right) = 2\kappa \overline{(T_{\mu\nu} n^\mu n^\nu)} + \kappa^2/4(S^{ij}S_{ij} - S^2/2). \tag{7.24} $$

These junction conditions are not all independent. As an addition to the Lancsoz equation, the conservation identity (7.21) and evolution identity (7.23) should be satisfied.

Now let us clarify an important ambiguity [81] that the orientation of the unit normals n_μ (7.9) of the shell Σ have an arbitrary sign denoted by $\varepsilon = \pm 1$. The Israel formalism requires that the normals in $\mathcal{M}$ to point from M^-_A to M^+_B where $A = 1, 2$ denotes the portion of M^- and $B = 1, 2$ denotes the portion of M^+ which $\mathcal{M}$ is formed by the union of these. For a timelike shell that we consider here, n_μ must be directed from M^- to M^+ (in the direction of increasing of a space-like coordinate related with the spacelike unit normal of $M^\pm$).

The most important advantage of the Darois-Israel thin shell formalism [79],[80] is that the junction conditions leave the interior and exterior coordinates free in general.

7.2. Matching Conditions and Energy Momentum Tensor of the Shell

In this section we will derive the energy momentum tensor of the static cylindrical thin shell. We will use the Darmois-Israel matching conditions since we want to treat the shell as general as possible and find out under what conditions these shell can be considered as photonic shells. Thus we chose the interior and the exterior regions of the shell represented by two static cylindrically symmetric vacuum Levi-Civita solutions with different parameters having different radial coordinates. We first write the Levi-Civita metric (4.7) in normal form by transforming the radius ρ into a proper radius r by defining $dr = \rho^{2\sigma(2\sigma-1)}d\rho$ which results in

$$ds^2 = -R^{4\sigma/N}dt^2 + dr^2 + R^{4\sigma(2\sigma-1)/N}dz^2 + \alpha^2 R^{2(1-2\sigma)/N}d\phi^2, \qquad (7.25)$$

where

$$\rho = R^{1/N}, \quad R = Nr, \quad N = 4\sigma^2 - 2\sigma + 1. \qquad (7.26)$$

In order to achieve the continuity we transform the metric in a more appropriate form since we can rescale the ignorable coordinates t by and z by $t = (Nr_0)^{-2\sigma/N}t'$ and $z = (Nr_0)^{-2\sigma(2\sigma-1)/N}z'$. We can not rescale ϕ if we want to keep its periodicity but we can define a new deficit parameter as $\alpha = (Nr_0)^{2\sigma-1}r_0\alpha'$. The metric (7.25) becomes:

$$ds^2 = -\left(\frac{r}{r_0}\right)^{4\sigma/N}dt^2 + dr^2 + \left(\frac{r}{r_0}\right)^{4\sigma(2\sigma-1)/N}dz^2 + \alpha^2 r_0^2\left(\frac{r}{r_0}\right)^{2(1-2\sigma)/N}d\phi^2, \qquad (7.27)$$

where we have removed primes for clarity. Now, let us denote the interior and the exterior radial coordinates as $r_\pm$ and the location of the shell at $(r_\pm = r_{0\pm})$. We want the induced metrics of both the interior and the exterior spacetimes to be continuous on the shell. The interior metric for $(r_- \leq r_{0-})$ is given by:

$$ds_-^2 = -\left(\frac{r_-}{r_{0-}}\right)^{\frac{4\sigma'}{N'}}dt^2 + dr_-^2 + \left(\frac{r_-}{r_{0-}}\right)^{\frac{4\sigma'(2\sigma'-1)}{N'}}dz^2 + \alpha_-^2 r_{0-}^2\left(\frac{r_-}{r_{0-}}\right)^{\frac{2(1-2\sigma')}{N'}}d\phi^2, \qquad (7.28)$$

and the exterior metric for $(r_+ \geq r_{0+})$ is given by

$$ds_+^2 = -\left(\frac{r_+}{r_{0+}}\right)^{\frac{4\sigma}{N}} dt^2 + dr_+^2 + \left(\frac{r_+}{r_{0+}}\right)^{\frac{4\sigma(2\sigma-1)}{N}} dz^2 + \alpha_+^2 r_{0+}^2 \left(\frac{r_+}{r_{0+}}\right)^{\frac{2(1-2\sigma)}{N}} d\phi^2. \tag{7.29}$$

The continuity at $r_- = r_{0-}$, $r_+ = r_{0+}$ requires:

$$\alpha_+ = \frac{\alpha_- r_{0-}}{r_{0+}}. \tag{7.30}$$

The shell $\Sigma^\pm$ is located at $r_\pm = r_{0\pm}$, thus $f_\pm = r_\pm - r_{0\pm}$ and from (7.8) the unit normals of the interior and exterior metrics are calculated as $n_{\mu\pm} = \delta_\mu^{r\pm}$. Since we use the metrics in Gaussian normal form, the extrinsic curvature tensor (7.12) has a simple form

$$K_{ij\pm} = -\frac{1}{2}\frac{\partial g_{ij\pm}}{\partial r_\pm}, \tag{7.31}$$

where $g_{ij\pm}$ is the (ij)th components of the interior and exterior metrics (7.28) and (7.29). The nonzero components of the extrinsic curvature tensor are calculated as:

$$K^t{}_{t+} = \frac{2\sigma}{Nr_+}, \qquad K^t{}_{t-} = \frac{2\sigma'}{N'r_-}, \tag{7.32}$$

$$K^z{}_{z+} = \frac{2\sigma(2\sigma-1)}{Nr_+}, \qquad K^z{}_{z-} = \frac{2\sigma'(2\sigma'-1)}{N'r_-}, \tag{7.33}$$

$$K^\phi{}_{\phi+} = \frac{(1-2\sigma)}{Nr_+}, \qquad K^\phi{}_{\phi-} = \frac{(1-2\sigma')}{N'r_-}, \tag{7.34}$$

$$K_+ = K^i{}_{i+} = \frac{1}{Nr_+}, \qquad K_- = K^i{}_{i-} = \frac{1}{N'r_-}. \tag{7.35}$$

The induced metric on the shell Σ is given by

$$ds^2|_\Sigma = -dt^2 + dz^2 + \alpha_-^2 r_{0-}^2\, d\phi^2, \tag{7.36}$$

and it is flat.

After evaluating (7.32)-(7.35) on the shell on Σ by taking $r_\pm = r_{0\pm}$ and replacing into the Lanczos equation (7.20), the energy-momentum tensor of the shell can be written as (we take $c = G = 1$ and $\kappa = 8\pi$):

$$8\pi {S^0}_0 = \frac{N - 2\sigma}{N r_{0+}} - \frac{N' - 2\sigma'}{N' r_{0-}}, \tag{7.37}$$

$$8\pi {S^z}_z = \frac{1}{N r_{0+}} - \frac{1}{N' r_{0-}}, \tag{7.38}$$

$$8\pi {S^\phi}_\phi = \frac{4\sigma^2}{N r_{0+}} - \frac{4\sigma'^2}{N' r_{0-}}. \tag{7.39}$$

For a given energy momentum tensor of the form $T^a_b = \mathrm{diag}(-\rho, p_i)$ where ρ is the energy density, p_i's are principal pressures, $i = 1, 2, 3$ for four dimensional spacetime and $i = 1, 2$ for a thin shell, we have three different energy conditions [82].

- Weak energy condition:

$$\rho \geq 0, \quad \rho + p_i \geq 0. \tag{7.40}$$

- Dominant energy condition:

$$\rho \geq 0, \quad -\rho \leq p_i \leq \rho. \tag{7.41}$$

- Strong energy condition

$$\rho \geq 0, \quad \rho + p_i \geq 0, \quad \rho + \sum_i p_i \geq 0. \tag{7.42}$$

7.3. Photonic Shells

The trace of the energy momentum tensor of the shell (7.37)-(7.39) is

$$4\pi\, S = \frac{1}{r_{0+}} - \frac{1}{r_{0-}}. \tag{7.43}$$

Thus, the vanishing of the trace of energy-momentum tensor of the shell requires $r_{0+} = r_{0-}$. Equivalently, we can perform a coordinate transformation on the inner or the outer independent radial coordinates r_- or r_+ to express one of them in terms of the other. In this case the condition can be expressed as follows: If there is no relative shift in the values of the interior and the exterior radial coordinates on the shell, than the shell is composed of massless particles. If there is a shift, than the shell is composed of massive particles. This fact is stressed previously in the works [31], [66],[65], [72], [77].

7.4. General Case: a Thin Shell with Counter Moving Photons Along a Helical Path

It is known that a static cylindrical thin shell with vanishing trace of the energy momentum tensor requires no relative shift for the interior and the exterior radial coordinates. Thus, we use same radial coordinates for the interior and the exterior regions of the shell, namely $r_- = r_+ = r$ and the location of the shell becomes $r_{0+} = r_{0-} = r_0$. Then the nonzero components of the stress energy tensor of the shell becomes:

$$8\pi S^0{}_0 = \frac{2\sigma'}{N' r_0} - \frac{2\sigma}{N r_0}, \tag{7.44}$$

$$8\pi S^z{}_z = \frac{1}{N r_0} - \frac{1}{N' r_0}, \tag{7.45}$$

$$8\pi S^\phi{}_\phi = \frac{4\sigma^2}{N r_0}. - \frac{4\sigma'^2}{N' r_0}. \tag{7.46}$$

We can also write the stress energy tensor of the shell using the parameters of the Kasner form of the metric by replacing $a = \frac{2\sigma}{N}$, $b = \frac{2\sigma(2\sigma-1)}{N}$ and $c = \frac{1-2\sigma}{N}$ for the exterior and their primed counterparts for the interior. This results:

$$8\pi S^0{}_0 = \frac{a' - a}{r_0}, \tag{7.47}$$

$$8\pi S^z{}_z = \frac{b' - b}{r_0}, \tag{7.48}$$

$$8\pi S^\phi{}_\phi = \frac{c' - c}{r_0}. \tag{7.49}$$

The above stress energy tensor can represent an anisotropic fluid with

$$S^i{}_j = \mathrm{diag}(-\rho, p_z, p_\phi), \tag{7.50}$$

where ρ is the energy density and p_i $(i = z, \phi)$ are the principal pressures [82]. The energy density and the pressures are

$$8\pi\rho = \frac{2\sigma}{Nr_0} - \frac{2\sigma'}{N'r_0} = \frac{a - a'}{r_0}, \tag{7.51}$$

$$8\pi p_z = \frac{1}{Nr_0} - \frac{1}{N'r_0} = \frac{b' - b}{r_0}, \tag{7.52}$$

$$8\pi p_\phi = \frac{4\sigma^2}{Nr_0} - \frac{4\sigma'^2}{N'r_0} = \frac{c' - c}{r_0}. \tag{7.53}$$

As we have said the stress energy tensor of the shell is traceless $S^i{}_i = 0$ and this result can be interpreted as an infinitely long thin shell along the z direction with radius r_0 composed of equal amount of oppositely moving massless particles along a helical direction. This helical motion gives rise to pressures in the ϕ and z directions with the equation of state $\rho = p_z + p_\phi$. Thus if one chooses the interior and the exterior metrics of the shell as Levi-Civita metrics with same radial coordinates then the shell is necessarily composed of massles particles.

The stress energy tensor of this solution can also represent two counter streams of pure radiation with

$$T_{\mu\nu} = [p_z(k_\mu k_\nu + l_\mu l_\nu) + p_\phi(m_\mu m_\nu + n_\mu n_\nu)]\,\delta(r - r_0), \tag{7.54}$$

where $p_z + p_\phi = \rho$ and k, l, m, n being null vectors satisfying

$$k^\mu k_\mu = l^\mu l_\mu = m^\mu m_\mu = n^\mu n_\mu = 0, \quad k^\mu l_\mu = m^\mu n_\mu = -1. \tag{7.55}$$

They are given by

$$k^{\mu} = \frac{1}{\sqrt{2}}(-1,0,1,0), \quad l^{\mu} = \frac{1}{\sqrt{2}}(-1,0,-1,0), \tag{7.56}$$

$$n^{\mu} = \frac{1}{\sqrt{2}}(-1,0,0,1), \quad m^{\mu} = \frac{1}{\sqrt{2}}(-1,0,0,-1). \tag{7.57}$$

The null vectors k^{μ} and l^{μ} represent two streams of photons counter moving in positive and negative z directions whereas n^{μ} and m^{μ} represent two counter rotating streams of photons.

Energy per unit length of the shell, according to Marder's definition[83] is given by :

$$\mu_M = 2\pi r_0\, \alpha\rho = \frac{\alpha}{4}(a-a') = \frac{\alpha}{2}\left(\frac{\sigma}{N} - \frac{\sigma'}{N'}\right), \tag{7.58}$$

whereas according to Israel's definition [84] it is given by:

$$\mu_I = 2\pi r_0\, \alpha(\rho + p_z + p_\phi) = 2\mu_M = \frac{\alpha}{2}(a-a') = \alpha\left(\frac{\sigma}{N} - \frac{\sigma'}{N'}\right). \tag{7.59}$$

and both are independent of r_0. Thus the shell is in neutral equilibrium with respect to changes of r_0. If we do not allow sources having negative energy, then the maximum value of energy per unit length is $\mu_M = 1/4$ and $\mu_I = 1/2$.

For $(a-a') > 0$ the shell has positive energy density. We have a line singularity at $r = 0$ in general. The singularity has positive energy density for a' positive and negative effective mass density for negative a'. The only singularity free metric is flat Minkowski metric and if we take interior metric (8.14) as Minkowski metric ($a' = b' = 0, c' = 1, \alpha = 1$) then our results are in accordance with [31] and [77]. In this case the shell has positive energy density for a positive and satisfies all energy conditions for $|a| > |b|$. Note that for $\alpha = 1, a' = 0$ μ_I gives the correct Newtonian limit of a cylindrical source since $\mu_I = \sigma$ for small σ hovewer the μ_M does not.

7.4.1. An Isotropic Photonic Thin Shell

By choosing $p_z = p_\phi$ for the energy momentum tensor of the previous shell solution, we find a cylindrical thin shell composed of photons which has isotropic pressures in z and ϕ directions. The exterior and the interior metric parameters σ and σ' should satisfy

$$\sigma = \frac{4\sigma' - 1}{4(\sigma' - 1)}. \tag{7.60}$$

The energy-momentum tensor of the shell becomes:

$$8\pi\rho = 16\pi p_z = 16\pi p_\phi = \frac{8\sigma'^2 - 18\sigma' + 2}{(12\sigma'^2 - 6\sigma' + 3)r_0}. \tag{7.61}$$

For flat interior ($\sigma' = 0$), we have $\sigma = 1/4$. and $8\pi\rho = 2/3r_0$. For $\sigma' < (1 - \sqrt{3}/2)$ or $\sigma' > (1 + \sqrt{3}/2)$ the shell has positive energy density and satisfies all energy conditions.

7.5. A Thin Shell with Counter Rotating Photons

Now we would like to discuss the case corresponding to a cylindrical shell composed of counter-rotating photons. We find a static cylindrical shell at $r = r_0$ with $\rho = p_\phi$ with other components of the energy momentum tensor vanishing. Taking $S_{zz} = 0$ in (7.45) yields $N = N'$ and this has only two solutions

$$\sigma' = \sigma \quad \text{or} \quad \sigma' = (1 - 2\sigma)/2. \tag{7.62}$$

For the Kasner parameters $S_{zz} = 0$ yields $b = b'$ and this yields two solutions

$$(a, b, c) = (a', b', c') \quad \text{or} \quad (a, b, c) = (c', b', a'). \tag{7.63}$$

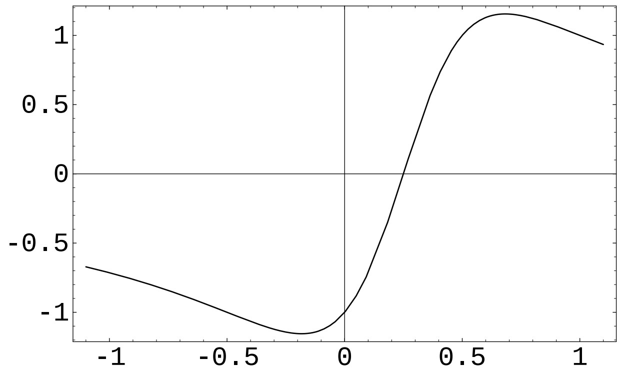

Figure 7.1. The energy density of the shell with counter rotating photons as a function of exterior metric parameter σ

Since for the first choice the shell disappears we have to consider the second cases. The interior and exterior metrics can be written in Kasner form as

$$ds_-^2 = -(\frac{r}{r_0})^{2c}dt^2 + dr^2 + (\frac{r}{r_0})^{2b}dz^2 + (\frac{r}{r_0})^{2a}\alpha^2\ r_0^2 d\phi^2, \quad (r < r_0), \tag{7.64}$$

and

$$ds_+^2 = -(\frac{r}{r_0})^{2a}dt^2 + dr^2 + (\frac{r}{r_0})^{2b}dz^2 + (\frac{r}{r_0})^{2c}\alpha^2\ r_0^2 d\phi^2. \quad (r > r_0). \tag{7.65}$$

The stress energy tensor satisfies

$$\rho = p_\phi = \frac{4\sigma - 1}{N r_0} = \frac{a - c}{r_0}. \tag{7.66}$$

For this case to have positive energy density on the shell we need $\sigma > 1/4$ or $a > c$. Thus for the exterior region of the shell the Levi-Civita parameter must be $\sigma > 1/4$. For this range of σ no circular null or timelike geodesics exist and the photons not moving pure radially are trapped and cannot escape to the radial infinity. For the interior region of the shell the situation is reverse and we need $\sigma' < 1/4$. Unless for $\sigma = 0$ there is a line singularity at the axis surrounded by the shell. The energy density of this cylindrical shell has a finite maximum value (Figure 7.1) at $\sigma = \frac{1+\sqrt{3}}{4}$. and this is consistent with previous static cylindrical shell solutions [31]. Note that the interior metric (7.64) always admits helical null geodesics but exterior one (7.65) does not. In the interior region the photons can reach to the shell and timelike circular geodesics or helical null geodesics present.

The stress energy tensor of this solution can be interpreted as two counter rotating pure radiation fields with

$$T_{\mu\nu} = p_\phi(m_\mu m_\nu + n_\mu n_\nu)\delta(r - r_0), \tag{7.67}$$

where m, n are null vectors satisfying

$$m^\mu m_\mu = n^\mu n_\mu = 0, m^\mu n_\mu = -1. \tag{7.68}$$

They are given by

$$n^\mu = \frac{1}{\sqrt{2}}(-1, 0, 0, 1), \quad m^\mu = \frac{1}{\sqrt{2}}(-1, 0, 0, -1). \tag{7.69}$$

The null vectors n^μ and m^μ represent two counter rotating streams of photons.

The energy per unit length of this shell has the expressions:

$$\mu_M = \frac{\alpha}{4}(a - c), \quad \mu_I = \frac{\alpha}{2}(a - c). \tag{7.70}$$

As we mentioned before, unless we choose globally Minkowski metric inside the shell, at $r = 0$ there is a line singularity. When the shell satisfies positive energy condition the singularity at the origin has positive energy density. The singularity free configuration for the interior metric gives an interesting case which requires further attention.

7.5.1. A Photonic Cylindrical Shell Embedded in Flat Spacetime

The choice $b = c = 0$, $a = 1$, $(\sigma = 1/2)$ for the parameters of the exterior and interior metrics (7.64) and (7.64) leads to a cylindrical shell solution with locally flat interior whereas taking also $\alpha = 1$ gives globally flat interior. The interior and the exterior metrics become:

$$ds_-^2 = -dt^2 + dr^2 + dz^2 + \alpha^2 r^2 d\phi^2, \quad (r \leq r_0), \tag{7.71}$$

$$ds_+^2 = -(\frac{r}{r_0})^2 dt^2 + dr^2 + dz^2 + \alpha^2 r_0^2 d\phi^2, \quad (r \geq r_0). \tag{7.72}$$

The interior metric is the flat Minkowski metric written in cylindrical coordinates. Exterior metric is also Riemann flat, hence it is Minkowski metric written in a different form. The transformations $T = r/r_0 \sinh t$, $X = r/r_0 \cosh t$, $Z = z$, $Y = \alpha r_0 \phi$, put this metric into standart Minkowski form:

$$ds^2 = -dT^2 + dX^2 + dY^2 + dZ^2. \tag{7.73}$$

The metric (7.72) is the Rindler metric [29]. It represents a uniform gravitational field. The gravitational field of a massive plane in Newtonian theory is also uniform. One may speculate that for this case the cylinder becomes an infinite plane for an observer at the outside of the shell. This speculation can be strenghened by the presence of a solution where the Rindler metric has been shown to represent the gravitational field of massive Newtonian plane [32]. However an observer at the inside of the shell sees that he is surrounded by a cylindrical shell. More interestingly, we have an energy distribution whose interior and exterior is flat and one might think that the presence of an energy distribution on the shell does not effect the exterior gravitational field, since it is flat. However, this is not the case, since, althought the interior and the exterior regions are flat, there is no global flat metric covering both the interior and the exterior regions.

After we present this solution in our papers [78] and [85], Zofka studied its properties in more detail [86]. Let us summarize his results. He studied the geodesics of test particles passing through the shell. They move along straight lines at a constant speed (in Mimkowski coordinates). When they cross the shell (plane) from outside, since the particle's four-velocity must be continuous (we neglect any interaction with the shell), the particle's direction of motion remains the same but the magnitude of its coordinate velocity changes. Since the flat interior region is surrounded by a cylinder with finite radius, they hit again the shell (cylinder) and leave the cylinder with the same angle they emerge. Studying the geodesics of outside region, he show that the free massive particles and photons leaving the shell fall back under the same angle they left the shell before after a finite proper time or affine parameter. Since the exterior region is characterized by $\sigma = 1/2$, only photons moving pure radially can espace to the radial infinity (see Section 3). If we use Minkowski coordinates for the exterior, then the geodesics lying inside the light cone (which are straight lines) leaving the shell at some point will intersect with the cylindrical surface again. Outside observers see the cylinder as an infinite planar wall falling on them at speed T/X. In this frame plane is not static. The plane will always hit them how matter they try. When they pass through the shell they realize that they are surrounded by a cylindrical shell. They can stay inside the shell by using a rocket for whatever time they want. However, they can always penetrate the outside of the cylinder and reemerge in their original spacetime.

The solution we have discussed has the property that we have a thin shell of matter composed of photons (counter) moving along circular orbits seperating two flat regions of spacetime. One can ask what holds photons together since there is no matter source inside the cylinder. For the metrics (7.71) (7.72) the shell is static. The centrifugal force acting on the particles of the shell is balanced by the gravitational force exerted on them by the other particles on the shell. If we take into account the two Minkowski metrics (7.71), (7.73), the outer shell surface is not static and accelerated towards free particles at rest in such a system and the centrifugal force acting on the shell is balanced by this effect.

Notice that there are similar thin shell solutions seperating two flat spacetimes.

Lemos presented a solution where an infinite thin disk composed of counter rotating photons whose both sides are flat [87]. Another example is a thin spherical shell seperating two identical regions of Minkowski spacetime having finite volume [88].

7.6. A Thin Shell with Counter Moving Photons in the z direction

In this case we find a shell with the equation of state $\rho = p_z$. Thus we need $c = c'$ ($4\sigma^2/N = 4\sigma'^2/N'$) in (8.22) which leads to $a = b'$ and $a' = b$ ($\sigma' = \sigma/(2\sigma - 1)$) then the interior and exterior metrics become:

$$ds_-^2 = -(\frac{r}{r_0})^{2b}dt^2 + dr^2 + (\frac{r}{r_0})^{2a}dz^2 + (\frac{r}{r_0})^{2c}\alpha^2\ r_0^2 d\phi^2 \quad (r < r_0), \tag{7.74}$$

and

$$ds_+^2 = -(\frac{r}{r_0})^{2a}dt^2 + dr^2 + (\frac{r}{r_0})^{2b}dz^2 + (\frac{r}{r_0})^{2c}\alpha^2\ r_0^2 d\phi^2, \quad (r > r_0). \tag{7.75}$$

and the only nonzero elements of the Einstein tensor are

$$G_{00} = G_{zz} = \frac{a-b}{r_0} = \frac{4\sigma(1-\sigma)}{Nr_0}. \tag{7.76}$$

The stress energy tensor of this solution can also represent two counter moving pure radiation in z direction with

$$T_{\mu\nu} = p_z(k_\mu k_\nu + l_\mu l_\nu)\delta(r - r_0), \tag{7.77}$$

where k, l are null vectors satisfying

$$k^\mu k_\mu = l^\mu l_\mu = 0, k^\mu l_\mu = -1. \tag{7.78}$$

They are given by

$$k^{\mu} = \frac{1}{\sqrt{2}}(-1, 0, 1, 0), \quad l^{\mu} = \frac{1}{\sqrt{2}}(-1, 0, -1, 0). \tag{7.79}$$

The null vectors k^{μ} and l^{μ} represents two counter propogating streams of photons.

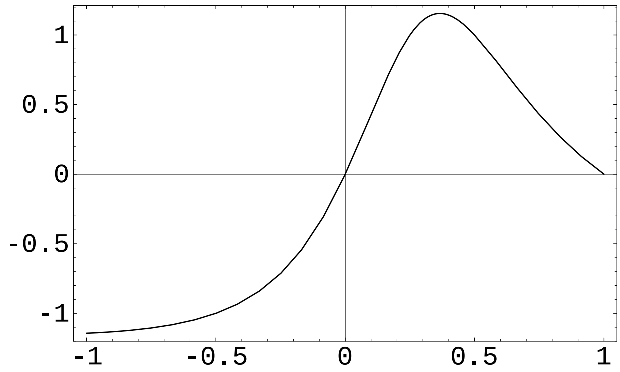

Figure 7.2. The energy density of the shell with photons counter-moving in z direction as a function of exterior metric parameter σ

To have a positive energy density of the shell we must have $a-b > 0$ or $0 < \sigma < 1$ where in this range the shell satisfies all energy conditions. Note that for this solution, unlike the first two cases, interior metric parameter is negative when the exterior metric parameter has the range $0 < \sigma < 1/2$ and the singularity at $r = 0$ has negative energy density. This might be expected since for this case the shell has no rotational pressure and we need an repelling force to keep this shel static. However for $\sigma > 1/2$ the shell satisfies the energy conditions but the interior metric parameter is positive impling the line singularity has positive mass density. However for this range of σ whether the metric describes a cylindrical source or not is not clear. Also if the possible change of the role of the coordinates z and ϕ occurs at $\sigma = 1/2$, then also for $\sigma > 1/2$ the interior metric parameter becomes negative. The energy density is finite (Figure 7.2). The exterior metric (7.75) of the physically acceptable shell may admit helical null geodesics but interior metric (7.74) does not. The energy per unit lenght of the shell is $\mu = \alpha(a-b)$. If the interior is chosen as Riemann flat $(a = b = 0, c = 1)$ then this shell disappears. Thus there is no photonic shell with axially moving photons surrounding a cosmic string.

8. MULTIPLE PHOTONIC SHELLS

Infinitely thin static shells composed of equal amounts of oppositely moving photons following circular or axial trajectories or their combinations around a line singularity [78] have been discussed in the previous section. In this section, we will generalize this solution to the solutions having multiple photonic shells around a line singularity [85]. Multiple photonic shells around a cosmic string will be obtained as a limiting case. We will show that the exterior region of the (multiple) shells may become an infinite plane wall.

In this section we consider cylindrically symmetric vacuum Levi-Civita metric in Kasner form

$$ds^2 = -r^{2a}dt^2 + dr^2 + r^{2b}dz^2 + r^{2c}\, r_0^2\, \alpha^2 d\phi^2, \tag{8.1}$$

Here α, a, b, c are real constants and r is the radial coordinate. The constant parameters satisfy

$$a + b + c = a^2 + b^2 + c^2 = 1, \tag{8.2}$$

so only one of the parameters a, b, c is free. For detailed discussion of the properties of this metric see Section 3. The metric reduces to the famous cosmic string metric [13] when $a = b = 0, c = 1$ keeping the angular defect parameter $\alpha < 1$. In this chapter we will again construct thin shells. However for this case the distributional formalism is more appropriate.

8.1. Distributional Formalism for Thin Shells

Our spacetime $\mathcal{M}$ consists of two different spacetimes (M^+, M^-) characterized by the coordinates (x_+^μ, x_-^ν) and the metrics $(g_{\mu\nu}^+, g_{\mu\nu}^-)$. For these manifolds we can consider some boundaries (Σ^+, Σ^-) having the intrinsic coordinates (ξ_+^i, ξ_-^i) with the metrics

(g_{ij}^+, g_{ij}^-). If we identify $\Sigma_+ = \Sigma_- = \Sigma$ we have a single manifold $M = M^+ \cup M^-$ which means that these two boundaries must be isometric to each other with same coordinates ($\xi_+ = \xi_- = \xi$). Thus, in order to glue these manifolds via a hypersurface we need

$$[g_{ij}] = 0. \tag{8.3}$$

where $[X]$ gives the jump of the quantity X at the hypersurface. Thus, the metrics induced on the hypersurface ($g_{ij}(\xi_\pm)$) must be same from the both sides of it. If the second fundamental forms or extrinsic curvatures are also continuous, namely

$$[K_{ij}] = 0, \tag{8.4}$$

on the hypersurface, we have a smooth matching of two manifolds via a common hypersurface. The conditions (8.3) and (8.4) are called the Darmois conditions [89]. For boundary surfaces these two conditions should be satisfied, but for a thin shell, the second condition may not be satisfied since the jump of the extrinsic curvature K_{ij} is related with the matter content of the shell and we have to consider the equation (7.20) instead of (8.4). For details see the previous chapter. Note that the condition (8.3) leaves the coordinates x_+^μ, x_-^μ free. This is an advantage of the Darmois-Israel method.

Let us assume a more strict condition for matching process, namely on the hypersurface

$$[g_{\mu\nu}] = 0, \tag{8.5}$$

which means that the metrics $g_{\mu\nu}^\pm$ with coordinates x_+^μ of $M^\pm$ are continuous at the hypersurface (if they are expressed with the same coordinates). To establish such a condition we have to express the metrics of the both spacetimes in both sides of the shell with the same coordinates x^μ. Thus we have to introduce a common coordinate system valid in both sides of the shell and transform both metrics to new metrics written in this new coordinate system. If the coordinates in either side of the shell are admissable coordinates, i. e., if they permit such transformations, then we can consider this matching condition (8.5). Together with the continuity of the first derivatives of the metric at the hypersurface

$$\left[\frac{\partial g_{ab}}{\partial x^c}\right] = 0, \tag{8.6}$$

these conditions (8.5) and (8.6) are called the Lichnerowicz conditions [90]. The first condition (8.5) also ensures that we do not have any derivatives of the dirac delta function in Einstein tensor since it contains up to second order derivatives of the metric. If the shell is only a boundary surface both conditions must be hold, but for a thin shell second condition may not be satisfied since the discontinuities of the derivatives of the metric will give rise to an infinitely thin shell. In the distributional method one must use the Lichnerowicz conditions. Although the Darmois conditions are more superior since one may have independent coordinate systems for the both side of the shell, for some cases, the calculations will be easier with the metrics satisfying Lichnerowicz conditions.

In distributional method, we have two different manifolds with two different sets of admissable coordinates. By a coordinate transformation using the boundary conditions arising from Lichnerowitz conditions we express whole manifold with same sets of coordinates x^μ. Then, the hypersurface, whose location is described by a function $f(x^\mu) = 0$, divides spacetime into two regions: Outer regions of the shell where $f(x^\mu) > 0$ and inner regions of the shell where $f(x^\mu) < 0$. The $\pm$, in this case, denotes the exterior and the interior regions of the shell, respectively. As we said, the metric should satisfy:

$$\overset{+}{g}_{\mu\nu}|_{(f=0)} = \overset{-}{g}_{\mu\nu}|_{(f=0)}. \tag{8.7}$$

The spacetime metric can be written in a local basis with the help of distributions as:

$$ds^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu, \tag{8.8}$$

where $\tilde{g}$ is the hybrid metric and is given by

$$\tilde{g}_{\mu\nu} = \theta(f(x^\mu))\ g^+_{\mu\nu} + \theta(-f(x^\mu))\ g^-_{\mu\nu}. \tag{8.9}$$

Here θ is the Heaviside step function. Let us take the first and second derivatives of the hybrid metric. The first derivatives does not result a delta dunction singularity since the metrics are continuous on the shell.

$$\tilde{g}^{'}_{\mu\nu} = \theta(f(x^\mu))\ g^{'+}_{\mu\nu} + \theta(-f(x^\mu))\ g^{'-}_{\mu\nu}. \tag{8.10}$$

The second derivatives may give rise to the terms involving delta functions:

$$\tilde{g}^{''}_{\mu\nu} = \theta(f(x^\mu))\ g^{''+}_{\mu\nu} + \theta(-f(x^\mu))\ g^{''-}_{\mu\nu} + f'(x^\mu)[g'_{\mu\nu}]\ \delta(f(x^\mu)), \tag{8.11}$$

where $[g'_{\mu\nu}] = g^{'+}_{\mu\nu} - g^{'-}_{\mu\nu}$ is the jump of derivatives of the interior and exterior metrics on the shell.

Calculating the Einstein tensor for the hybrid metric (8.8) and considering the Einstein equations one can find the energy momentum tensor of the whole spacetime as:

$$T_{\mu\nu} = T^+_{\mu\nu}\theta(f(x^\mu)) + T^-_{\mu\nu}\theta(-f(x^\mu)) + T^{(0)}_{\mu\nu}\delta(f(x^\mu)). \tag{8.12}$$

Here $T^0_{\mu\nu}$ is the (four dimentional) energy momentum tensor of the shell. An advantage of this formalism is that since we have considered full four dimensional Einstein equations, unlike Darmois-Israel formalism, we do not need to worry about any constraint equation.

We can get proper three dimensional stress energy tensor of the shell by multiplying the $T^0_{\mu\nu}$ with the corrresponding vielbeins e^μ_i given in (7.10):

$$S_{ij} = e^\mu_i e^\nu_j T^{(0)}_{\mu\nu}. \tag{8.13}$$

For a more detailed discussion of distributional formalism we refer [91], where the equivalance of both formalisms is also demonstrated in that work.

8.2. Multiple Photonic Thin Shells Surrounding a Line Singularity

Let us first re-derive the photonic shell solution around a line singularity using distributional method. We choose the interior and the exterior regions of the infinitely long thin cylindrical shell with radius r_1 to be two different Levi-Civita metrics in Kasner form with the metrics:

$$ds_-^2 = -(\frac{r}{r_1})^{2a}dt^2 + dr^2 + (\frac{r}{r_1})^{2b}dz^2 + (\frac{r}{r_1})^{2c}\alpha^2 r_1^2 d\phi^2 \quad (r < r_1), \tag{8.14}$$

and

$$ds_+^2 = -(\frac{r}{r_1})^{2a'}dt^2 + dr^2 + (\frac{r}{r_1})^{2b'}dz^2 + (\frac{r}{r_1})^{2c'}\alpha^2 r_1^2 d\phi^2 \quad (r > r_1), \tag{8.15}$$

where a, b, c and a', b', c' satisfy the relations (8.2). Since we study photonic shells we have to use same radial coordinate, r, as shown in previous chapter, for the both sides of the shell. The function representing the location of the shell is $f(x^\mu) = r - r_1$. We can define an infinitely thin and long shell if the first Lichnerowitz condition (8.5) holds. Then the first derivatives may be discontinuous at $r = r_1$. These discontunitues give rise to an infinitely thin shell. We can combine (8.14) and (8.15) in the form :

$$ds^2 = -A^2(r)dt^2 + dr^2 + B^2(r)dz^2 + C^2(r)d\phi^2, \tag{8.16}$$

with

$$A(r) = (r/r_1)^a \ \theta(r_1 - r) + (r/r_1)^{a'} \ \theta(r - r_1), \tag{8.17}$$

$$B(r) = (r/r_1)^b \ \theta(r_1 - r) + (r/r_1)^{b'} \ \theta(r - r_1), \tag{8.18}$$

and

$$C(r) = [(r/r_1)^c \ \theta(r_1 - r) + (r/r_1)^{c'} \ \theta(r - r_1)] r_1 \alpha, \tag{8.19}$$

where $\theta(x - x_0)$ is the Heaviside step function.

If we put these functions into the Einstein tensor (4.9-4.12) of the metric (8.16), since the exterior and interior regions are vacuum, we see that the only surviving terms are the terms which contain Dirac delta functions giving the energy momentum tensor of the shell. The nonzero elements of $G^0_{\mu\nu}$ in an orthonormal basis are:

$$G^0_{00} = \frac{a' - a}{r_1} \ \delta(r - r_1) = 8\pi T_{00} \ \delta(r - r_1), \tag{8.20}$$

$$G^0_{22} = \frac{b - b'}{r_1} \ \delta(r - r_1) = 8\pi T_{22} \ \delta(r - r_1), \tag{8.21}$$

$$G^0_{33} = \frac{c - c'}{r_1} \ \delta(r - r_1) = 8\pi T_{33} \ \delta(r - r_1). \tag{8.22}$$

For the Einstein equation

$$G_{\mu\nu} = 8\pi \ T_{\mu\nu}, \tag{8.23}$$

the energy momentum tensor of the shell has the form

$$T^{\mu(0)}_{\nu} = \mathrm{diag}(-\rho, 0, p_z, p_\phi), \tag{8.24}$$

where ρ is the energy density and p_z, p_ϕ are the principal pressures. Notice that the formalim ensures that p_r is necessarily zero. As we have shown in the previous chapter the energy momentum tensor of the shell satisfies the condition

$$T^{\mu(0)}_{\mu} = T = 0, \tag{8.25}$$

and this result has been interpreted as an infinitely long thin shell along the z direction with radius r_1 composed of equal amount of oppositely moving photons along a helical direction around a line singularity. The corresponding energy momentum tensor in the form of pure radiation have also been presented in previous chapter. This combined motion of photons gives rise to pressures in the ϕ and z directions with the equation of state $\rho = p_z + p_\phi$. Thus if one chooses the interior and the exterior metrics of the shell as Levi-Civita metrics in Kasner form (8.14),(8.15) with same radial coordinate, then the shell is necessarily composed of massless particles. Since we have $a, a' > 0$ if $a' > a$ the shell has positive energy density. The singularity at $r = 0$ has positive energy density. Choosing $b = b'$ in (8.21) give rise to the solution $\rho = p_\phi$ with other components of the $T^{(0)}_{\mu\nu}$ vanishing. This can be interpreted as equal amount of oppositely rotating photons along a circular path and the relations between the parameters of the interior and exterior metrics (8.14),(8.15) become $a = c', b = b'$ and $c = a'$. The shell has positive energy density for $a' > a$ and the line singularity has again positive effective mass density. We can find the solution where the shell with photons counter moving along the z direction with choosing $c = c'$ in (8.22) where the nonzero components of the $T^{(0)}_{\mu\nu}$ are $\rho = p_z$ and the relation between the parameters of the interior and exterior metrics (8.14),(8.15) are $a = b', b = a'$ and $c = c'$. But this time, either shell or line singularity has negative energy density [78].

Now, let us discuss the two concentric thin shells around a line singularity. The two shells seperate spacetime into three vacuum regions which can be characterized by three different Levi-Civita metrics in Kasner form (8.1) with the metric parameters $a, b, c, a', b', c', a'', b'', c''$ which satisfy the relations (8.2). These metrics can be combined as (8.16) where the metric is continuous but its first derivatives are discontinuous with:

$$A(r) = \theta(r_1 - r)(\frac{r}{r_1})^a + \theta(r - r_1)\theta(r_2 - r)(\frac{r}{r_1})^{a'} + \theta(r - r_2)(\frac{r}{r_2})^{a''}(\frac{r_2}{r_1})^{a'}. \tag{8.26}$$

To obtain $B(r)$ replace a with b and to obtain $C(r)$ replace a with c and multiply with αr_1 in (8.26). The nonzero components of the Einstein tensor are:

$$G_{00}^{(0)} = (\frac{a' - a}{r_1})\ \delta(r - r_1) + (\frac{a'' - a'}{r_2})\ \delta(r - r_2), \tag{8.27}$$

$$G_{zz}^{(0)} = (\frac{b - b'}{r_1})\ \delta(r - r_1) + (\frac{b' - b''}{r_2})\ \delta(r - r_2), \tag{8.28}$$

$$G_{\phi\phi}^{(0)} = (\frac{c - c'}{r_1})\ \delta(r - r_1) + (\frac{c' - c''}{r_2})\ \delta(r - r_2). \tag{8.29}$$

Since the energy-momentum tensor satisfies (8.25), we have two infinitely thin photonic cylindrical shells with radii r_1 and r_2 where photons counter moving along a helical path around a line singularity. The first and second shells have positive energy density for $a' > a$ and $a'' > a'$. Thus if we have $a'' > a' > a$ both of the two photonic shells have positive energy density.

We can construct three photonic shells around a line singularity by choosing the continuous function $A(r)$ as:

$$\begin{aligned} A(r) = &\ \theta(r_1 - r)(\frac{r}{r_1})^a + \theta(r - r_1)\ \theta(r_2 - r)\ (\frac{r}{r_1})^{a'} \\ &+\theta(r - r_2)\theta(r_3 - r)(\frac{r}{r_2})^{a''}(\frac{r_2}{r_1})^{a'} + \theta(r - r_3)(\frac{r}{r_3})^{a'''}(\frac{r_3}{r_2})^{a''}(\frac{r_2}{r_1})^{a'}. \end{aligned} \tag{8.30}$$

To obtain $B(r)$ replace a with b and to obtain $C(r)$ replace a with c and multiply with αr_1 in (8.30). The nonzero components of the Einstein tensor are calculated as:

$$G_{00}^{(0)} = (\frac{a' - a}{r_1})\delta(r - r_1) + (\frac{a'' - a'}{r_2})\delta(r - r_2) + (\frac{a''' - a''}{r_3})\delta(r - r_3), \tag{8.31}$$

$$G_{zz}^{(0)} = (\frac{b - b'}{r_1})\delta(r - r_1) + (\frac{b' - b''}{r_2})\delta(r - r_2) + (\frac{b'' - b'''}{r_3})\delta(r - r_3), \tag{8.32}$$

$$G_{00}^{(0)} = (\frac{c-c'}{r_1})\delta(r-r_1) + (\frac{c'-c''}{r_2})\delta(r-r_2) + (\frac{c''-c'''}{r_3})\delta(r-r_3). \tag{8.33}$$

For this case $T_{\mu\nu}^{(0)}$ again satisfy (8.25) thus we have three photonic shells around a line singularity with photons moving along helical path. If we have $a''' > a'' > a' > a$ both of the three photonic shells have positive energy density.

Thus, using this method, one can construct a solution where a line singularity (for $a = b = 0, c = 1, \alpha < 1$ a cosmic string) at $r = 0$ is surrounded by n infinitely thin cylindrical photonic shells with photons counter moving along a helical path. All shells satisfy the positive energy condition if the metric parameters satisfy the condition

$$a^{(n)} > \ldots > a''' > a'' > a' > a. \tag{8.34}$$

Due to the relations (8.2) the maximum value that $a^{(n)}$ can have is 1 and in this case the exterior region of the multiple shells ($r > r_n$) is determined by the metric:

$$ds^2 = -r^2dt^2 + dr^2 + dz^2 + d\phi^2. \tag{8.35}$$

This metric is the Rindler's metric [29] which describes static plane symmetric vacuum spacetime [32]. It corresponds to a uniform gravitational field and test particles are uniformly accelerated in this field whereas the Riemann tensor is identically zero. Thus the line singularity may be surrounded by multiple photonic shells until they all appear as an infinitely long static plane wall.

We can also construct n photonic shells with counter rotating photons ($p_z = 0$) or counter moving photons along axial direction ($p_\phi = 0$) around a line singularity but if one shell has positive energy density then the next one has negative energy density or vice versa thus these solutions cannot be physically relevant. Notice that if interior singularity is a cosmic string ($a, b = 0, c = 1$), for a shell with counter rotating photons, the exterior metric reduces to (8.35) since for a counter rotating shell $a' = c,\ b' = b,\ c' = a$.

Our results can be summarized as follows. When a cosmic string is surrounded by a single photonic shell composed of circular counter rotating photons, from the outside the photonic shell looks like a plane wall. However when the photons are helical, multiple photonic shells are possible. There is no photonic shell with axial photons around a cosmic string. For a general line singularity a single photonic shell is always possible. Multiple photonic shells, however, require that the photons are helical, i.e. multiple photonic shells are not possible if photons move purely circularly or axially. In all cases the photonic shells may be terminated by an outermost photonic shell, which, from the outside looks like a planar wall.

9. A CYLINDRICAL THICK SHELL AROUND THE LINE SINGULARITY: AN APPROXIMATE PHOTONIC SOLUTION

In this Chapter, we will try to replace the infinitely thin shell solution composed of rotating photons around a line singularity with a solution with a smooth hollow cylindrical material with inner radius m and outer radius n around a line singularity [78]. We again choose the metrics of the inside and the outside of the cylinder to be the same as that of our thin shell solution (8.14), (8.15) except rescaling the coordinates and adding some constants to satisfy the continuity. In between we use a third metric which may represent the cylindrical material.

$$ds_-^2 = -r^{2c}dt^2 + dr^2 + r^{2b}dz^2 + \alpha^2 r^{2a}d\phi^2 \quad (r < m), \tag{9.1}$$

$$ds_0^2 = -e^2(r)dt^2 + dr^2 + f^2(r)dz^2 + \alpha^2 g(r)^2 d\phi^2 \ (m \le r \le n), \tag{9.2}$$

$$ds_+^2 = -C^2 r^{2a}dt^2 + dr^2 + r^{2b}dz^2 + \alpha^2 F^2 r^{2c}d\phi^2 \quad (n < r). \tag{9.3}$$

We thus want the cylindrical material to be smooth so the metric and its first derivatives should be continuous at $r = m$ and $r = n$. Then we find following boundary conditions:

$$e(m) = m^c, \quad f(m) = m^b, \quad g(m) = m^a, \tag{9.4}$$

$$e(n) = Cn^a, \quad f(n) = n^b, \quad g(n) = Fn^c, \tag{9.5}$$

$$e_r(m) = cm^{c-1}, \quad f_r(m) = bm^{b-1}, \quad g_r(m) = am^{a-1}, \tag{9.6}$$

$$e_r(n) = nCn^{a-1}, \quad f_r(n) = bn^{b-1}, \quad g_r(n) = cFn^{c-1}, \tag{9.7}$$

where C and F are constants to be determined from these boundary conditions. Since we have to match the metrics and their first derivatives at the boundaries we simply take $e(r)$ and $g(r)$ to be quadratic functions whose coefficients are determined by the boundary conditions. Since the dz^2 part of the metric is the same inside and outside we also choose $f = r^b$. We use the following ansatz for $e(r), f(r)$, and $g(r)$:

$$e(r) = cm^{c-1}r + A + \frac{(r-m)^2}{(n-m)^2}(Can^{a-1} - (cm^{c-1}n + A) + B), \tag{9.8}$$

$$f(r) = r^b, \tag{9.9}$$

$$g(r) = Fcn^{c-1}r + E + \frac{(r-n)^2}{(m-n)^2}G, \tag{9.10}$$

where

$$A = m^c(1-c), \tag{9.11}$$

$$B = Cn^{a-1}(n-a), \tag{9.12}$$

$$C = \frac{cm^{c-1}(n-m) + 2m^c}{an^{a-1}(m-n) + 2n^a}, \tag{9.13}$$

$$E = Fn^c(1-c), \tag{9.14}$$

$$G = -F(cn^{c-1}(m-n) + n^c) - m^a, \tag{9.15}$$

and

$$F = -n\frac{2m^a + am^{a-1}(n-m)}{cn^{c-1}(n-m) - 2n^2}. \tag{9.16}$$

Next we calculate the nonzero elements of the Einstein tensor between $m < r < n$. present a series of graphs showing the components of the Einstein tensor as a function of r for five different values of b (Figure 9.1-9.5) for $m = 1$ and $n = 2$. We consider the energy momentum tensor of this thick shell in the form (8.24). Note that for the thin shell solution we have only $\rho = p_\phi \neq 0$. For the thick shell case, since we present an approximate solution, the pressure in other directions also contribute to the energy-momentum tensor. We show that their contribution is small compared to the energy density and azimuthal stress. The radial pressure p_r should vanish at the boundaries.

Note that for $-1/3 < b \leq -0.235$ and $0.315 \leq b \leq 1$ our solution satisfies only the weak energy condition since at some region $p_\phi > \rho$. For $-0.235 < b < 0.315$ all energy conditions are satisfied. As b goes from -1/3 to 0 the energy density increases and reaches a finite maximum value at $b = 0$. Then as b increases to 1 the energy decreases towards zero. Having a maximum finite energy density with b is an expected behaviour since previous shell solutions have this behaviour too. If we accept the dominant energy condition as physically relevant energy condition, because otherwise the local speed of sound can be greater than the speed of light [82] when $p_i > \rho$, then our solution is physically acceptable only for $-0.235 < b < 0.315$.

Now let us present the expressions of the nonzero components of the Einstein tensor of the metric (9.2) for $r_1 = 1$, $r_2 = 2$ for different values of the parameters a, b, c.

For $b = -0.3$, $a = 0.83$, $c = 0.47$ (Figure 9.1):

$$\begin{aligned}
G_{00} &= \frac{-1.8r^2 + 0.52r - 0.095}{r^2(r-0.044)(r-5.6)}, \\
G_{rr} &= \frac{2.8\,(r+2.1)\,(r-1)(r-2)}{r\,(r-0.045)\,(r-5.6)\,(r^2+3.3\,r+7)},
\end{aligned}$$

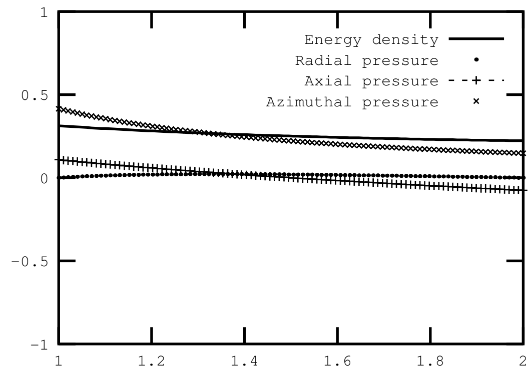

Figure 9.1. The nonzero components of the Einstein tensor for $b = -0.3$

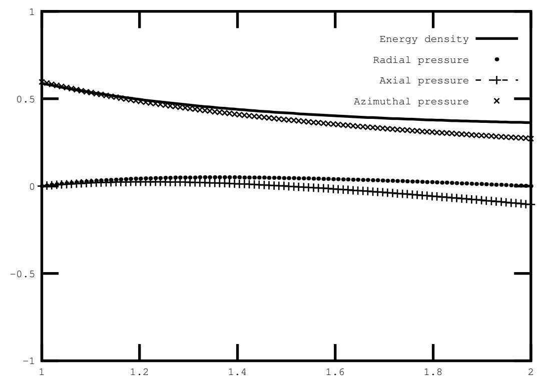

Figure 9.2. The nonzero components of the Einstein tensor for $b = -0.23$

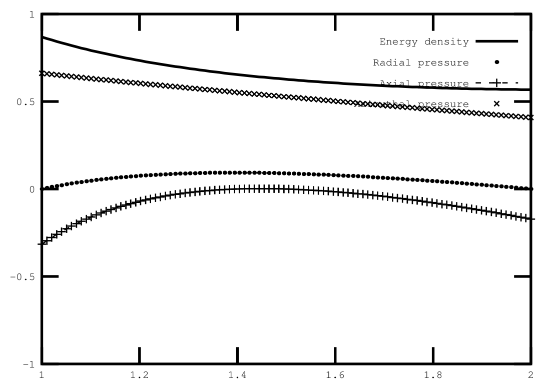

Figure 9.3. The nonzero components of the Einstein tensor for $b = -0.1$

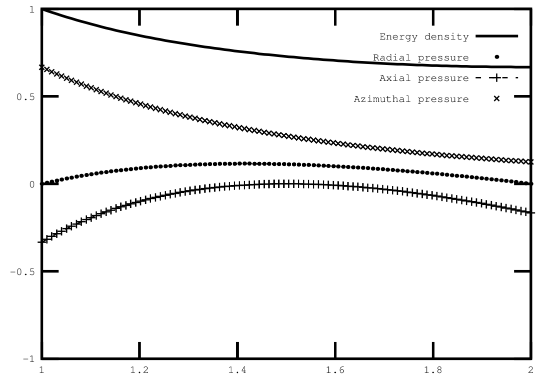

Figure 9.4. The nonzero components of the Einstein tensor for $b = 0$

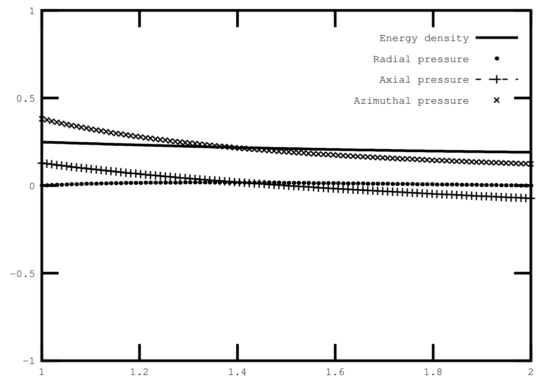

Figure 9.5. The nonzero components of the Einstein tensor for $b = 0.5$

$$
\begin{aligned}
G_{zz} &= \frac{-4.2 - 9.2\ r + 8\ r^2}{(r^2 + 3.3r + 7.0)(r - 0.044)(r - 5.6)}, \\
G_{\phi\phi} &= \frac{2.6 + 0.29\ r + 1.8\ r^2}{r^2(r^2 + 3.3\ r + 7.0)}.
\end{aligned} \tag{9.17}
$$

For $b = -0.23$, $a = 0.92$, $c = 0.32$ (Figure 9.2):

$$
\begin{aligned}
G_{00} &= \frac{-1.8\, r^2 + 0.26\, r - 0.22}{r^2\,(r - 0.17)(r - 4.6)}, \\
G_{rr} &= \frac{3.1\,(r + 0.85)\,(r - 1)\,(r - 2)}{r\,(r - 0.17)\,(r - 4.6)\,(r^2 - 0.32\, r + 4.6)}, \\
G_{zz} &= \frac{8\,(r - 1)\,(r - 1.5)}{(r - 0.17)\,(r - 4.6)\,(r^2 - 0.32\, r + 4.6)}, \\
G_{\phi\phi} &= \frac{25 - 0.34\ r + 35\ r^2}{(87 - 6\ r + 19\ r^2)r^2}.
\end{aligned} \tag{9.18}
$$

For $b = -0.1$, $a = 0.99$, $c = 0.11$ (Figure 9.3):

$$
\begin{aligned}
G_{00} &= \frac{-22 + 8.7\ r - 430\ r^2}{r^2\ (200 - 940\ r + 230\ r^2)}, \\
G_{rr} &= \frac{3.6\,(r + 0.26)(r - 1)(r - 2)}{r\,(r - 0.25)\,(r - 3.9)(r^2 - 1.6\, r + 4.1)}, \\
G_{zz} &= \frac{8\,(r - 1.4)\,(r - 1.5)}{(r - 0.25)\,(r - 3.9)(r^2 - 1.6\, r + 4.1)}, \\
G_{\phi\phi} &= \frac{13 - 0.47\ r + 55\ r^2}{r^2\ (120 - 47\ r + 29\ r^2)}.
\end{aligned} \tag{9.19}
$$

For $b = 0$, $a = 1$, $c = 0$ (Figure 9.4):

$$
\begin{aligned}
G_{00} &= \frac{-2}{1 - 4\ r + r^2}, \\
G_{rr} &= \frac{4(r - 1)(r - 2)}{(4 - 2r + r^2)(1 - 4r + r^2)}, \\
G_{zz} &= \frac{18 - 24r + 8r^2}{(4 - 2r + r^2)(1 - 4r + r^2)}, \\
G_{\phi\phi} &= \frac{2}{4 + r^2 - 2r}.
\end{aligned} \tag{9.20}
$$

For $b = 0.5$, $a = 0.81$, $c = -0.31$ (Figure 9.5):

$$
\begin{aligned}
G_{00} &= \frac{-5.2 + 120\ r - 360\ r^2}{r^2\ (13 - 1200\ r + 200\ r^2)}, \\
G_{rr} &= \frac{2.8\ (r + 2.8)\ (r - 1)(r - 2)}{r\ (r - 0.011)\ (r - 6)\ (r^2 + 5.5r + 8.5)}, \\
G_{zz} &= \frac{8.2\ (r + 1.3)\ (r - 1.5)}{(r - 0.022)\ (r - 6)\ (r^2 + 5.5\ r + 8.5)}, \\
G_{\phi\phi} &= \frac{1.8\ r^2 + 0.52\, r + 3.4}{r^2\ (r^2 + 5.5\, r + 8.5)}.
\end{aligned}
\tag{9.21}
$$

10. GRAVITATIONAL FIELD OF A BEAM OF LIGHT

10.1. Review of Gravitational Field of a Pencil of Light

The gravitational field of a steady beam of light (pure radiation) directed to a certain direction is given in the linearized approximation by Tolman [92] and for full theory by Peres [93] and Bonnor [94]. The linearized solution was derived using the corresponding Einstein-Maxwell solutions. The linearized solution can also be derived by boosting the cylindrical vacuum solution with the velocity of light as shown by Peres [93]. Let us consider the Levi-Civita solution:

$$ds^2 = -\rho^{4\sigma} dt^2 + \rho^{4\sigma(2\sigma-1)}(d\rho^2 + dz^2) + Q^2 \rho^{2(1-2\sigma)} d\phi^2, \tag{10.1}$$

which describes a static infinite cylindrical source. If we translate the cylinder in its own direction by applying the following transformations:

$$t = t' \cosh a + z' \sinh a, \tag{10.2}$$

$$z = z' \sinh a + t' \cosh a, \tag{10.3}$$

and let $Q \to 1$, $\sigma \to 0$, $a^2 \to \infty$ but $\sigma e^{2a} = 2M =$ constant we obtain:

$$ds^2 = -dt'^2 + dr^2 + r^2 d\phi^2 + dz'^2 - 8M \ln r (dt - dz)^2. \tag{10.4}$$

This metric was interpreted as describing a pencil of light by Tolman in linear approximation, if diffraction effects are ignored. Here M is related with the intensity of the light beam. Actually one can generalize this solution to describe several parallel pencils of light by considering the metric in the form:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 - 4 \sum_n M_n \ln \left[(x - x_n)^2 + (y - y_n)^2 \right] (dt - dz)^2, \tag{10.5}$$

which satisfies vacuum equations except at $x = x_n$, $y = y_n$. This solution also verifies that the paralel light beams shining in the same direction have no gravitational interaction.

The general relativistic solution corresponding to a light beam is given by:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 - A(x, y, v)(dt - dz)^2, \tag{10.6}$$

where x, y, z, t can be taken as cartesian coordinates. The metric function A depends in general on the coordinates x, y and the null coordinate $v = t - z$. Note that this form of the metrics are also called as the metrics obeying Kerr-Schild ansatz [95],[17]. If A satisfies

$$A_{xx} + A_{yy} = 0, \tag{10.7}$$

this solution describes a vacuum solution. If A does not satisfy this equation, then nonzero components of the Einstein tensor yield:

$$G_{tt} = -G_{tz} = G_{zz} = \frac{A_{xx} + A_{yy}}{2} = \rho. \tag{10.8}$$

Thus, the energy momentum tensor of this solution corresponds to null dust with:

$$T_{\mu\nu} = \rho k_\mu k_\nu; \quad k^\mu = (-1, 0, 0, 1). \tag{10.9}$$

We can have a stationary beam of light if we take $A = A(x, y)$. Different choices of A lead to different energy densities for the beam. For example choosing A as a sum of two polynomials up to second order and each depending only on x and y gives a beam with constant density profile.

Let us investigate the Newmann-Penrose Weyl, Ricci and spin coefficients by employing a null tetrad. We first transform the metric into a more suitable form by introducing two null coordinates

$$u = t + z, \quad v = t - z. \tag{10.10}$$

The metric (10.6) becomes:

$$ds^2 = -dudv + dx^2 + dy^2 - A(x, y, v)dv^2. \tag{10.11}$$

Now let us introduce a null tetrad

$$\mathbf{l} = \frac{dv}{2}, \quad \mathbf{n} = \frac{1}{2}(du + Adv), \quad \mathbf{m} = \frac{1}{2}(dx + idy), \tag{10.12}$$

with the metric

$$ds^2 = \mathbf{l} \otimes \mathbf{n} + \mathbf{m} \otimes \bar{\mathbf{m}}. \tag{10.13}$$

In this frame the all spin coefficinets vanish except:

$$\nu = \frac{-A_x + iA_y}{\sqrt{2}}, \quad \bar{\nu} = \frac{-A_x - iA_y}{\sqrt{2}}. \tag{10.14}$$

Hence in this spacetime the null congruences are shearfree, nonexpanding and have vanishing twist. The nonvanishing Ricci and Weyl scalars are:

$$\Phi_{22} = \frac{A_{xx} + A_{yy}}{2}, \tag{10.15}$$
$$\Psi_4 = \frac{1}{2}\left(A_{xx} - A_{yy} + 2iA_{xy}\right). \tag{10.16}$$

We have a transverse wave component in $\mathbf{n}$ direction. This spacetime is Petrov Type N, and for vanishing Ψ_4 (constant A) it is Petrov type O. Petrov type N solutions admitting a shearfree, non-expanding, non-twisting null congruence of the form (10.6) or (10.11) are called plane-fronted gravitational waves with parallel rays (pp-waves) [17]. They were first discovered by Brinkmann [96]. A comprehensive geometrical treatment was presented by Ehlers and Kundt [97]. As we summarized above, Peres and Bonnor demonstrated that parallel light beams can be sources of these plane waves by considering several sources generating these wave solutions. An example of this form is a beam of light having cylindrical cross section [94].

10.1.1. A Beam of Photons with Cylindrical Cross Section

The field of a steady uniform beam symmetrical about z axis and shining in the positive z direction is presented by Bonnor[94]. By employing cylindrical coordinates satisfying $r = (x^2 + y^2)^{1/2}$ the metric (10.6) can be written as:

$$ds^2 = -dt^2 + dr^2 + r^2 d\phi^2 - A(r)(dt - dz)^2. \tag{10.17}$$

The solutions of this metric represent a beam of photons in z direction. The energy density becomes:

$$\rho = \frac{1}{2}\left(A_{rr} + \frac{A_r}{r}\right). \tag{10.18}$$

For a beam with constant energy density, the exterior and the interior regions are given by [94]:

$$A_e = 8M \log r/a + 4M, \quad (r \geq a), \tag{10.19}$$

$$A_i = 4Mr^2/a^2, \quad (r \leq a). \tag{10.20}$$

Here $A_{e(i)}$ denotes the exterior (interior) metric function, M denotes the constant energy density per unit length. The interior solution has constant energy density and smoothly matches to the exterior vacuum metric at the boundary three surface $r = a$.

10.1.2. A Plane of Photons Moving in z Direction

Here we present a solution where a plane at $x = 0$ composed of photons moving along z direction in (y, z) plane with constant energy density in cartesian coordinates. Let us consider the metric (10.6). We will choose the function $F(x, y)$ such that the metric has mirror symmetry about x axis. Thus we will consider the following ansatz for A:

$$A(x, y) = x\,[\theta(x) - \theta(-x)]\,K[y]. \tag{10.21}$$

In order that this solution to represent an infinitely thin plane, K must be at most linear in x but we see that we have to choose K as a constant; otherwise the energy density of the plane blows up as $y \to \infty$. Thus we choose

$$K = a. \tag{10.22}$$

The energy momentum tensor becomes

$$T_{tt} = T_{zz} = -T_{tz} = \kappa a \delta(x), \tag{10.23}$$

where energy density is constant.

10.2. Gravitational Field of a Circulating Light Beam

The gravitational field of incoherent light or flux of photons is induced by the energy momentum tensor $T_{\mu\nu} = m k_\mu k_\nu$ in geometric optics limit where m is the energy density and k^μ is a null vector representing the propagation direction of the null dust. In a recent paper [98], Mallet presented a solution where a cylindrically symmetric beam of light is circulating around the axis and forming a cylinder. This solution has an interesting property that it admits closed timelike curves (CTC). This property was later studied [99] by Olum and Everett. They found that the radius of the CTC must be far greater than current radius of the universe. Also the solution is singular at the axis hence one cannot construct a time machine starting from Minkowski spacetime.

In this chapter a solution of this form is investigated. The existence of circular null geodesics in cylindrically symmetric vacuum spacetimes suggest that a beam of light can follow circular path and due to the symmetry, it forms a cylinder of photons. Mallet presented such a solution by solving full Einstein equations. When solving them second derivative of a combination of metric functions with respect to radial coordinate vanish. Mallet chose this combination proportional to the radial coordinate and found the exterior and interior solutions. Here we will chose another possibility that it can be a constant, which was also suggested in [99]. This leads to a solution with one free function and also, this solution is similar to the pp-wave solutions of the Kerr-Schild form.

10.2.1. Field Equations and the Solution

Let us start with the general cylindrically symmetric stationary nondiagonal metric:

$$ds^2 = -f\,dt^2 + 2w\,dt\,d\phi + l\,d\phi^2 + e^{\mu}\,(dr^2 + dz^2), \tag{10.24}$$

where f, k, l, w are the functions of the radial coordinate r. This metric can also be written in an orthonormal basis as follows:

$$ds^2 = \eta_{ab}\,e^a\,e^b, \quad e^0 = A(dt - Bd\phi),\ e^1 = Cdr,\ e^2 = Cdz,\ e^3 = Dd\phi, \tag{10.25}$$

where $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$ is the Minkowski metric and

$$A = \sqrt{f},\ B = \frac{w}{f},\ C = e^{\mu/2}, D = \sqrt{\frac{lf + w^2}{f}}. \tag{10.26}$$

In this basis, the field equations read:

$$R_{00} = \left(\frac{A''}{A} + \frac{A'D'}{AD} + \frac{A^2B'^2}{2D^2}\right)\frac{1}{C^2}, \tag{10.27}$$

$$R_{11} = \left(-\frac{A''}{A} - \frac{C''}{C} - \frac{D''}{D} + \frac{A'C'}{AC} + \frac{A'D'}{AD} + \frac{A^2B'^2}{2D^2} + \frac{C'^2}{C^2}\right)\frac{1}{C^2}, \tag{10.28}$$

$$R_{22} = \left(-\frac{C''}{C} - \frac{A'C'}{AC} - \frac{C'D'}{CD} + \frac{C'^2}{C^2}\right)\frac{1}{C^2}, \tag{10.29}$$

$$R_{33} = \left(-\frac{D''}{D} - \frac{A'D'}{AD} + \frac{A^2B'^2}{2D^2}\right)\frac{1}{C^2}, \tag{10.30}$$

$$R_{03} = \left(B'' + \frac{3A'B'}{A} - \frac{B'D'}{D} +\right)\frac{A}{2C^2D}. \tag{10.31}$$

We will try to find solutions satisfying the following relations:

$$R_{11} = R_{22} = 0, -R^0{}_0 = R^3{}_3 = R^0{}_3. \tag{10.32}$$

This constraints give a solution with vanishing Ricci scalar R. The relation $R_{00} = R_{33}$ yields:

$$(AD)_{rr} = 0. \tag{10.33}$$

This equation has two solutions $AD \sim r$ or $AD = a =$ const. Mallet has chosen $AD = \Delta = r$ and found an exact solution [98]. Here we choose the other possibility:

$$A\,D = c_0, \tag{10.34}$$

where c_0 is a constant. Using this, $R_{22} = 0$ yields

$$C = e^{c_1 r}, \tag{10.35}$$

and from $R_{11} = 0$ we find

$$B = \frac{\pm c_0}{A^2} + c_2. \tag{10.36}$$

Here $\pm$ signs determine the direction of rotation and we have to choose minus sign to satisfy (10.32). For the metric (10.25), the solutions (10.34-10.36) satisfies the relations (10.32) for a given $A(r)$, and the nonzero components of the Einstein tensor become:

$$-R^0{}_0 = R^0{}_3 = R^3{}_3 = \rho = \left(\frac{A_{rr}}{A} + \frac{A_r^2}{A^2}\right) e^{-2c_1 r}. \tag{10.37}$$

Thus for a given A if the terms inside the parantesis are nonvanishing then this gives the desired result. If they vanish then we have an exterior vacuum solution. We can either for a given $A(r)$ find the energy density or for a given energy density find the function A. The metric reads:

$$ds^2 = -A^2 dt^2 + e^{2c_1 r}\left(dr^2 + dz^2\right) + c_2(2c_0 - c_2 A^2)d\phi^2 - 2(c_0 - c_2 A^2)d\phi dt. \tag{10.38}$$

Let us analyze some properties of the metric (10.38) using a Newmann-Penrose tetrad as follows:

$$ds^2 = \mathbf{l} \otimes \mathbf{n} + \mathbf{m} \otimes \bar{\mathbf{m}}, \tag{10.39}$$

$$\sqrt{2}\mathbf{l} = \mathbf{e}^0 + \mathbf{e}^3, \quad \sqrt{2}\mathbf{n} = \mathbf{e}^0 - \mathbf{e}^3, \quad \sqrt{2}\mathbf{m} = \mathbf{e}^1 + i\mathbf{e}^2, \tag{10.40}$$

$$e^0 = A\mathbf{dt} - AB\sqrt{2}\mathbf{d}\phi, \quad \mathbf{e}^1 = e^{c_1 r}\mathbf{dr}, \quad \mathbf{e}^2 = e^{c_1 r}\mathbf{dz}, \mathbf{e}^3 = D\mathbf{d}\phi. \tag{10.41}$$

The nonzero components of Spin, Ricci and Weyl scalars are given by:

$$\nu = \bar{\nu} = \frac{\sqrt{2}A_r e^{-c_1 r}}{A}, \quad \alpha = \bar{\alpha} = \frac{(A_r + c_1 A)}{2\sqrt{2}A} e^{-c_1 r}, \tag{10.42}$$

$$\beta = \bar{\beta} = \frac{(A_r - c_1 A)}{2\sqrt{2}A} e^{-c_1 r}, \quad \Phi_{22} = \left(\frac{A_{rr}}{A} + \frac{A_r^2}{A^2}\right) e^{-2c_1 r}, \tag{10.43}$$

$$\Psi_4 = \left(\frac{A_{rr}}{A} + \frac{A_r^2}{A^2} - 2c_1\frac{A_r}{A}\right) e^{-2c_1 r}. \tag{10.44}$$

This shows that this spacetime is Petrov type N and it admits a shear free nonexpanding nontwisting null congruence. There is only a wave conponent in n direction. This analysis shows that the choice $\Delta =$ constant leads to a solution whose metric is in Kerr-Schild form adapted to cylindrical symmetry.

Note that we can further simplify the metric (10.38), since we can take $c_0 = c_2 = -1$ without loss of generality. This results,

$$ds^2 = -dt^2 + e^{2c_1 r}(dr^2 + dz^2) + d\phi^2 - F(dt - d\phi)^2, \tag{10.45}$$

where $F(r) = 1 - A^2(r)$. Also, for $c_1 = 0$ this metric reduces to a metric in Kerr-Schild form

$$ds^2 = -du dv + dr^2 + dz^2 - F dv^2, \tag{10.46}$$

where $u = t + \phi$, $v = t - \phi$.

10.2.2. Interior Solutions and Regularity Conditions

Now we present solutions having a cylinder with finite radius r_1 composed of this circulating light. The exterior region is represented by vacuum solution of the metric (10.45) and for this case the metric function $F(r)$ is given by:

$$F(r) = a + br. \tag{10.47}$$

For the interior cylinder we can choose several density profiles for the beam. For the metric (10.45) the energy density becomes

$$\rho = \frac{1}{2} e^{-2c1} F_{rr}. \tag{10.48}$$

In order to accept a cylindrical solution as an interior solution containing the symmetry axis, it must satisfy some regularity conditions as $r \to 0^+$ where we suppose that $r = 0$ is the symmetry axis. Hereafter we only consider $c_1 = 0$ case with the metric (10.46). Let us first derive these regularity conditions for the metric (10.46) by comparing it with the Minkowski metric in cylindrical coordinates:

$$ds^2 = -dt^2 + dr^2 + dz^2 + r^2 d\phi^2. \tag{10.49}$$

For a regular axis, the $g_{\phi\phi}$ term should proportional to r^2, which gives that it must vanish, also its first derivative with resect to radial coordinate should vanish but its second derivative should be equal to 2. The other terms g_{tt}, g_{rr} and g_{zz} should be constant and their first derivatives must vanish. For the metric under consideration (10.46) these conditions are equivalent to the conditions given below:

$$F(0) \quad = 1, \tag{10.50}$$

$$F'(0) \quad = 0, \tag{10.51}$$

$$F''(0) \quad = -2. \tag{10.52}$$

The solution should also satisfy one of the energy conditions and for this problem all conditions give

$$F''(r) \geq 0. \tag{10.53}$$

Note that the positive energy condition is already conflicting with (10.52) on the axis, which might imply that the solutions having positive density cannot be regular on the axis. However, this is not the case, since some solutions failing to satisfy this condition are sometimes accepted [17] as solutions with regular axis such as Gott-Hicshock-Linet interior cosmic string solutions [100]-[102] but with the possibility of having a (regular interior solution representing a) cosmic string at the axis. Thus, for these cases we can have a cosmic string type regular object on the axis in general.

Apart from these regularity and energy conditions, there is another condition that the metric representing the solution must have correct signature. Since for (10.46),

$g_{tt} \leq 0$ requires $(F \geq -1)$ and $g_{\phi\phi} \geq 0$ requires $F \leq 1$. Thus we must have

$$-1 < F < 1. \tag{10.54}$$

Now we show that near the axis the regularity conditions are incompatible with (10.54). The solutions satisfying the regularity conditions (10.50)-(10.51) and (10.53) cannot satisfy (10.54) for $0 < r \leq \epsilon$ where $\epsilon > 0$ is a small constant. The first condition (10.50) gives $F(0) = 1$ on the axis. The condition (10.51) gives $F'(0) = 0$, which says that the function $F(r)$ stays constant near the axis. Thus the function $F(r)$ is equal to 1 for $r \in [0, \epsilon)$. The condition $F''(0) \geq 0$ (10.53) says that F must be an increasing function of r for $r \in [0, \epsilon)$. This completes the proof since F exceeds the range from above for $r \in [0, \epsilon)$. Thus if the solution satisfies the regularity conditions and has positive energy, than near the axis we must have $g_{\phi\phi} < 0$, and the metric does not have correct signature.

10.2.3. A Particular Solution with Regular Interior

Here we present a solution where the metric (10.46) is matched to a (regular) cosmic string solution containing the axis. The solution is also matched to exterior vacuum solution smoothly. The exterior metric has the form (10.38) and it necessarily contains closed timelike curves.

10.2.3.1. Matching to a Cosmic String from Interior. Here by matching the solution (10.46) to an interior cosmic string solution, we can present a regular solution where circulating light can surround the string. We suppose that we have an interior cosmic string solution on the axis with radius r_0 with the metric[100],[101]:

$$ds_-^2 = -dt^2 + dz^2 + dr^2 + b^2 \sin^2 cr d\phi^2. \tag{10.55}$$

For $r > r_0$ we have the metric

$$ds_+^2 = -(1+F)dt^2 + dz^2 + dr^2 + (1-F)d\phi^2 + 2F dt d\phi, \tag{10.56}$$

describing a cylindrical thick shell composed of light rotating around the axis.

The nonvanishing components of the energy momentum tensor of the interior string solution satisfies

$$T^0{}_0 = T^z{}_z = -c^2. \tag{10.57}$$

The first junction conditions require the inside and outside metrics reduce to same metric on $r = r_0$ and this is satisfied if the conditions given below hold:

$$F(r_0) = 0, \quad b = 1/\sin cr_0. \tag{10.58}$$

The second junction conditions demand the first derivatives of the metrics should be the same at $r = r_0$ and these are satisfied if

$$F'(r_0) = 0, \quad c = \frac{n\pi}{2r_0}, n = 1, 3, 5, ... \tag{10.59}$$

Thus, for a suitable choosen function $F(r)$ satisfying the conditions (10.53), (10.54) for $r_0 \leq r \leq r_1$ and the junction conditions $F(r_0) = F'(r_0) = 0$ we have a regular solution with above interpretation.

10.2.3.2. A Thick Cylinder of Light with Constant Density and Vacuum Exterior. In (10.2.3.1) we have demonstrated that the solution (10.46) can be matched to a regular interior cosmic string solution at $r = r_0$, for any $F(r)$ satisfying the above conditions. Now we present a particular choice of $F(r)$ representing constant energy density:

$$F(r) = d + er + \mu r^2. \tag{10.60}$$

where $\rho = \mu \geq 0$ is constant energy density of the cylinder of light. Using the matching conditions $F(r_0) = F'(r_0)$ we can determine d and e as

$$d = -er_0 - r_0^2, \quad e = -2r_0\mu, \tag{10.61}$$

and F becomes

$$F(r) = \mu(r - r_0)^2. \tag{10.62}$$

Note that the solution is valid for the range

$$r_0 \leq r \leq r_0 + \frac{1}{\sqrt{\mu}}, \tag{10.63}$$

and the maximum radius of the cylinder of light is

$$r_c = \frac{1}{\sqrt{\mu}}. \tag{10.64}$$

Hence, when the energy density of the solution increases its radius decreases as $\mu^{-1/2}$.

Now let us match this solution to the exterior metric (10.38) at $r = r_1$ where $r_0 < r_1 < r_c$ and $F(r_1) = x$ where $x \in (0, 1)$. In terms of x we have

$$r_1 = r_0 + \sqrt{\frac{x}{\mu}}. \tag{10.65}$$

The junction conditions for the exterior metric with parameter $F_{ext} = p + qr$ and F_{int} is given in (10.62) are satisfied if the following conditions hold:

$$p = x - q\left(r_0 + \sqrt{\frac{x}{\mu}}\right), \quad q = 2\sqrt{x\mu}. \tag{10.66}$$

Thus the exterior metric function becomes

$$F(r) = 2\sqrt{x\mu}\,(r - r_0) - x. \tag{10.67}$$

In summary, we have a cosmic string on the axis with radius r_0, then for $r_0 < r < r_1$ a cylinder of rotating light with thickness $r_1 - r_0$ and for $r > r_1$ a vacuum exterior admitting gravitational waves. The exterior region necessarily contains CTCs for large $r > r_2$ since we have $g_{\phi\phi} < 0$. The distance of the minimum radius of CTC's to the axis (r_2) can be found by equating (10.67) to one:

$$r_2 = r_0 + \frac{1+x}{2\sqrt{x\mu}}. \tag{10.68}$$

The distance to the r_2 to cylinder of light is

$$r_2 - r_1 = \frac{1-x}{2\sqrt{x\mu}}. \tag{10.69}$$

Thus for a given x, when the density increases the distance of CTC's to the cylinder of light and also to the symmetry axis decrease and when the density decreases these distances increase. Note that the minimum value of r_2 is for $x = 1$ and it is

$$r_{2min} = r_0 + \frac{1}{\sqrt{\mu}} = r_{1max}. \tag{10.70}$$

Hence the spacetime metric admits CTC's just after the cylinder of light.

Let us compare our results with Mallet's solution[98]. His $g_{\phi\phi} = l$ term has the expression:

$$l = r\alpha(1 - \lambda \ln r/\alpha), \tag{10.71}$$

where λ, α are constants. Here λ is proportional to μ, the energy density of the radiation and α is proportional to the radius of a cylinder. The region where $l > 0$ is

$$r_M > \alpha\, e^{1/\lambda}. \tag{10.72}$$

To compare his results with our results we calculate the ratio r_{2min}/r_M for a light beam with same radius. Hence we take $\alpha \approx r_{2min}$. Then the ratio is

$$\frac{r_{2min}}{r_M} \approx \frac{1}{e^{1/\mu}}. \tag{10.73}$$

For very small μ this ratio is nearly zero and although r_2 is very large for both models, their model is far larger because of the exponantial dependence of μ. For large μ both models give comparable results but our model gives more smaller values for the minimum radius of CTCs compared to their results, for light beams with same radii. Another difference of our model is that we do not have a curvature singularity on the axis, and axis can be made regular by matching the solution to an appropriate regular interior solution.

Since in both models, $g_{\phi\phi}$ becomes negative for sufficiently large r, ϕ becomes a timelike coordinate. If one tranverses $z = r = constant$ circle in positive ϕ direction and returns to his starting point, he transverses a CTC and returns to the same point in earlier time. However, the regions containing CTC's is not compact in these solutions and extends to infinity, resulting a CTC's without negative energy densities required by the violation of weak energy condition as the theorems of and Tipler [103] and Hawking [104] demands. Although, we have a noncompact solution admitting CTC's, their theorems rule out compact solutions approximate to our solutions having CTC's with finite size.

11. THE GRAVITATIONAL FIELD OF A CYLINDRICAL THICK SHELL COMPOSED OF COUNTER CIRCULATING LIGHT

In a recent paper, Kramer [105] presented a solution corresponding to two counter propagating pure radiation fields in z direction. This solution was extended to the other cases by Von der Gonna and Kramer [106] and Ivanov [107] where the pure radiation is not directed along the principle eigendirections. In this chapter we will investigate a similar problem with light following a circular path [108].

There are several reasons for studying this problem. First of all it is well known that cylindrically symmetric vacuum solutions admit circular null geodesics for a specific value of one of their parameters [10, 109]. Another property is that for a certain range of this parameter the photons not moving in the purely radial direction can be trapped and cannot escape to radial infinity due to the gravitational field of the source generating these vacuum solutions [11]. The existence of the thin shell solutions composed of counter rotating photons [77], [78],[85] implies the possibility of the existence of corresponding thick shells and cylinders.

The gravitational field of incoherent light or flux of photons is induced by the energy momentum tensor $T_{\mu\nu} = mk_\mu k_\nu$ in geometric optics limit where m is the energy density and k^μ is a null vector representing the propagation direction of the null dust. If we have two beams of same intensity but opposite direction of propagation, their energy momentum tensor and tangent vectors satisfy:

$$T_{\mu\nu} = m\,(k_\mu k_\nu + l_\mu l_\nu), \quad k^\mu k_\mu = l^\mu l_\mu = 0, \quad k^\mu l_\mu = -1. \tag{11.1}$$

Let us consider the cylindrically symmetric static metric:

$$ds^2 = e^{2K}(dx^2 - dt^2) + e^{-2U}W^2 d\phi^2 + e^{2U}(dz + A d\phi)^2, \tag{11.2}$$

where the functions $K, U, W,$ and A are functions of the radial coordinate x where $x^\mu = (t, x, z, \phi)$. This metric can be put in stationary form by a complex substitution $t \to iz$, $z \to it$, $A \to iA$. Starting from this metric Kramer found the exact solution representing counter moving light beams in z direction [105]. His metric can be written in normal form as:

$$\begin{aligned} ds^2 &= dL^2 + V^{2/3}\left[dz^2 - dt^2 + L^2(1 + \frac{4}{3}\gamma^2 L^2)V^{-2}d\phi^2 - \frac{4\gamma L^2}{\sqrt{3}V}dz d\phi\right], & (11.3) \\ V &= 1 + \gamma^2 L^2, & (11.4) \end{aligned}$$

where γ is a free parameter and L is a new radial coordinate. This solution is regular on the axis ($L = 0$) and has no curvature singularities. The directions of the null radiation are given by the null vectors

$$k^m = \frac{V^{-1/3}}{\sqrt{2}}(1, 0, 1, 0), \quad l^m = \frac{V^{-1/3}}{\sqrt{2}}(1, 0, -1, 0). \tag{11.5}$$

These null vectors are geodesic, shear free, non expanding but twisting. Smooth matching of this solution to exterior Levi-Civita metric is also presented in [105].

Now, starting from the same metric (11.2), we present the solution corresponding to counter rotating pure radiation. We assume that the null vectors k^μ and l^μ have zero radial and z components, $k^x = l^x = k^z = l^z = 0$. Then the set of field equations reduce to:

$$R = 0, \quad R_{11} = 0, \quad R^{22} = 0, \quad R^{23} = 0, \tag{11.6}$$

where the consistency condition $R^{22}R^{33} - (R^{23})^2 = 0$ is automatically satisfied.

We will consider the static metric (11.2) and after a redefinition of the radial coordinate x, and renaming the metric functions, this metric can be written as:

$$ds^2 = -A(r')^2 dt^2 + K(r')^2 dr'^2 + M(r')^2 \left(dz + N(r')\, d\phi\right)^2 + L(r')^2\, d\phi^2. \tag{11.7}$$

This metric can also be written in the following form by introducing a new radial coordinate and regrouping the metric functions:

$$ds^2 = -A(r)^2 dt^2 + dr^2 + C(r)^2 dz^2 + D(r)^2 (d\phi + E(r) dz)^2. \tag{11.8}$$

Here we try to find a solution satisfying (11.6).We employ an orthonormal basis as follows:

$$e^0 = A dr, \quad e^1 = dr, \quad e^2 = C dz, \quad e^3 = D(d\phi + E dz). \tag{11.9}$$

In this basis the nonzero components of the Ricci scalar and Ricci tensor become:

$$R = -2\left(\frac{A_{rr}}{A} + \frac{C_{rr}}{C} + \frac{D_{rr}}{D} + \frac{A_r C_r}{AC} + \frac{A_r D_r}{AD} + \frac{C_r D_r}{CD} + \frac{D^2 E_r^2}{4C^2}\right), \tag{11.10}$$

$$R_{00} = \frac{A_{rr}}{A} + \frac{A_r C_r}{AC} + \frac{A_r D_r}{AD}, \tag{11.11}$$

$$R_{11} = R_{rr} = -\frac{A_{rr}}{A} - \frac{C_{rr}}{C} - \frac{D_{rr}}{D} - \frac{D^2 E_r^2}{2C^2}, \tag{11.12}$$

$$R_{22} = R_{zz} = -\frac{C_{rr}}{C} - \frac{A_r C_r}{AC} - \frac{C_r D_r}{CD} - \frac{D^2 E_r^2}{2C^2}, \tag{11.13}$$

$$R_{33} = R_{\phi\phi} = -\frac{D_{rr}}{D} - \frac{A_r D_r}{AD} - \frac{C_r D_r}{CD} + \frac{D^2 E_r^2}{2C^2}, \tag{11.14}$$

$$R_{23} = R_{z\phi} = -\frac{D}{2C}\left(E_{rr} + E_r\left(\frac{A_r}{A} - \frac{C_r}{C} + \frac{3D_r}{D}\right)\right). \tag{11.15}$$

In this basis, the constraint equations (11.6) can also be written as

$$R = 0, \;\; R_{11} = 0, \;\; R_{22} = 0, \;\; R_{23} = 0. \tag{11.16}$$

Taking E_r in (11.12) and putting in (11.10) gives:

$$(ACD)_{rr} = 0, \quad ACD = c_1 r + c_2. \tag{11.17}$$

Without loss of generality we can choose $c_1 = 1$, $c_2 = 0$ and

$$ACD = r. \tag{11.18}$$

Remaining three equations are:

$$E_r = \frac{kC}{AD^3}, \tag{11.19}$$
$$\frac{A_{rr}}{A} + \frac{C_{rr}}{C} + \frac{D_{rr}}{D} + \frac{A_r C_r}{AC} + \frac{A_r D_r}{AD} + \frac{C_r D_r}{CD} + \frac{D^2 E_r^2}{4C^2} = 0, \tag{11.20}$$
$$\frac{C_{rr}}{C} + \frac{A_r C_r}{AC} + \frac{C_r D_r}{CD} + \frac{D^2 E_r^2}{2C^2} = 0. \tag{11.21}$$

Using (11.19) and using (11.18) we can eliminate D and E in (11.20) and (11.21). Then we have two equations for two unknowns A and C:

$$\frac{A_r^2}{A^2} + \frac{C_r^2}{C^2} - \frac{A_r}{rA} - \frac{C_r}{rC} + \frac{A_r C_r}{AC} + \frac{k^2 A^2 C^4}{4r^4} = 0, \tag{11.22}$$
$$-\frac{C_{rr}}{C} + \frac{A_r^2}{A^2} - \frac{A_r}{rA} + 2\frac{C_r^2}{C^2} - 2\frac{C_r}{rC} + \frac{A_r C_r}{AC} - \frac{k^2 A^2 C^4}{4r^4} = 0. \tag{11.23}$$

We can simplify (11.23) by adding (11.22) and (11.23) which gives:

$$2\frac{C_{rr}}{C} - 2\frac{C_r^2}{C^2} + 2\frac{C_r}{rC} + \frac{k^2A^2C^4}{r^4} = 0. \tag{11.24}$$

Let us consider the following transformations

$$A(r) = \exp\int \frac{a(r)}{r}dr, \quad C(r) = \exp\int \frac{c(r)}{r}dr, \tag{11.25}$$

then (11.22) and (11.24) becomes

$$4r^2\left(a^2 - a + c^2 - c + ac\right) + k^2\left(\exp\int \frac{(2a+4c)}{r}\,dr\right) = 0, \tag{11.26}$$

$$2r^3c_r = -k^2\left(\exp\int \frac{(2a+4c)}{r}\,dr\right). \tag{11.27}$$

Differentiating (11.27) one finds

$$a(r) = \frac{1}{2}\left(3 - 4c + \frac{rc_{rr}}{c_r}\right), \tag{11.28}$$

and using (11.27) and (11.28), (11.26) becomes:

$$\left(\frac{rc_{rr}}{c_r}\right)^2 - (6c-4)\frac{rc_{rr}}{c_r} - 2rc_r + 12c^2 - 14c + 3 = 0. \tag{11.29}$$

The transformation

$$r = e^\rho, \tag{11.30}$$

puts (11.29) in the form:

$$\left(\frac{c_{\rho\rho}}{c_\rho}\right)^2 + 2(1-3c)\frac{c_{\rho\rho}}{c_\rho} - 2c_\rho + 12c^2 - 8c = 0. \tag{11.31}$$

Let us denote $c_\rho = p$ which gives $c_{\rho\rho} = pp_c$ and puts the equation in the form

$$p_c^2 + 2(1-3c)p_c - 2p + 12c^2 - 8c = 0, \tag{11.32}$$

which can also be written in the form

$$(p_c + (1-3c))^2 = 2p - 3c^2 + 2c + 1. \tag{11.33}$$

This gives two first order differential equations for $p(c)$:

$$p_c = 3c - 1 \pm \sqrt{2p(c) - 3c^2 + 2c + 1}. \tag{11.34}$$

We are able to find two solutions of above equation. They are:

$$p(c) = \frac{3}{2}\,c^2 - c - \frac{1}{2}; \quad p(c) = 2c^2 - c - \frac{1}{2} - c_1 c + \frac{1}{2}c_1^2. \tag{11.35}$$

Now we have to integrate back all these functions to find the metric functions. Since second solution yields a vacuum spacetime, let us only consider the first solution:

$$p(c) = \frac{3}{2}\,c^2 - c - \frac{1}{2}. \tag{11.36}$$

We can find $\rho(c)$ by integrating back $p(c)$:

$$\rho(c) = \int \frac{dc}{p(c)} = \int \frac{dc}{\frac{3}{2}\,c^2 - c - \frac{1}{2}} = \ln\sqrt{\frac{c-1}{3c+1}} + \ln h, \tag{11.37}$$

where h is an i ntegration constant. Using this and by considering (11.30) we can find r as a function of c:

$$r(c) = e^\rho = \sqrt{\frac{c-1}{3c+1}}\,h. \tag{11.38}$$

By inverting this relation we find:

$$c(r) = \frac{h^2 + r^2}{h^2 - 3r^2}. \tag{11.39}$$

We can find $a(r)$ by considering (11.28):

$$a(r) = \frac{2r^2}{3r^2 - h^2}. \tag{11.40}$$

The equation (11.27) gives the constant h in terms of the constant k as:

$$h = \pm\frac{ik}{4}. \tag{11.41}$$

Using the transformations (11.25), the equations (11.18) and (11.19) and choosing another constant parameter $\alpha = k/4$, we finally find the metric functions:

$$A(r) = (\alpha^2 + 3r^2)^{1/3}, \tag{11.42}$$
$$C(r) = \frac{r}{(\alpha^2 + 3r^2)^{2/3}}, \tag{11.43}$$
$$D(r) = (\alpha^2 + 3r^2)^{1/3}, \tag{11.44}$$
$$E(r) = \frac{-2\alpha}{3(\alpha^2 + 3r^2)}. \tag{11.45}$$

Using these functions the spacetime metric representing the solution can be written as:

$$ds^2 = dr^2 + A^{2/3}\left[\left(d\phi - \frac{2\lambda\, dz}{\sqrt{3}A}\right)^2 + \frac{r^2 dz^2}{A^2} - dt^2\right], \quad A = \lambda^2 + r^2. \tag{11.46}$$

When writing this metric we introduce a new parameter, $\lambda = \sqrt{3}\alpha$, and rescaled the ignorable coordinates. The null vectors k and l have the contravariant components

$$k^\mu = \frac{1}{\sqrt{2}}A^{-1/3}(-1,0,0,1), \quad l^\mu = \frac{1}{\sqrt{2}}A^{-1/3}(-1,0,0,-1), \tag{11.47}$$

and they have no coordinate components in z direction. However they have covariant components in z direction since the metric is nondiagonal. The energy density of the solution is:

$$m = \frac{4\lambda^2}{3\kappa(\lambda^2 + r^2)^2}, \tag{11.48}$$

which is a monotonically decreasing function of the radial coordinate. Energy density is maximum at $r = 0$ with $m_0 = 4/(3\kappa\lambda^2)$. The curvature scalars are well behaved everywhere including $r = 0$ since

$$K = R_{\mu\nu\kappa\delta}R^{\mu\nu\kappa\delta} = \frac{64(3\lambda^2 + r^2)}{27(\lambda^2 + r^2)^3}. \tag{11.49}$$

Let us first discuss the asymptotic behaviour of the solution. For large values of r where $r \gg \lambda$ the metric (11.46) becomes:

$$ds^2 \approx -r^{4/3}dt^2 + dr^2 + r^{-2/3}dz^2 + r^{4/3}d\phi^2. \tag{11.50}$$

This metric corresponds to the vacuum Levi-Civita solution [5, 10]

$$ds^2 = -r^{4\sigma/\Sigma}dt^2 + dr^2 + r^{4\sigma(2\sigma-1)/\Sigma}dZ^2 + \alpha^2 r^{2(1-2\sigma)/\Sigma}d\Phi^2, \quad \Sigma = 4\sigma^2 - 2\sigma + 1, \tag{11.51}$$

with $\sigma = 1/4$ where σ is the parameter related with energy density of the source generating this vacuum solution and α is the conicity parameter. This result justifies the ϕ as an angular coordinate since for $|\sigma| < 1/2$ there is an agreement [6] that this metric describes a cylindrically symmetric exterior vacuum solution [24]-[28],[83] with coordinates have their usual meanings. What is amazing about this solution is that only for this value of σ, static Levi-Civita [10] and stationary Lewis [109] metrics admit circular null geodesics!

Let us now discuss the behaviour of the solution near the axis. Actually the exchange of the role of the coordinates $z \leftrightarrow \phi$ in the metric (11.2) is locally permissable since ∂_z and ∂_ϕ are Killing vectors of cylindrical symmetry and Einstein equations are local. However, for this transformation to be acceptable globally, the derived solution should be regular on the axis. The vanishing of the scalars constructed from curvature components show that the solution (11.46) is not singular at $r = 0$, contrary to our expectations. We expect the axis to be singular representing a possibly regular source whose gravitational field causes the photons to follow circular paths, since the vacuum solution admitting null circular geodesics is also singular on the axis.

Although the solution is not singular at $r = 0$, it does not satisfy the regularity conditions of cylindrical symmetry since $g_{\phi\phi}$ does not vanish ($g_{\phi\phi} \sim \lambda^{4/3} \neq 0$) when $r \to 0^+$, which might imply that $r = 0$ is not the axis of symmetry. To get the correct behaviour we can try to introduce a new radial coordinate where (11.49) is singular with $\rho = (\lambda^2 + r^2)^{1/3}$. The metric becomes:

$$ds^2 = -\rho^2 dt^2 + \frac{9\rho^4 \, d\rho^2}{4(\rho^3 - \lambda^2)} + \rho^2 d\phi^2 + \frac{\lambda^2 + 3\rho^3}{3\rho^4} \, dz^2 - \frac{2\lambda}{\sqrt{3}\rho} \, dz d\phi. \tag{11.52}$$

Here the axis is at $\rho = 0$. The curvature scalars are singular there, and the $g_{\rho\rho}$ term changes sign for $\rho < \rho_0 = \lambda^{2/3}$ corresponding to $r < 0$. Thus, the solution (11.52) does not include the symmetry axis since it it not valid there. An interesting observation is that the minimum possible value of the interior radius where the solution is valid is directly proportional to the inverse of the maximum energy density of the shell $\sim \lambda^{-2}$. Thus when the energy density of the shell or intensity of the photons increases, the region of validity of the solution can be closer to the symmetry axis.

In order to have a globally well behaving solution, we have to match the solution smoothly either to a regular solution or a vacuum solution from interior. This can be done since the solution (11.3) can be matched smoothly [105] to cylindrically symmetric vacuum Levi-Civita solution for any value of the radial coordinate. One can show that this is also valid for (11.46) by repeating the calculations by exchanging $z \leftrightarrow \phi$. Smooth matching requires the metrics and their first derivatives to be continuous at the boundary $r = r_0$. To match the metrics (11.46) and (11.51) we have to perform a linear transformation on (11.51) $Z = q\, z + \Omega\, \phi, \quad \Phi = p\, \phi + \omega\, z$, with the constant coefficients p, q, ω, Ω. We have also the freedom to rescale the ignorable coordinates t and z and together with α and r_0 we have eight unknowns for eight equations. Thus all these unknowns can be determined by the junction conditions uniquely in terms of σ and λ. Matching for locally flat interior $\sigma = 0$ is not possible. The boundary radius r_0 can be calculated in terms of σ and λ

$$r_0 = \frac{\lambda^2 \sigma}{\sqrt{1 - 5\sigma + 4\sigma^2}}. \tag{11.53}$$

This expression shows that for a given σ the inner radius r_0 is inversely proportional to the energy density parameter λ^{-1} and matching is possible for the range $0 < \sigma < 1/4 \ \cup \ \sigma > 1$.

By considering these, our solution can be interpreted as a thick shell composed of counter rotating photons with vacuum interior, where the photons are trapped by the gravitational field of the source generating the interior Levi-Civita solution. The Levi-Civita solution is singular on the axis in general. However there exist several physically acceptable perfect fluid [6] [24]-[28], or shell [31], [65], [75]-[78],[85] solutions, regular at the axis and smoothly matching to the Levi-Civita solution from exterior. Thus, we can have a cylindrically symmetric well behaved solution from axis to radial infinity with the interpretation given above. The solution asymptotically goes to a particular cylindrical vacuum solution which admits circular null geodesics. This solution can be derived from [105] by reinterpreting the coordinates.

12. CONCLUSIONS

In this thesis we have investigated certain solutions of Einstein field equations admitting cylindrical symmetry. The energy momentum tensor of some of these solutions has the form of pure radiation. Pure radiation or null dust fields represent a flux of photons in certain direction in geometric optics limit.

After reviewing properties of pure radiation in Section 2, the cylindrical symmetry and vacuum solutions in Section 3 and the static vacuum Levi-Civita solution in Section 4, in Section 5 we have extended the Levi-Civita metric to include Kasner type time dependence where, for every value of the Levi-Civita parameter σ we have two corresponding time dependent metrics in general, except for special cases. We have analyzed the conformal structure of this spacetime using the Newman-Penrose formalism. We have also analyzed other physical parameters such as the existence of spacetime singularities by analyzing the scalars constructed from curvature elements, the radial acceleration of a test particle at rest and the existence of circular geodesics. As an application, we have presented a time dependent cosmological solution having stiff fluid equation of state, by a small modification to the Kasner generalized Levi-Civita metric.

In Section 6, using the solution presented in the previous section, we have introduced a time dependent solution representing a null radiation emitted from a cylindrical source. The physical properties mentioned in the above paragraph are also studied for this case, and compared to the static or Kasner generalized vacuum solutions. A radiating nonstatic cosmic string solution is also presented as an application to this solution.

Sections 7 and 8 are devoted to the static cylindrical thin shells composed of counter propagating pure radiation. In order to treat as general as possible, we have chosen the interior and the exterior regions of the shell as represented by two different Levi-Civita metrics. Thus, the shell in general surrounds a line singularity located on the axis but shells with flat interior are also included as a spacial case. Actually, the flat interior shell solutions are well known and introduced as a source generating exterior vaccum solution before, although in most of those works the concentration is on the exterior vacuum solution rather than the shell itself. The matter content of the shell actually determines whether values of the radial coordinates of the interior and exterior metrics agree or not. If they have the same value then the shell is necesserely composed of massless particles. The anisotropic fluid forming the shell can be interpreted as photons with either counter rotating or counter moving in axial direction or both of them. We need counter propogating photons since we only consider static vacuum exterior. The solutions satisfy certain energy conditions for certain ranges of the interior and exterior metric parameters. For the flat interior case, the counter helically moving or the counter rotating solutions satisfy energy conditions whereas the counter propogating photons in z directions do not. Also when the shell with counter moving photons in z direction has positive energy, the energy density of the interior singularity is negative. This may be due to the fact that there is no rotational pressure on the shell for this case and there is no force to keep the shell static that might require a repelling force due to interior source. The energy densities of the shell solutions have a finite maximum.

In these studies for the counterrotating shell solution we face with an interesting solution. When the interior region of the shell is (locally) flat, the exterior region of the shell is also represented by a flat metric, the Rindler metric, describing an exterior of a plane. We concluded that for an observer exterior to the shell, the cylinder becomes an infinite plane.

The multiple photonic shells are also possible if the following conditions hold. The shells must be composed of two counter streams of photons counter-propagating in z and ϕ directions. Multiple shells composed of either counter rotating or counter moving photons in axial direction is not possible. Also for each shell the exterior Levi-Civita parameter σ must increase compared to the interior one. By choosing the interior of the innermost shell as flat, one can have a well behaving spacetime on the axis. Since the parameter σ must increase for each new shell with increasing radius, the exterior metric of the outer most shell can be at most the Rindler metric. This metric describes gravitational field of a plane. Thus, by increasing the energy density of the interior spacetime by adding new shells of this form, we end up with a plane. This might be another evidence that the $\sigma = 1/2$ case actually describes a plane symmetric metric and the source generating this solution is actually a plane. Although we can choose the interior flat, the innermost region can also contain a cosmic string at the axis if we choose interior as locally flat or a general line singularity if we choose innermost region as described by another Levi-Civita metric. Then we see that, line singularities including cosmic strings may be screened by photonic shells until they appear as a planar wall.

The following sections were devoted to thick solutions composed of pure radiation. We have first introduced an approximate thick shell solution which might be interpreted as a shell composed of counter rotating photons with small contribution of an anisotropic fluid. The interior and exterior regions are chosen as two Levi-Civita metrics in Kasner form and they smoothly match to the shell. We choose that, as in the thin shell with counter-rotating photons, the parameters corresponding to the metric components g_{tt} and $g_{\phi\phi}$ interchange for interior and exterior metrics. We have introduced several figures representing the Einstein tensor for different choices of the metric parameters.

In the Section 10 we have reviewed the solution describing a pencil of light. This solution has the Kerr-Schild type of metric and it was shown by Peres [93] and Bonnor [94] that a light beam can produce the exterior gravitational field which is of the form of plane gravitational waves. It was also shown by Bonnor that when two light beams shine in same direction they do not interact whereas when they shine in opposite direction they interact nonlinearly. After reviewing these, we have presented a solution with a cylinder composed of circulating light beam. Same problem was studied by Mallet before, but we have completed his analysis by choosing a combination of metric functions as constant. This choice leads to a solution with same form as in Peres-Bonnor solution. The conditions that the solution must hold near the axis shows that the solution cannot regular near the axis and we need an interior regular solution matching smoothly to this solution. We have presented a particular model where an interior cosmic string solution is surrounded by a cylinder composed of circulating light whose exterior region is vacuum admitting gravitational waves. The closed timelike curves are necessarily obtained in this solution whose minimum distance from the axis is inversely proportional to the energy density of the cylinder of light.

As a last application, we have introduced a solution corresponding to a cylindrical thick shell composed of two counter rotating light beams. We see that for large values of radial coordinate, the solution asymptotically goes to the only Levi-Civita metric admitting circular null geodesics. For this solution the curvature scalars are not singular near the axis. However, this solution does not satisfy the regularity conditions of cylindrical symmetry, indicating that the axis is not covered by the solution, since it is not valid there. By transforming the radial coordinate we obtain a metric which is both singular at the axis as we expect, and also has wrong signature when the radial coordinate has smaller values then a critical radius proportional to the maximum energy density of the light beams. This implies that the shell of radiation cannot contain the axis. The existence of singularity indicates that there is actually a cylindrical source (posibbly physically acceptable) which can cause the photons to counter rotate around it. The shell solution can match to a vacuum from the interior and also there are regular solutions which can match to this vacuum from exterior. Thus we can have a source distribution which is regular everywhere starting from axis to radial infinity.

In this thesis we have presented some cylindrically symmetric solutions having pure radiation equation of state. As a future research, one can investigate some of these solutions in other theories such as Brans-Dicke theory [110], brane world scenarios [111], supergravity and string theories [112],[113] and compare with general relativistic solutions. For example, parallel beams of light interact linearly whereas anti parallel beams interact nonlinearly in general relativity. Whether these properties are modified in these theories or not could be interesting since the field equations are modified in these theories compared to general relativity. Another problem under investigation is to study the dynamics and stability of cylindrical shells where we consider more general settings which we believe to help to clarify the conflicting solutions previous studies present.

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